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Oscillatory Properties of Third-Order Neutral Delay Differential Equations with Noncanonical Operators

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Abstract: In the paper, we study the oscillatory and asymptotic properties of solutions to a class of third-order linear neutral delay differential equations with noncanonical operators. Via the application of comparison principles with associated first and second-order delay differential inequalities, we offer new criteria for the oscillation of all solutions to a given differential equation. Our technique essentially simplifies the process of investigation and reduces the number of conditions required in previously known results. The strength of the newly obtained results is illustrated on the Euler type equations.

Keywords: linear differential equation; delay; third-order; noncanonical operators; oscillation

MSC: 34C10; 34K11

1. Introduction

This paper deals with the oscillatory behavior of solutions to a third-order linear neutral delay differential equation of the form

$$\left(r_2(t) \left(r_1(t) y'(t) \right)' \right)' + q(t) x(\sigma(t)) = 0, \quad t \geq t_0 > 0, \quad (1)$$

where $y(t) = x(t) + p(t)x(\tau(t))$. We make the following assumptions:

Hypothesis 1 (H1). $r_i \in C^{(3-i)}([t_0, \infty), (0, \infty))$, $i = 1, 2$;

Hypothesis 2 (H2). $p, q \in C([t_0, \infty), [0, \infty))$, q does not vanish eventually, and $p(t) \geq p_0 > 1$;

Hypothesis 3 (H3). $\tau \in C^1([t_0, \infty), \mathbb{R})$ is strictly increasing, $\tau(t) \leq t$, and $\lim_{t \rightarrow \infty} \tau(t) = \infty$; $\sigma \in C([t_0, \infty), \mathbb{R})$, $\sigma(t) \leq t$, and $\lim_{t \rightarrow \infty} \sigma(t) = \infty$;

Hypothesis 4 (H4). $h(t) := \tau^{-1}(\sigma(t)) \leq t$.

For the sake of simplicity, we define the operators

$$L_0 y = y, \quad L_1 y = r_1 y', \quad L_2 y = r_2 (r_1 y')', \quad L_3 y = \left(r_2 (r_1 y')' \right)',$$

and assume without further mention that L_3y is of noncanonical type, that is,

$$\int_{t_0}^{\infty} \frac{ds}{r_1(s)} < \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{ds}{r_2(s)} < \infty. \quad (2)$$

Under a solution of Equation (1), we mean a nontrivial function $x \in C^1([T_y, \infty), \mathbb{R})$ with $T_y \geq t_0$, which has the property $L_i y \in C^1([T_y, \infty), \mathbb{R})$, $i = 0, 1, 2$, and satisfies (1) on $[T_y, \infty)$. We restrict our attention to only those solutions of (1) which exist on some half-line $[T_y, \infty)$ and satisfy the condition

$$\sup\{|y(t)| : T \leq t < \infty\} > 0 \quad \text{for any} \quad T \geq T_y.$$

We tacitly assume that (1) admits such a solution. A solution y of (1) is called oscillatory if it has arbitrarily large zeros and nonoscillatory otherwise. The equation itself is termed oscillatory if all its solutions oscillate. Following the seminal work of Kiguradze and Kondrat'ev (see, e.g., [1]), we say that (1) has *property A* if any solution x of (1) is either oscillatory or satisfies $\lim_{t \rightarrow \infty} x(t) = 0$.

Being aware of numerous indications of the practical importance of third-order differential equations as well as a number of mathematical problems involved [2], the area of the qualitative theory for such equations has attracted a large portion of research interest in the last three decades. The asymptotic properties of equations of type (1) with $p \equiv 0$ were extensively investigated in the literature, see, e.g., [3–14] and the references cited therein. Most of the papers have been devoted to the examination of so-called canonical equations, where conditions opposite to (2) hold, namely,

$$\int_{t_0}^{\infty} \frac{ds}{r_1(s)} = \int_{t_0}^{\infty} \frac{ds}{r_2(s)} = \infty. \quad (3)$$

The advantage and usefulness of a noncanonical representation of linear disconjugate operators in the study of the oscillatory and asymptotic behavior of (1) was recently shown in [15]. In 2018, Džurina and Jadlovská [16] considered a particular case of Equation (1) in noncanonical form with $p \equiv 0$ and established various oscillation criteria for Equation (1). Their method simplifies the process and reduces the number of conditions required in previously known results. A further improvement of these results was presented in [17].

Depending on various ranges of p , a variety of results for property A of (1), its generalizations or particular cases, exist in the literature, see, e.g., [13,18–27] and the references cited therein. Among them, only the work in [13] considered (1) in noncanonical form with $0 \leq p < 1$.

The question concerning the study of sufficient conditions which guarantee that all solutions of (1) in the presence of a neutral term oscillate was open until recently. First such results were established in [28] for (1) in canonical form, under the crucial requirement of $0 \leq p(t) \leq p_0 < \infty$ and the commutativity of the delayed arguments $\tau \circ \sigma = \sigma \circ \tau$. To the best of the authors' knowledge, there is nothing known regarding property A or the oscillation of all solutions of (1) under the assumptions (H_1) – (H_4) . Motivated by this observation, we attempt to fill this gap by extending the ideas exploited in [16], based on comparisons with associated first and second-order delay differential inequalities, for more general neutral equations in noncanonical form.

Hopefully, our contribution should be of interest to the reader as, contrary to the majority of results in the literature, we attain the oscillation of all solutions of (1), and find the conditions of all theorems very simple and easy to verify.

2. Main Results

For the reader's convenience, we list the functions to be used in the paper. That is, for $t \geq t_1 \geq t_0$, we put

$$\begin{aligned} R_1(t) &:= \int_{t_1}^t \frac{ds}{r_1(s)}, & R_2(t) &:= \int_{t_1}^t \frac{ds}{r_2(s)}, & R(t) &:= \int_{t_1}^t \frac{R_2(s)}{r_1(s)} ds, \\ \pi_1(t) &:= \int_t^\infty \frac{ds}{r_1(s)}, & \pi_2(t) &:= \int_t^\infty \frac{ds}{r_2(s)}, & \pi(t) &:= \int_t^\infty \frac{\pi_1(s)}{r_2(s)} ds, \\ R(t, t_1) &:= \int_{t_1}^t \frac{1}{r_1(s)} \int_s^t \frac{du}{r_2(u)} ds, & \tilde{R}(t, t_1) &:= \int_{t_1}^t \frac{1}{r_1(s)} \int_s^t \frac{du}{r_2(u) u^\gamma} ds, \end{aligned}$$

where γ is a constant satisfying

$$0 \leq \frac{p_0 \gamma}{p_0 - 1} \leq \frac{t q(t) R(t, h(t))}{p(h(t))}. \quad (4)$$

Here, we remark that the function h from (H_4) is nondecreasing with $\lim_{t \rightarrow \infty} h(t) = \infty$, which follows from $h(t) \geq \sigma(t)$ and

$$h'(t) = \frac{\sigma'(t)}{\tau'(\tau^{-1}(\sigma(t)))} \geq 0,$$

respectively. As usual, all functional inequalities are assumed to hold eventually, that is, they are satisfied for all t large enough.

We start with the following auxiliary result, which can be considered as a slight extension of [16] (Lemma 1) given for (1) with $p(t) = 0$.

Lemma 1. Suppose that (H_1) – (H_3) are satisfied and x is an eventually positive solution of Equation (1). Then

$$y(t) > x(t) \geq \frac{1}{p(\tau^{-1}(t))} \left(y(\tau^{-1}(t)) - \frac{y(\tau^{-1}(\tau^{-1}(t)))}{p(\tau^{-1}(\tau^{-1}(t)))} \right) \quad (5)$$

and the corresponding function y belongs to one of the following classes

$$\begin{aligned} y(t) \in \mathcal{S}_1 &\iff y > 0, \quad L_1 y < 0, \quad L_2 y < 0, \\ y(t) \in \mathcal{S}_2 &\iff y > 0, \quad L_1 y < 0, \quad L_2 y > 0, \\ y(t) \in \mathcal{S}_3 &\iff y > 0, \quad L_1 y > 0, \quad L_2 y > 0, \\ y(t) \in \mathcal{S}_4 &\iff y > 0, \quad L_1 y > 0, \quad L_2 y < 0, \end{aligned}$$

eventually.

Proof. Choose $t_1 \in [t_0, \infty)$ such that $x(\sigma(t)) > 0$ and $x(\tau(t)) > 0$ on $[t_1, \infty)$. From the definition of y , we see that $y(t) > x(t) > 0$ and

$$x(t) = \frac{y(\tau^{-1}(t)) - x(\tau^{-1}(t))}{p(\tau^{-1}(t))} \geq \frac{1}{p(\tau^{-1}(t))} \left(y(\tau^{-1}(t)) - \frac{y(\tau^{-1}(\tau^{-1}(t)))}{p(\tau^{-1}(\tau^{-1}(t)))} \right)$$

for $t \geq t_1$. Obviously, $L_3 y$ is nonincreasing on $[t_1, \infty)$, since

$$L_3 y(t) = -q(t)x(\sigma(t)) \leq 0.$$

Hence, $L_1 y$ and $L_2 y$ are of one sign eventually, which implies that four cases $\mathcal{S}_1$ – $\mathcal{S}_4$ are possible for y . $\square$

Next, we state an auxiliary criterion for the nonexistence of positive increasing solutions of (1). As will be shown later, this condition is already included in those eliminating solutions from the class S_1 . In the proof, we will take advantage of the useful fact

$$\lim_{t \rightarrow \infty} \frac{R(\tau^{-1}(t))}{R(t)} = \lim_{t \rightarrow \infty} \frac{R_1(\tau^{-1}(t))}{R_1(t)} = 1, \quad (6)$$

which follows from (2).

Lemma 2. Suppose that (H_1) – (H_3) are satisfied. If

$$\int_{t_0}^{\infty} \frac{\pi_2(s)q(s)}{p(h(s))} ds = \infty, \quad (7)$$

then $S_3 = S_4 = \emptyset$.

Proof. For the sake of contradiction, let (7) be satisfied but $y \in S_3 \cup S_4$. Choose $t_1 \in [t_0, \infty)$ such that $x(t) > 0$, $x(\sigma(t)) > 0$, and $x(\tau(t)) > 0$ on $[t_1, \infty)$.

Assume first that $y \in S_3$. Since L_2y is decreasing, we have

$$L_1y(t) \geq \int_{t_1}^t \frac{1}{r_2(s)} L_2y(s) ds \geq L_2y(t)R_2(t).$$

Thus,

$$\left(\frac{L_1y(t)}{R_2(t)} \right)' = \frac{L_2y(t)R_2(t) - L_1y(t)}{r_2(t)R_2^2(t)} \leq 0.$$

Therefore, L_1y/R_2 is nonincreasing on $[t_1, \infty)$ and moreover, this fact yields

$$y(t) \geq \int_{t_1}^t \frac{R_2(t)}{r_1(s)R_2(t)} L_1y(s) ds \geq \frac{L_1y(t)}{R_2(t)} R(t)$$

for $t \geq t_1$. Consequently, y/R is nonincreasing on $[t_1, \infty)$ as well, since

$$\left(\frac{y(t)}{R(t)} \right)' = \frac{L_1y(t)R(t) - y(t)R_2(t)}{r_1(t)R^2(t)} \leq 0.$$

From $\tau^{-1}(\tau^{-1}(t)) \geq \tau^{-1}(t)$, we have

$$y(\tau^{-1}(\tau^{-1}(t))) \leq \frac{R(\tau^{-1}(\tau^{-1}(t)))}{R(\tau^{-1}(t))} y(\tau^{-1}(t)). \quad (8)$$

Using this in (5), we find that

$$x(t) \geq \frac{y(\tau^{-1}(t))}{p(\tau^{-1}(t))} \left(1 - \frac{R(\tau^{-1}(\tau^{-1}(t)))}{R(\tau^{-1}(t))p(\tau^{-1}(\tau^{-1}(t)))} \right) \quad t \geq t_1.$$

By virtue of (H_2) and (6), there is a $t_2 \geq t_1$ such that for any constant $\varepsilon \in (0, p_0 - 1)$ and $t \geq t_2$,

$$\frac{R(\tau^{-1}(\tau^{-1}(t)))}{R(\tau^{-1}(t))p(\tau^{-1}(\tau^{-1}(t)))} \leq \frac{1 + \varepsilon}{p_0},$$

which implies

$$x(t) \geq \frac{y(\tau^{-1}(t))}{p(\tau^{-1}(t))} \left(1 - \frac{1 + \varepsilon}{p_0} \right) > 0. \quad (9)$$

Combining (9) with (1), we have

$$\begin{aligned} 0 &\geq L_3 y(t) + \left(1 - \frac{1+\varepsilon}{p_0}\right) \frac{q(t)}{p(h(t))} y(h(t)) \\ &\geq L_3 y(t) + k \left(1 - \frac{1+\varepsilon}{p_0}\right) \frac{q(t)}{p(h(t))}, \end{aligned} \quad (10)$$

where we used the fact that y and h are increasing, and set $k := y(\tau^{-1}(\sigma(t_2))) < y(h(t))$. Integrating (10) from t_2 to t , we obtain

$$L_2 y(t) \leq L_2 y(t_2) - k \left(1 - \frac{1+\varepsilon}{p_0}\right) \int_{t_2}^t \frac{q(s)}{p(h(s))} ds. \quad (11)$$

On the other hand, from (2) and (7), it follows that

$$\int_{t_0}^{\infty} \frac{q(s)}{p(h(s))} ds = \infty,$$

which in view of (11) contradicts the positivity of $L_2 y$.

Now, assume that $y \in \mathcal{S}_4$ for $t \geq t_1$. Using the monotonicity of $L_1 y$, we find

$$y(t) \geq \int_{t_1}^t \frac{1}{r_1(s)} L_1 y(s) ds \geq L_1 y(t) R_1(t).$$

Thus, one can see that

$$\left(\frac{y(t)}{R_1(t)}\right)' = \frac{L_1 y(t) R_1(t) - y(t)}{r_1(t) R_1^2(t)} \leq 0,$$

which implies that y/R_1 is nonincreasing on $[t_1, \infty)$. Hence,

$$y(\tau^{-1}(\tau^{-1}(t))) \leq \frac{R_1(\tau^{-1}(\tau^{-1}(t)))}{R_1(\tau^{-1}(t))} y(\tau^{-1}(t)).$$

As in the first case, we use (6) to arrive at (10), which holds for any $\varepsilon > 0$ and $t \geq t_2$ for $t_2 \geq t_1$ sufficiently large. Integrating (10) from t_2 to t , we have

$$-(L_1 y(t))' \geq k \left(1 - \frac{1+\varepsilon}{p_0}\right) \frac{1}{r_2(t)} \int_{t_2}^t \frac{q(s)}{p(h(s))} ds.$$

Integrating the above inequality again from t_2 to t , we find that

$$L_1 y(t) \leq L_1 y(t_2) - k \left(1 - \frac{1+\varepsilon}{p_0}\right) \int_{t_2}^t \frac{1}{r_2(u)} \int_{t_2}^u \frac{q(s)}{p(h(s))} ds du.$$

Letting t to ∞ , changing the order of integration, and using (7), we obtain

$$\begin{aligned} 0 \leq L_1 y(\infty) &\leq L_1 y(t_2) - k \left(1 - \frac{1+\varepsilon}{p_0}\right) \int_{t_2}^{\infty} \frac{1}{r_2(u)} \int_{t_2}^u \frac{q(s)}{p(h(s))} ds du \\ &= L_1 y(t_2) - k \left(1 - \frac{1+\varepsilon}{p_0}\right) \int_{t_2}^{\infty} \frac{\pi_2(s) q(s)}{p(h(s))} ds = -\infty, \end{aligned}$$

a contradiction. The proof is complete. $\square$

Theorem 1. Suppose that (H_1) – (H_3) are satisfied. If

$$\int_{t_0}^{\infty} \frac{\pi(s) q(s)}{p(h(s))} ds = \infty, \quad (12)$$

then (1) has property A.

Proof. Assume that x is a nonoscillatory solution of (1). Without loss of generality, we may make it eventually positive, i.e., we suppose that $x(t) > 0$, $x(\sigma(t)) > 0$ and $x(\tau(t)) > 0$ on $[t_1, \infty)$, where $t_1 \geq t_0$ is large enough. By conclusion of Lemma 1, $y \in \mathcal{S}_i$, $i = 1, 2, 3, 4$ for $t \geq t_1$. First, it is easy to see that in view of (2), condition (12) implies

$$\int_{t_0}^{\infty} \frac{\pi_2(s)q(s)}{p(h(s))} ds = \int_{t_0}^{\infty} \frac{q(s)}{p(h(s))} ds = \infty.$$

Thus, by Lemma 2, $\mathcal{S}_3 = \mathcal{S}_4 = \emptyset$ and so either $y \in \mathcal{S}_1$ or $y \in \mathcal{S}_2$. Using (H_2) and the fact that y is decreasing in (5), we have

$$x(t) \geq \frac{y(\tau^{-1}(t))}{p(\tau^{-1}(t))} \left(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))} \right) \geq \left(1 - \frac{1}{p_0} \right) \frac{y(\tau^{-1}(t))}{p(\tau^{-1}(t))}. \quad (13)$$

On the other hand, since $y' < 0$, there is a constant $\ell > 0$ such that

$$\lim_{t \rightarrow \infty} y(t) = \ell < \infty.$$

If $\ell > 0$, there exists a $t_2 \geq t_1$ such that $y(t) \geq \ell$ for $t \geq t_2$. Hence, from (13), we see that

$$x(t) \geq \frac{\ell(p_0 - 1)}{p_0} \frac{1}{p(\tau^{-1}(t))}, \quad t \geq t_2.$$

Using this in (1), we find

$$L_3 y(t) + \frac{\ell(p_0 - 1)}{p_0} \frac{q(t)}{p(h(t))} \leq 0, \quad t \geq t_2. \quad (14)$$

If we assume that $y \in \mathcal{S}_1$, then by integrating (14) from t_2 to t , we obtain

$$-(L_1 y(t))' \geq \frac{\ell(p_0 - 1)}{p_0} \frac{1}{r_2(t)} \int_{t_2}^t \frac{q(s)}{p(h(s))} ds.$$

Integrating the above inequality over $[t_2, t]$ again, we have

$$-y'(t) \geq \frac{\ell(p_0 - 1)}{p_0} \frac{1}{r_1(t)} \int_{t_2}^t \frac{1}{r_2(u)} \int_{t_2}^u \frac{q(s)}{p(h(s))} ds du. \quad (15)$$

Integrating (15) over $[t_2, t]$, letting t to ∞ and changing the order of integration in the resulting inequality, and taking (12) into account, we obtain

$$\begin{aligned} \ell = y(\infty) &\leq y(t_2) - \frac{\ell(p_0 - 1)}{p_0} \int_{t_2}^{\infty} \frac{1}{r_1(v)} \int_{t_2}^v \frac{1}{r_2(u)} \int_{t_2}^u \frac{q(s)}{p(h(s))} ds du dv \\ &= y(t_2) - \frac{\ell(p_0 - 1)}{p_0} \int_{t_2}^{\infty} \frac{\pi(s)q(s)}{p(h(s))} ds = -\infty, \end{aligned} \quad (16)$$

a contradiction. Hence, $\ell = 0$ in this case.

If we take $y \in \mathcal{S}_2$, then by integrating (14) from t_2 to t and using (12), we arrive at

$$L_2 y(t) \leq L_2 y(t_2) - \frac{\ell(p_0 - 1)}{p_0} \int_{t_2}^t \frac{q(s)}{p(h(s))} ds \rightarrow -\infty \quad \text{as } t \rightarrow \infty, \quad (17)$$

which contradicts the positivity of $L_2 y$ and so $\ell = 0$. Since $y(t) \geq x(t)$, we find $\lim_{t \rightarrow \infty} x(t) = 0$. The proof is complete. $\square$

In the following auxiliary result, we present a criterion for nonexistence of $\mathcal{S}_1$ type solutions, based on comparison of the studied Equation (1) with an associated first-order delay differential inequality. The given criterion also excludes solutions from classes $\mathcal{S}_3$ and $\mathcal{S}_4$.

Lemma 3. Suppose that (H_1) – (H_4) are satisfied. If

$$\liminf_{t \rightarrow \infty} \int_{h(t)}^t \frac{q(s)\pi(s)}{p(h(s))} ds > \frac{p_0}{(p_0 - 1)e}, \quad (18)$$

then $\mathcal{S}_1 = \mathcal{S}_3 = \mathcal{S}_4 = \emptyset$.

Proof. For the sake of contradiction, let (18) be satisfied but $y \in \mathcal{S}_1 \cup \mathcal{S}_3 \cup \mathcal{S}_4$. Choose $t_1 \in [t_0, \infty)$ such that $x(t) > 0$, $x(\sigma(t)) > 0$, and $x(\tau(t)) > 0$ on $[t_1, \infty)$.

Assume first that $y \in \mathcal{S}_1$. As in the proof of Theorem 1, we arrive at (13), which in view of (1) yields

$$L_3 y(t) + \frac{p_0 - 1}{p_0} \frac{q(t)}{p(h(t))} y(h(t)) \leq 0. \quad (19)$$

Define the function

$$w(t) := \pi_1(t) L_1 y(t) + y(t). \quad (20)$$

From

$$y(t) \geq - \int_t^\infty \frac{1}{r_1(s)} L_1 y(s) ds \geq -L_1 y(t) \pi_1(t) \quad (21)$$

and

$$w'(t) = \pi_1(t) (L_1 y(t))' = \frac{\pi_1(t)}{r_2(t)} L_2 y(t) < 0,$$

we see that $w(t)$ is a strictly decreasing eventually positive function, say on $[t_2, \infty)$, for $t_2 \geq t_1$. Using the definition of w in (19), we have

$$\left(\frac{r_2(t)}{\pi_1(t)} w'(t) \right)' + \frac{p_0 - 1}{p_0} \frac{q(t)}{p(h(t))} y(h(t)) \leq 0.$$

Hence, w is a solution of the second-order delay differential inequality

$$\left(\frac{r_2(t)}{\pi_1(t)} w'(t) \right)' + \frac{p_0 - 1}{p_0} \frac{q(t)}{p(h(t))} w(h(t)) \leq 0. \quad (22)$$

Similarly as before, we define the function u by

$$u(t) := \pi(t) \frac{r_2(t)}{\pi_1(t)} w'(t) + w(t).$$

From

$$u'(t) = \left(\frac{r_2(t)}{\pi_1(t)} w'(t) \right)' \pi(t) = L_3 y(t) \pi(t) \leq 0$$

and

$$w(t) \geq - \int_t^\infty \frac{\pi_1(s)}{r_2(s)} \frac{r_2(s)}{\pi_1(s)} w'(s) ds \geq - \frac{r_2(t)}{\pi_1(t)} w'(t) \pi(t), \quad (23)$$

we conclude that u is eventually positive and nonincreasing. Using the definition of u in (22), it is easy to see that u satisfies the first-order delay differential inequality

$$u'(t) + \frac{p_0 - 1}{p_0} \frac{q(t)\pi(t)}{p(h(t))} u(h(t)) \leq 0. \quad (24)$$

However, by [29] (Theorem 2.1.1), condition (18) ensures that the above inequality does not possess a positive solution, which is a contradiction.

To show that also $\mathcal{S}_3 = \mathcal{S}_4 = \emptyset$, it suffices to note that (12) is necessary for validity of (18) since otherwise, the left-hand side of (18) would equal zero. The conclusion then immediately follows from Theorem 1. The proof is complete. $\square$

Using the Riccati transformation technique, we propose a result serving as an alternative of Lemma 3, which is applicable also in the case when $h(t) = t$.

Lemma 4. Suppose that (H_1) – (H_4) are satisfied and (7) holds. If, for any $t_1 \geq t_0$ large enough,

$$\limsup_{t \rightarrow \infty} \int_{t_1}^t \left(\frac{\pi(s)q(s)}{p(h(s))} - \left(\frac{p_0}{p_0 - 1} \right) \frac{\pi_1(s)}{4\pi(s)r_2(s)} \right) ds > \frac{p_0}{p_0 - 1}, \quad (25)$$

then $\mathcal{S}_1 = \mathcal{S}_3 = \mathcal{S}_4 = \emptyset$.

Proof. For the sake of contradiction, let (18) be satisfied but $y \in \mathcal{S}_1 \cup \mathcal{S}_3 \cup \mathcal{S}_4$. Choose $t_1 \in [t_0, \infty)$ such that $x(t) > 0$, $x(\sigma(t)) > 0$, and $x(\tau(t)) > 0$ on $[t_1, \infty)$.

Assume first that $y \in \mathcal{S}_1$. We proceed as in the proof of Lemma 3 to obtain (22), where w is given by (20). Consider the function ρ defined by

$$\rho(t) = \frac{r_2(t)w'(t)}{\pi_1(t)w(t)}, \quad t \geq t_1. \quad (26)$$

Clearly, $\rho < 0$ on $[t_1, \infty)$. From (23), it is easy to see that

$$-1 \leq \pi(t)\rho(t) < 0. \quad (27)$$

Differentiating ρ and using (22) together with (26), we have

$$\begin{aligned} \rho'(t) &= \left(\frac{r_2(t)}{\pi_1(t)} w'(t) \right)' \frac{1}{w(t)} - \frac{r_2(t)(w'(t))^2}{\pi_1(t)w^2(t)} \\ &\leq -\frac{p_0 - 1}{p_0} \frac{q(t)}{p(h(t))} \frac{w(h(t))}{w(t)} - \frac{\pi_1(t)}{r_2(t)} \rho^2(t) \\ &\leq -\frac{p_0 - 1}{p_0} \frac{q(t)}{p(h(t))} - \frac{\pi_1(t)}{r_2(t)} \rho^2(t). \end{aligned} \quad (28)$$

Multiplying both sides of (28) by $\pi(t)$ and integrating the resulting inequality from t_1 to t , we have

$$\begin{aligned} \pi(t)\rho(t) &\leq \pi(t_1)\rho(t_1) - \int_{t_1}^t \frac{\pi_1(s)\rho(s)}{r_2(s)} ds - \frac{p_0 - 1}{p_0} \int_{t_1}^t \frac{\pi(s)q(s)}{p(h(s))} ds - \int_{t_1}^t \frac{\pi(s)\pi_1(s)}{r_2(s)} \rho^2(s) ds \\ &= \pi(t_1)\rho(t_1) - \frac{p_0 - 1}{p_0} \int_{t_1}^t \frac{\pi(s)q(s)}{p(h(s))} ds - \int_{t_1}^t \frac{\pi(s)\pi_1(s)}{r_2(s)} \left[\left(\rho(s) + \frac{1}{2\pi(s)} \right)^2 - \frac{1}{4\pi^2(s)} \right] ds \\ &\leq -\left(\frac{p_0 - 1}{p_0} \right) \int_{t_1}^t \left(\frac{\pi(s)q(s)}{p(h(s))} - \left(\frac{p_0}{p_0 - 1} \right) \frac{\pi_1(s)}{4\pi(s)r_2(s)} \right) ds. \end{aligned}$$

However, in view of (27), this inequality contradicts (25). Hence, $\mathcal{S}_1 = \emptyset$. By Lemma 2, $\mathcal{S}_3 = \mathcal{S}_4 = \emptyset$ due to (7). The proof is complete. $\square$

Corollary 1. Suppose that (H_1) – (H_3) are satisfied and (7) holds. If there is a constant c_* such that

$$\frac{\pi^2(t)q(t)r_2(t)}{p(h(t))\pi_1(t)} \geq c_* > \frac{p_0}{4(p_0 - 1)}, \quad (29)$$

then $\mathcal{S}_1 = \mathcal{S}_3 = \mathcal{S}_4 = \emptyset$.

To attain the oscillation of all solutions, it remains to eliminate the solutions of $\mathcal{S}_2$ type.

Lemma 5. Suppose that (H_1) – (H_4) are satisfied. If

$$\limsup_{t \rightarrow \infty} \int_{h(t)}^t \frac{q(s)R(h(t), h(s))}{p(h(s))} ds > \frac{p_0}{p_0 - 1}, \quad (30)$$

then $\mathcal{S}_2 = \emptyset$.

Proof. For the sake of contradiction, let (30) be satisfied but $y \in \mathcal{S}_2$. Choose $t_1 \in [t_0, \infty)$ such that $x(t) > 0$, $x(\sigma(t)) > 0$, and $x(\tau(t)) > 0$ on $[t_1, \infty)$. Using (13) in (1), we obtain

$$L_3 y(t) + \frac{p_0 - 1}{p_0} \frac{q(t)}{p(h(t))} y(h(t)) \leq 0. \quad (31)$$

Using the monotonicity of $L_2 y$, one can easily verify that

$$-L_1 y(u) \geq L_1 y(v) - L_1 y(u) = \int_u^v \frac{L_2 y(s)}{r_2(s)} ds \geq L_2 y(v) \int_u^v \frac{ds}{r_2(s)} \quad (32)$$

for $v \geq u \geq t_1$. Integrating the latter inequality from u to $v \geq u$, we obtain

$$y(u) \geq L_2 y(v) \int_u^v \frac{1}{r_1(s)} \int_s^v \frac{1}{r_2(x)} dx ds = L_2 y(v) R(v, u). \quad (33)$$

Setting $u = h(s)$ and $v = h(t)$, $t \geq s \geq t_1$ in (33), we find

$$y(h(s)) \geq L_2 y(h(t)) R(h(t), h(s)). \quad (34)$$

On the other hand, integrating (31) from $h(t)$ to t and using (34), we see that

$$\begin{aligned} L_2 y(h(t)) &\geq L_2 y(h(t)) - L_2 y(t) \geq \frac{p_0 - 1}{p_0} \int_{h(t)}^t \frac{q(s)}{p(h(s))} y(h(s)) ds \\ &\geq \frac{p_0 - 1}{p_0} L_2 y(h(t)) \int_{h(t)}^t \frac{q(s)R(h(t), h(s))}{p(h(s))} ds. \end{aligned}$$

Dividing the above inequality by $L_2 y(h(t))$ and taking the $\limsup$ on both sides of the resulting inequality as $t \rightarrow \infty$, we obtain a contradiction with (30). The proof is complete. $\square$

Here, a further improvement of Lemma 5 is made.

Lemma 6. Suppose that (H_1) – (H_4) are satisfied and let γ be a constant satisfying (4) eventually. If

$$\limsup_{t \rightarrow \infty} h^\gamma(t) \int_{h(t)}^t \frac{q(s)\tilde{R}(h(t), h(s))}{p(h(s))} ds > \frac{p_0}{p_0 - 1}, \quad (35)$$

then $\mathcal{S}_2 = \emptyset$.

Proof. Setting $u = h(t)$ and $v = t$ in (33), we obtain

$$y(h(t)) \geq L_2 y(t) R(t, h(t)). \quad (36)$$

By (4), (31), and (36), we have

$$\begin{aligned}(t^\gamma L_2 y(t))' &= \gamma t^{\gamma-1} L_2 y(t) + t^\gamma L_3 y(t) \leq \gamma t^{\gamma-1} L_2 y(t) - \frac{p_0 - 1}{p_0} \frac{t^\gamma q(t)}{p(h(t))} y(h(t)) \\ &\leq \gamma t^{\gamma-1} L_2 y(t) - \frac{p_0 - 1}{p_0} \frac{t^\gamma q(t) R(t, h(t))}{p(h(t))} L_2 y(t) \\ &= t^{\gamma-1} L_2 y(t) \left(\gamma - \frac{p_0 - 1}{p_0} \frac{t q(t) R(t, h(t))}{p(h(t))} \right) \leq 0.\end{aligned}$$

That is, $t^\gamma L_2 y(t)$ is eventually nonincreasing. From here, we obtain that

$$-L_1 y(u) \geq L_1 y(v) - L_1 y(u) = \int_u^v \frac{L_2 y(s) s^\gamma}{s^\gamma r_2(s)} ds \geq L_2 y(v) v^\gamma \int_u^v \frac{ds}{s^\gamma r_2(s)} \quad (37)$$

for $v \geq u \geq t_1$. Proceeding as in the proof of Lemma 5 with (32) replaced by (37), one arrives at a contradiction with (35). The proof is complete. $\square$

As a main result of the paper, we have the following oscillatory criterion for (1), which is a simple consequence of Lemmas 3–6.

Theorem 2. Suppose that (H_1) – (H_4) are satisfied. If (18) (or (25)) and (30) (or (35)) hold, then (1) is oscillatory.

Example 1. Consider the third-order neutral differential equation

$$\left(t^\beta \left(t^\alpha (x(t) + p_0 x(\tau(t)))' \right)' \right)' + q_0 t^\delta x(\sigma(t)) = 0, \quad t \geq 1, \quad (38)$$

where $\alpha > 1$, $\beta > 1$, $\delta > 0$, $q_0 > 0$, $p_0 > 1$, and the delay functions τ and σ satisfy (H_3) . Using that

$$\pi(t) = \frac{1}{(\alpha - 1)(\alpha + \beta - 2)t^{\alpha+\beta+2}},$$

it is easy to verify that condition (12) reduces to

$$\int_1^\infty \frac{q_0}{p_0(\alpha - 1)(\alpha + \beta - 2)} \frac{1}{t^{\alpha+\beta-2-\delta}} dt = \infty.$$

Hence, if

$$\delta \geq \alpha + \beta - 3,$$

all assumptions of Theorem 1 are satisfied and (38) has property A.

Example 2. Consider the third-order neutral differential equation

$$\left(t^2 \left(t^2 (x(t) + k t^\alpha x(\lambda_1 t))' \right)' \right)' + q_0 t^{\alpha+1} x(\lambda_2 t) = 0, \quad t \geq 1, \quad (39)$$

where $k > 0$, $\alpha \geq 0$, $\lambda_1 \in (0, 1]$, $\lambda_2 \in (0, 1]$, and $h(t) := \lambda_2/\lambda_1 t \leq t$. Here,

$$\pi(t) = \frac{1}{2t^2} \quad \text{and} \quad R(h(t), h(s)) = \frac{1}{2} \left(\frac{\lambda_1}{\lambda_2} \right)^2 \left(\frac{1}{t^2} - \frac{2}{ts} + \frac{1}{s^2} \right).$$

Since (12) is satisfied, by Theorem 1, (39) has property A. By some computations, conditions (18) and (25) reduce to

$$\frac{q_0}{2} \left(\frac{\lambda_1}{\lambda_2} \right)^\alpha \ln \left(\frac{\lambda_1}{\lambda_2} \right) > \begin{cases} \frac{k^2}{(k-1)e} & \text{for } \alpha = 0 \text{ and } k > 1 \\ \frac{k}{e} & \text{for } \alpha > 0 \end{cases} \quad (40)$$

and

$$q_0 \left(\frac{\lambda_1}{\lambda_2} \right)^\alpha > \begin{cases} \frac{k^2}{(k-1)} & \text{for } \alpha = 0 \text{ and } k > 1 \\ k & \text{for } \alpha > 0, \end{cases} \quad (41)$$

respectively. Finally, condition (30) takes the form

$$\frac{q_0}{2} \left(\frac{\lambda_1}{\lambda_2} \right)^{2+\alpha} \left(\frac{1 - \left(\frac{\lambda_2}{\lambda_1} \right)^2}{2} - 2 \left(1 - \left(\frac{\lambda_2}{\lambda_1} \right) \right) + \ln \left(\frac{\lambda_1}{\lambda_2} \right) \right) > \begin{cases} \frac{k^2}{k-1} & \text{for } \alpha = 0 \text{ and } k > 1 \\ k & \text{for } \alpha > 0. \end{cases} \quad (42)$$

By Theorem 2, we conclude that (39) is oscillatory if (40) (or (41)) and (42) hold.

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