mathematics

Article

Bivariate α, q-Bernstein–Kantorovich Operators and GBS Operators of Bivariate α, q-Bernstein–Kantorovich Type

Qing-Bo Cai [1], **Wen-Tao Cheng** [2,*] and **Bayram Çekim** [3]

1 School of Mathematics and Computer Science, Quanzhou Normal University, Quanzhou 362000, China; qbcai@qztc.edu.cn
2 School of Mathematics and Computation Sciences, Anqing Normal University, Anhui 246133, China
3 Department of Mathematics, Faculty of Science, Gazi University, Beşevler, Ankara 06500, Turkey; bayramcekim@gazi.edu.tr
* Correspondence: chengwentao@aqnu.edu.cn; Tel.: +86-139-6699-9238

Received: 27 August 2019; Accepted: 14 November 2019; Published: 2 December 2019

Abstract: In this paper, we introduce a family of bivariate α, q-Bernstein–Kantorovich operators and a family of GBS (Generalized Boolean Sum) operators of bivariate α, q-Bernstein–Kantorovich type. For the former, we obtain the estimate of moments and central moments, investigate the degree of approximation for these bivariate operators in terms of the partial moduli of continuity and Peetre's K-functional. For the latter, we estimate the rate of convergence of these GBS operators for B-continuous and B-differentiable functions by using the mixed modulus of smoothness.

Keywords: α, q-Bernstein–Kantorovich operators; GBS operators; B-continuous functions; moduli of continuity; B-differentiable functions; mixed modulus of smoothness; q-integers

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1. Introduction

Since the famous Bernstein polynomial was proposed in 1912, the study of Bernstein type operators has not ceased. In 2017, Chen et al. [1] introduced and studied the monotonic, convex properties and also some other important properties of a new generalized positive linear Bernstein operators with parameter α which are defined as

$$T_{n,\alpha}(f;x) = \sum_{i=0}^{n} f_i p_{n,i}^{(\alpha)}(x), \tag{1}$$

where $f \in C_{[0,1]}$, $x \in [0,1]$, $n \in \mathbb{N}$, $f_i = f\left(\frac{i}{n}\right)$, $n \in \mathbb{N}$, $\alpha \in [0,1]$, and $p_{n,i}^{\alpha}(x)$ is defined by

$$\begin{cases} p_{1,0}^{(\alpha)}(x) & = 1-x,\ p_{1,1}^{(\alpha)}(x) = x, \\ p_{n,i}^{(\alpha)}(x) & = \left[\binom{n-2}{i}(1-\alpha)x + \binom{n-2}{i-2}(1-\alpha)(1-x) \right. \\ & \quad \left. + \binom{n}{i}\alpha x(1-x)\right] x^{i-1}(1-x)^{n-1-i}, \quad (n \geq 2). \end{cases} \tag{2}$$

In the same year, Mohiuddine et al. [2] constructed the Kantorovich type of these family of Bernstein operators (1). These operators they introduced are

$$K_{n,\alpha}(f;x) = (n+1)\sum_{i=0}^{n} p_{n,i}^{(\alpha)}(x)\int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} f(t)dt,$$

where $\alpha \in [0,1]$, $p_{n,i}^{(\alpha)}(x)$ for $i = 0, 1, ..., n$ are defined in Equation (2).

Very recently, Cai and Xu [3] proposed the α, q–Bernstein operators as

$$T_{n,q,\alpha}(f;x) = \sum_{i=0}^{n} p_{n,q,i}^{(\alpha)}(x) f\left(\frac{[i]_q}{[n]_q}\right), \tag{3}$$

where $\alpha \in [0,1]$, $q \in (0,1]$, $x \in [0,1]$, $f \in C[0,1]$ and

$$\begin{cases} p_{1,q,0}^{(\alpha)}(x) = 1-x, \; p_{1,q,1}^{(\alpha)}(x) = x, \\ p_{n,q,i}^{(\alpha)}(x) \\ = \left(\begin{bmatrix} n-2 \\ i \end{bmatrix}_q (1-\alpha)x + \begin{bmatrix} n-2 \\ i-2 \end{bmatrix}_q (1-\alpha) q^{n-i-2}\left(1-q^{n-i-1}x\right) \right. \\ \left. \quad + \begin{bmatrix} n \\ i \end{bmatrix}_q \alpha x \left(1-q^{n-i-1}x\right) \right) x^{i-1}(1-x)_q^{n-i-1}, \; (n \geq 2). \end{cases} \tag{4}$$

Note that the first term on the right-hand side of the above equation equals 0 when $i = n-1, n$, and the second term on the right-hand side of the above equation equals 0 when $i = 0, 1$. As we know, the application of q-integers in approximation theory has been a hot topic in recent decades. Even recently, there have also been many papers mentioned about the q-analogue of Bernstein type operators, such as [4–14].

Motivated by the research above, in the next section, we will introduce bivariate α, q-Bernstein–Kantorovich operators $K_{n_1,n_2,q_1,q_2}^{\alpha_1,\alpha_2}(f)$ and GBS operators of bivariate α, q-Bernstein–Kantorovich type $UK_{n_1,n_2,q_1,q_2}^{\alpha_1,\alpha_2}(f)$. In Section 3, we compute the moments and central moments of $K_{n_1,n_2,q_1,q_2}^{\alpha_1,\alpha_2}(f)$. In Section 4, we investigate the degree of approximation for $K_{n_1,n_2,q_1,q_2}^{\alpha_1,\alpha_2}(f)$. In Section 5, we estimate the convergence of $UK_{n_1,n_2,q_1,q_2}^{\alpha_1,\alpha_2}(f)$ for B-continuous and B-differentiable functions.

We evoke some definitions based on q-integers; details can be seen in [15,16]. q-integers by $[s]_q$ are denoted for any fixed real number $0 < q \leq 1$ and each nonnegative integer s, where

$$[s]_q := \begin{cases} \frac{1-q^s}{1-q}, & q \neq 1, \\ s, & q = 1. \end{cases}$$

In addition, q-factorial and q-binomial coefficients are defined as follows:

$$[s]_q! := \begin{cases} [s]_q[s-1]_q...[1]_q, & s = 1, 2, ..., \\ 1, & s = 0, \end{cases}$$

$$\begin{bmatrix} n \\ s \end{bmatrix}_q := \frac{[n]_q!}{[s]_q![n-s]_q!} \quad (n \geq s \geq 0).$$

$(1+x)_q^n$ is defined by $(1+x)_q^n := (1+x)(1+qx)...(1+q^{n-1}x) = \prod_{s=0}^{n-1}(1+q^s x)$. The q-Jackson integral on $[a,b]$ is defined as

$$\int_a^b f(x)d_qx := (1-q)\sum_{j=0}^{\infty}\left[bf\left(q^jb\right) - af\left(q^ja\right)\right]q^j.$$

2. Construction of Operators

For the convenience, we denote $q_i := \{q_{n_i}\}$, $(i=1,2)$.

We introduce the bivariate α, q-Bernstein–Kantorovich operators as follows: for $f \in C(I^2)$, $I^2 = [0,1]\times[0,1]$, $0 < q_1, q_2 < 1$ and α_1, α_2 are any fixed real numbers in $[0,1]$,

$$\begin{aligned} K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f;x,y) &= [n_1+1]_{q_1}[n_2+1]_{q_2}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} p_{n_1,q_1,k_1}^{(\alpha_1)}(x)p_{n_2,q_2,k_2}^{(\alpha_2)}(y)q_1^{-k_1}q_2^{-k_2} \\ &\quad\times\int_{\frac{[k_1]_{q_1}}{[n_1+1]_{q_1}}}^{\frac{[k_1+1]_{q_1}}{[n_1+1]_{q_1}}}\int_{\frac{[k_2]_{q_2}}{[n_2+1]_{q_2}}}^{\frac{[k_2+1]_{q_2}}{[n_2+1]_{q_2}}} f(t,s)d_{q_1}td_{q_2}s, \end{aligned} \tag{5}$$

where $p_{n_i,q_i,k_i}^{(\alpha_i)}(\cdot)$, $(i=1,2)$ are defined in Equation (4), $x,y\in[0,1]$.

The GBS operators of the bivariate α, q-Bernstein–Kantorovich type are as in the below:

$$\begin{aligned} & UK_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f(t,s);x,y) \\ =\; & K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f(x,s)+f(t,y)-f(t,s);x,y) \\ =\; & [n_1+1]_{q_1}[n_2+1]_{q_2}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} p_{n_1,q_1,k_1}^{(\alpha_1)}(x)p_{n_2,q_2,k_2}^{(\alpha_2)}(y)q_1^{-k_1}q_2^{-k_2} \\ & \times\int_{\frac{[k_1]_{q_1}}{[n_1+1]_{q_1}}}^{\frac{[k_1+1]_{q_1}}{[n_1+1]_{q_1}}}\int_{\frac{[k_2]_{q_2}}{[n_2+1]_{q_2}}}^{\frac{[k_2+1]_{q_2}}{[n_2+1]_{q_2}}}[f(x,s)+f(t,y)-f(t,s)]\,d_{q_1}td_{q_2}s, \end{aligned} \tag{6}$$

where $\alpha_1,\alpha_2\in[0,1]$, $p_{n_i,q_i,k_i}^{(\alpha_i)}(\cdot)$, $(i=1,2)$ are defined in Equation (4), $x,y\in[0,1]$.

3. Auxiliary Results

In order to prove the main conclusion of this paper, the following lemmas are given:

Lemma 1. *(See [3]) The following equalities hold:*

$$\begin{aligned} & T_{n,q,\alpha}(1;x) = 1,\ T_{n,q,\alpha}(t;x) = x, \\ & T_{n,q,\alpha}\left(t^2;x\right) = x^2 + \frac{x(1-x)}{[n]_q} + \frac{(1-\alpha)q^{n-1}[2]_q x(1-x)}{[n]_q^2}. \end{aligned}$$

Lemma 2. *Let $e_{i,j} = t^is^j$, $i,j = 0,1,2$, and we give the following equalities:*

$$K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(e_{00};x,y) = 1; \tag{7}$$

$$K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(e_{10};x,y) = x + \frac{1-[2]_{q_1}q_1^{n_1}x}{[2]_{q_1}[n_1+1]_{q_1}}; \tag{8}$$

$$K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(e_{01};x,y) = y + \frac{1-[2]_{q_2}q_2^{n_2}y}{[2]_{q_2}[n_2+1]_{q_2}}; \tag{9}$$

$$K^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}(e_{20};x,y) = x^2 - \frac{2q_1^{n_1}}{[n_1+1]_{q_1}}x^2 + \frac{x(1-x)}{[n_1+1]_{q_1}} + \frac{(1+2q_1)x}{[3]_{q_1}[n_1+1]_{q_1}} + \frac{1-q_1^{n_1}(1+2q_1)x}{[3]_{q_1}[n_1+1]^2_{q_1}} + \frac{q_1^{2n_1}x^2+q_1^{n_1-1}x(1-x)\big(1-\alpha_1[2]_{q_1}\big)}{[n_1+1]^2_{q_1}}; \tag{10}$$

$$K^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}(e_{02};x,y) = y^2 - \frac{2q_2^{n_2}}{[n_2+1]_{q_2}}y^2 + \frac{y(1-y)}{[n_2+1]_{q_2}} + \frac{(1+2q_2)y}{[3]_{q_2}[n_2+1]_{q_2}} + \frac{1-q_2^{n_2}(1+2q_2)y}{[3]_{q_2}[n_2+1]^2_{q_2}} + \frac{q_2^{2n_2}y^2+q_2^{n_2-1}y(1-y)\big(1-\alpha_2[2]_{q_2}\big)}{[n_2+1]^2_{q_2}}. \tag{11}$$

Proof. From Equation (5), we have

$$\begin{aligned}
&K^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}(e_{00};x,y)\\
&= [n_1+1]_{q_1}[n_2+1]_{q_2}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} p^{(\alpha_1)}_{n_1,q_1,k_1}(x)p^{(\alpha_2)}_{n_2,q_2,k_2}(y)q_1^{-k_1}q_2^{-k_2}\int_{\frac{[k_1]_{q_1}}{[n_1+1]_{q_1}}}^{\frac{[k_1+1]_{q_1}}{[n_1+1]_{q_1}}}\int_{\frac{[k_2]_{q_2}}{[n_2+1]_{q_2}}}^{\frac{[k_2+1]_{q_2}}{[n_2+1]_{q_2}}} d_{q_1}td_{q_2}s\\
&= [n_1+1]_{q_1}\sum_{k_1=0}^{n_1} p^{(\alpha_1)}_{n_1,q_1,k_1}(x)q_1^{-k_1}\frac{q_1^{k_1}}{[n_1+1]_{q_1}}[n_2+1]_{q_2}\sum_{k_2=0}^{n_2} p^{(\alpha_2)}_{n_2,q_2,k_2}(y)q_2^{-k_2}\frac{q_2^{k_2}}{[n_2+1]_{q_2}}\\
&= T_{n_1,q_1,\alpha_1}(1;x)T_{n_2,q_2,\alpha_2}(1;y) = 1.
\end{aligned}$$

Next, with the help of q-Jackson integrals, we obtain

$$\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} td_qt = (1-q)\sum_{j=0}^{\infty}\left(\frac{[k+1]_q^2}{[n+1]_q^2}-\frac{[k]_q^2}{[n+1]_q^2}\right)q^{2j} = \frac{2q^k[k]_q+q^{2k}}{[2]_q[n+1]_q^2}.$$

Using the above Equation (5), and Lemma 1, we obtain

$$\begin{aligned}
&K^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}(e_{10};x,y)\\
&= [n_1+1]_{q_1}[n_2+1]_{q_2}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} p^{(\alpha_1)}_{n_1,q_1,k_1}(x)p^{(\alpha_2)}_{n_2,q_2,k_2}(y)q_1^{-k_1}q_2^{-k_2}\\
&\quad\times\int_{\frac{[k_1]_{q_1}}{[n_1+1]_{q_1}}}^{\frac{[k_1+1]_{q_1}}{[n_1+1]_{q_1}}}\int_{\frac{[k_2]_{q_2}}{[n_2+1]_{q_2}}}^{\frac{[k_2+1]_{q_2}}{[n_2+1]_{q_2}}} td_{q_1}td_{q_2}s\\
&= [n_1+1]_{q_1}\sum_{k_1=0}^{n_1} p^{(\alpha_1)}_{n_1,q_1,k_1}(x)q_{n_1}^{-k_1}\frac{2q_1^{k_1}[k_1]_{q_1}+q_1^{2k_1}}{[2]_{q_1}[n_1+1]^2_{q_1}}T_{n_2,q_2,\alpha_2}(1;y)\\
&= \frac{2[n_1]_{q_1}}{[2]_{q_1}[n_1+1]_{q_1}}\sum_{k_1=0}^{n_1} p^{(\alpha_1)}_{n_1,q_1,k_1}(x)\frac{[k_1]_{q_1}}{[n_1]_{q_1}} + \frac{1}{[2]_{q_1}[n_1+1]_{q_1}}\sum_{k_1=0}^{n_1} p^{(\alpha_1)}_{n_1,q_1,k_1}(x)q_1^{k_1}\\
&= \frac{2[n_1]_{q_1}T_{n_1,q_1,\alpha_1}(t;x)}{[2]_{q_1}[n_1+1]_{q_1}} + \frac{1}{[2]_{q_1}[n_1+1]_{q_1}}\sum_{k_1=0}^{n_1} p^{(\alpha_1)}_{n_1,q_1,k_1}(x)\left[1-(1-q_1)[k_1]_{q_1}\right]\\
&= \frac{2[n_1]_{q_1}x}{[2]_{q_1}[n_1+1]_{q_1}} + \frac{T_{n_1,q_1,\alpha_1}(1;x)}{[2]_{q_1}[n_1+1]_{q_1}} - \frac{[n_1]_{q_1}(1-q_1)}{[2]_{q_1}[n_1+1]_{q_1}}T_{n_1,q_1,\alpha_1}(t;x)\\
&= \frac{2[n_1]_{q_1}x}{[2]_{q_1}[n_1+1]_{q_1}} + \frac{1}{[2]_{q_1}[n_1+1]_{q_1}} - \frac{[n_1]_{q_1}(1-q_1)x}{[2]_{q_1}[n_1+1]_{q_1}}\\
&= \frac{[n_1]_{q_1}x}{[n_1+1]_{q_1}} + \frac{1}{[2]_{q_1}[n_1+1]_{q_1}} = x + \frac{1-[2]_{q_1}q_1^{n_1}x}{[2]_{q_1}[n_1+1]_{q_1}}.
\end{aligned}$$

Similarly, we get Equation (9) easily. Finally, by via q-Jackson integrals and $[k+1]_q = 1 + q[k]_q$, we have

$$\begin{aligned}
\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} t^2 d_q t &= (1-q)\sum_{j=0}^{\infty}\left(\frac{[k+1]_q^3}{[n+1]_q^3} - \frac{[k]_q^3}{[n+1]_q^3}\right) q^{3j} \\
&= \frac{\left([k+1]_q - [k]_q\right)\left([k+1]_q^2 + [k]_q[k+1]_q + [k]_q^2\right)}{[3]_q[n+1]_q^3} \\
&= \frac{q^k\left([3]_q[k]_q^2 + (1+2q)[k]_q + 1\right)}{[3]_q[n+1]_q^3}.
\end{aligned}$$

From the above Equation (5), and Lemma 1, we get

$$\begin{aligned}
& K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(e_{20};x,y) \\
= \;& [n_1+1]_{q_1}[n_2+1]_{q_2}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} p_{n_1,q_1,k_1}^{(\alpha_1)}(x) p_{n_2,q_2,k_2}^{(\alpha_2)}(y) q_1^{-k_1} q_2^{-k_2} \\
& \times \int_{\frac{[k_1]_{q_1}}{[n_1+1]_{q_1}}}^{\frac{[k_1+1]_{q_1}}{[n_1+1]_{q_1}}} \int_{\frac{[k_2]_{q_2}}{[n_2+1]_{q_2}}}^{\frac{[k_2+1]_{q_2}}{[n_2+1]_{q_2}}} t^2 d_{q_1} t d_{q_2} s \\
= \;& [n_1+1]_{q_1}\sum_{k_1=0}^{n_1} p_{n_1,q_1,k_1}^{(\alpha_1)}(x) q_1^{-k_1}\frac{q_1^{k_1}\left([3]_{q_1}[k_1]_{q_1}^2 + (1+2q_1)[k_1]_{q_1} + 1\right)}{[3]_{q_1}[n_1+1]_{q_1}^3} \\
= \;& \frac{[n_1]_{q_1}^2}{[n_1+1]_{q_1}^2} T_{n_1,q_1,\alpha_1}\left(t^2;x\right) + \frac{(1+2q_1)[n_1]_{q_1}}{[3]_{q_1}[n_1+1]_{q_1}^2} T_{n_1,q_1,\alpha_1}(t;x) + \frac{1}{[3]_{q_1}[n_1+1]_{q_1}^2} \\
= \;& \frac{[n_1]_{q_1}^2}{[n_1+1]_{q_1}^2}\left[x^2 + \frac{x(1-x)}{[n_1]_{q_1}} + \frac{(1-\alpha_1)q_1^{n_1-1}[2]_{q_1}x(1-x)}{[n_1]_{q_1}^2}\right] + \frac{(1+2q_1)[n_1]_{q_1}x}{[3]_{q_1}[n_1+1]_{q_1}^2} \\
& + \frac{1}{[3]_{q_1}[n_1+1]_{q_1}^2} \\
= \;& x^2 - \frac{2q_1^{n_1}}{[n_1+1]_{q_1}}x^2 + \frac{q_1^{2n_1}x^2 + (1-\alpha_1)q_1^{n_1-1}[2]_{q_1}x(1-x)}{[n_1+1]_{q_1}^2} + \frac{1}{[3]_{q_1}[n_1+1]_{q_1}^2} \\
& + \frac{[n_1]_{q_1}\left[[3]_{q_1}x(1-x) + (1+2q_1)x\right]}{[3]_{q_1}[n_1+1]_{q_1}^2} \\
= \;& x^2 - \frac{2q_1^{n_1}}{[n_1+1]_{q_1}}x^2 + \frac{x(1-x)}{[n_1+1]_{q_1}} + \frac{(1+2q_1)x}{[3]_{q_1}[n_1+1]_{q_1}} + \frac{1 - q_1^{n_1}(1+2q_1)x}{[3]_{q_1}[n_1+1]_{q_1}^2} \\
& + \frac{q_1^{2n_1}x^2 + q_1^{n_1-1}x(1-x)\left(1-\alpha_1[2]_{q_1}\right)}{[n_1+1]_{q_1}^2}.
\end{aligned}$$

We can obtain Equation (11) using the same method. Lemma 2 is proved. □

Corollary 1. *With the help of Lemma 2, we obtain*

$$K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(t-x;x,y) = \frac{1 - [2]_{q_1}q_1^{n_1}x}{[2]_{q_1}[n_1+1]_{q_1}};$$

$$K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(s-y;x,y) = \frac{1 - [2]_{q_2}q_2^{n_2}y}{[2]_{q_2}[n_2+1]_{q_2}};$$

$$\begin{aligned} K^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}\left((t-x)^2;x,y\right) &= \tfrac{x(1-x)}{[n_1+1]_{q_1}} - \tfrac{2x}{[2]_{q_1}[n_1+1]_{q_1}} + \tfrac{(1+2q_1)x}{[3]_{q_1}[n_1+1]_{q_1}} \\ &+ \tfrac{1-q_1^{n_1}(1+2q_1)x}{[3]_{q_1}[n_1+1]^2_{q_1}} + \tfrac{q_1^{2n_1}x^2+q_1^{n_1-1}x(1-x)\left(1-\alpha_1[2]_{q_1}\right)}{[n_1+1]^2_{q_1}} := \delta^{(\alpha_1)}_{n_1}(x); \end{aligned} \tag{12}$$

$$\begin{aligned} K^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}\left((s-y)^2;x,y\right) &= \tfrac{y(1-y)}{[n_2+1]_{q_2}} - \tfrac{2y}{[2]_{q_2}[n_2+1]_{q_2}} + \tfrac{(1+2q_2)y}{[3]_{q_2}[n_2+1]_{q_2}} \\ &+ \tfrac{1-q_2^{n_2}(1+2q_2)y}{[3]_{q_2}[n_2+1]^2_{q_2}} + \tfrac{q_2^{2n_2}y^2+q_2^{n_2-1}y(1-y)\left(1-\alpha_2[2]_{q_2}\right)}{[n_2+1]^2_{q_2}} := \delta^{(\alpha_2)}_{n_2}(y). \end{aligned} \tag{13}$$

Lemma 3. *(See [3])*

(i) *The* (α, q)*-Bernstein operators may be expressed as follows:*

$$T_{n,q,\alpha}(f;x) = \sum_{r=0}^{n}\left((1-\alpha)\begin{bmatrix} n-1 \\ r \end{bmatrix}_q \triangle^r_q g_0 + \alpha \begin{bmatrix} n \\ r \end{bmatrix}_q \triangle^r_q f_0\right)x^r,$$

where $\begin{bmatrix} n-1 \\ n \end{bmatrix}_q = 0$, $\triangle^r_q f_j = \triangle^{r-1}_q f_{j+1} - q^{r-1}\triangle^{r-1}_q f_j$, $r \geq 1$, *with* $\triangle^0_q f_j = f_j = f\left(\frac{[j]_q}{[n]_q}\right)$.

(ii) *The higher-order forward difference of* g_i *may be expressed as follows:*

$$\triangle^r_q g_i = \left(1 - \frac{q^{n-i-1}[i]_q}{[n-1]_q}\right)\triangle^r_q f_i + \frac{q^{n-i-1-r}[i+r]_q}{[n-1]_q}\triangle^r_q f_{i+1},$$

where $\triangle^0_q g_i = g_i = \left(1 - \frac{q^{n-1-i}[i]_q}{[n-1]_q}\right)f_i + \frac{q^{n-1-i}[i]_q}{[n-1]_q}f_{i+1}$.

Lemma 4. *For* $T_{n,q,\alpha}(f;x)$*, the following equalities are hold*

$$\begin{aligned} & T_{n,q,\alpha}\left(t^3;x\right) \\ = & \left(\tfrac{q^3[n-1]_q[n-2]_q[n-3]_q}{[n]^3_q} + \tfrac{q^{n+2}[n-2]_q[n-3]_q}{[n]^3_q} + \tfrac{\alpha[2]_q q^n[n-2]_q}{[n]^2_q}\right)x^3 \\ & + \left\{\tfrac{[n-1]_q[n-2]_q\left(2q+q^2\right)}{[n]^3_q} + \tfrac{q^n[n-2]_q\left(3+2q+q^2\right)}{[n]^3_q}\right. \\ & \left. + \tfrac{\alpha q^{n-1}\left[(q-1)[n-1]_q+3+2q+q^2\right]}{[n]^3_q}\right\}x^2 \\ & + \left[\tfrac{[n-1]_q}{[n]^3_q} + \tfrac{q^{n-1}\left(3+3q+q^2\right)}{[n]^3_q} - \tfrac{\alpha q^{n-1}\left(2+3q+q^2\right)}{[n]^3_q}\right]x, \end{aligned} \tag{14}$$

$$\begin{aligned}
&T_{n,q,\alpha}\left(t^4;x\right)\\
=\;&\left[\frac{q^6[n-1]_q[n-2]_q[n-3]_q[n-4]_q}{[n]_q^4}+\frac{\alpha[3]_q q^{n+2}[n-2]_q[n-3]_q}{[n]_q^3}\right.\\
&\left.+\frac{q^{n+5}[n-2]_q[n-3]_q[n-4]_q}{[n]_q^4}+(1-\alpha)\frac{10q^{n+7}[n-2]_q[n-3]_q[n-4]_q}{[2]_q[3]_q[4]_q[n]_q^4}\right]x^4\\
&+\left[\frac{q^3\left(3+2q+q^2\right)[n-1]_q[n-2]_q[n-3]_q}{[n]_q^4}\right.\\
&+\frac{\alpha q^n\left(3+5q+6q^2+3q^3+q^4\right)[n-1]_q[n-2]_q}{[n]_q^4}\\
&\left.+(1-\alpha)\frac{q^{n+2}\left(4+3q+2q^2+q^3\right)[n-2]_q[n-3]_q}{[n]_q^4}\right]x^3\\
&+\left[\frac{[n-1]_q[n-2]_q\left(3q+3q^2+q^3\right)}{[n]_q^4}+\frac{\alpha\left(3+3q+q^2\right)q^{n-1}[2]_q[n-1]_q}{[n]_q^4}\right.\\
&\left.+(1-\alpha)\frac{q^n\left(6+8q+7q^2+3q^3+q^4\right)[n-2]_q}{[n]_q^4}\right]x^2\\
&+\left[\frac{[n-1]_q+q^{n-1}\alpha}{[n]_q^4}+(1-\alpha)\frac{q^{n-1}\left(4+6q+4q^2+q^3\right)}{[n]_q^4}\right]x.
\end{aligned}\tag{15}$$

Proof. For $f(t)=t^3$, by using Lemma 3, we have the following statements:

$$\triangle_q^0 f_0=0,\ \triangle_q^0 f_1=\frac{1}{[n]_q^3},\ \triangle_q^0 f_2=\frac{[2]_q^3}{[n]_q^3},\ \triangle_q^0 f_3=\frac{[3]_q^3}{[n]_q^3},\ \triangle_q^0 f_4=\frac{[4]_q^3}{[n]_q^3};\tag{16}$$

$$\left\{\begin{aligned}
\triangle_q^1 f_0&=f_1-f_0=\frac{1}{[n]_q^3},\\
\triangle_q^1 f_1&=f_2-f_1=\frac{3q+3q^2+q^3}{[n]_q^3},\\
\triangle_q^1 f_2&=f_3-f_2=\frac{3q^2+6q^3+6q^4+3q^5+q^6}{[n]_q^3},\\
\triangle_q^1 f_3&=f_4-f_3=\frac{3q^3+6q^4+9q^5+9q^6+6q^7+3q^8+q^9}{[n]_q^3};
\end{aligned}\right.\tag{17}$$

$$\left\{\begin{aligned}
\triangle_q^2 f_0&=\triangle_q^1 f_1-q\triangle_q^1 f_0=\frac{2q+3q^2+q^3}{[n]_q^3},\\
\triangle_q^2 f_1&=\triangle_q^1 f_2-q\triangle_q^1 f_1=\frac{3q^3+5q^4+3q^5+q^6}{[n]_q^3},\\
\triangle_q^2 f_2&=\triangle_q^1 f_3-q\triangle_q^1 f_2=\frac{3q^5+6q^6+5q^7+3q^8+q^9}{[n]_q^3};
\end{aligned}\right.\tag{18}$$

$$\left\{\begin{aligned}
\triangle_q^3 f_0&=\triangle_q^2 f_1-q^2\triangle_q^2 f_0=\frac{q^3+2q^4+2q^5+q^6}{[n]_q^3},\\
\triangle_q^3 f_1&=\triangle_q^2 f_2-q^2\triangle_q^2 f_1=\frac{q^6+2q^7+2q^8+q^9}{[n]_q^3};
\end{aligned}\right.\tag{19}$$

$$\left\{\begin{aligned}
\triangle_q^0 g_0&=\triangle_q^0 f_0=0,\\
\triangle_q^1 g_0&=\triangle_q^1 f_0+\frac{q^{n-2}}{[n-1]_q}\triangle_q^1 f_1=\frac{1}{[n]_q^3}+\frac{q^{n-2}\left(3q+3q^2+q^3\right)}{[n-1]_q[n]_q^3},\\
\triangle_q^2 g_0&=\triangle_q^2 f_0+\frac{q^{n-3}}{[n-1]_q}\triangle_q^2 f_1=\frac{2q+3q^2+q^3}{[n]_q^3}+\frac{q^{n-3}\left(3q^3+5q^4+3q^5+q^6\right)}{[n-1]_q[n]_q^3},\\
\triangle_q^3 g_0&=\triangle_q^3 f_0+\frac{q^{n-4}}{[n-1]_q}\triangle_q^3 f_1=\frac{q^3+2q^4+2q^5+q^6}{[n]_q^3}+\frac{q^{n-4}\left(q^6+2q^7+2q^8+q^9\right)}{[n-1]_q[n]_q^3}.
\end{aligned}\right.\tag{20}$$

By (i) of Lemma 3, Equations (16)–(20), and some necessary computations, we have

$$\begin{aligned}
&T_{n,q,\alpha}\left(t^3;x\right)\\
=\;&\sum_{r=0}^{3}\left[(1-\alpha)\begin{bmatrix}n-1\\ r\end{bmatrix}_q\triangle_q^r g_0+\alpha\begin{bmatrix}n\\ r\end{bmatrix}_q\triangle_q^r f_0\right]x^r
\end{aligned}$$

$$
\begin{aligned}
&= \left[(1-\alpha)\triangle_q^0 g_0 + \alpha\triangle_q^0 f_0\right] + \left[(1-\alpha)[n-1]_q\triangle_q^1 g_0 + \alpha[n]_q\triangle_q^1 f_0\right] x \\
&\quad + \left[(1-\alpha)\frac{[n-1]_q[n-2]_q}{[2]_q}\triangle_q^2 g_0 + \alpha\frac{[n]_q[n-1]_q}{[2]_q}\triangle_q^2 f_0\right] x^2 \\
&\quad + \left[(1-\alpha)\frac{[n-1]_q[n-2]_q[n-3]_q}{[3]_q[2]_q}\triangle_q^3 g_0 + \alpha\frac{[n]_q[n-1]_q[n-2]_q}{[3]_q[2]_q}\triangle_q^3 f_0\right] x^3 \\
&= \left[\frac{[n-1]_q}{[n]_q^3} + \frac{q^{n-2}\left(3q+3q^2+q^3\right)}{[n]_q^3} - \frac{\alpha q^{n-1}\left(2+3q+q^2\right)}{[n]_q^3}\right] x \\
&\quad + \left\{\frac{[n-1]_q[n-2]_q\left(2q+q^2\right)}{[n]_q^3} + \frac{q^n[n-2]_q\left(3+2q+q^2\right)}{[n]_q^3}\right. \\
&\quad \left. + \frac{\alpha q^{n-1}\left[(q-1)[n-1]_q + 3 + 2q + q^2\right]}{[n]_q^3}\right\} x^2 \\
&\quad + \left[\frac{q^3[n-1]_q[n-2]_q[n-3]_q}{[n]_q^3} + \frac{q^{n+2}[n-2]_q[n-3]_q}{[n]_q^3} + \frac{\alpha[2]_q q^n[n-2]_q}{[n]_q^2}\right] x^3.
\end{aligned}
$$

Hence, we obtain Equation (14), using the similar methods, we can get Equation (15). Lemma 3 is proved. □

Lemma 5. *Let $e_{i,j} = t^i s^j$, $i, j = 3, 4$ be the bivariate test functions. Then, we have the following equalities:*

$$
\begin{aligned}
& K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(e_{30}; x, y) \\
=\ & \left(\frac{q_1^3[n_1-1]_{q_1}[n_1-2]_{q_1}[n_1-3]_{q_1}}{[n_1+1]_{q_1}^3} + \frac{q_1^{n_1+2}[n_1-2]_{q_1}[n_1-3]_{q_1}}{[n_1+1]_{q_1}^3}\right. \\
& \left. + \frac{\alpha_1[2]_{q_1}q_1^{n_1}[n_1-2]_{q_1}[n_1]_{q_1}}{[n_1+1]_{q_1}^3}\right) x^3 + \left\{\frac{[n_1-1]_{q_1}[n_1-2]_{q_1}\left(2q_1+q_1^2\right)}{[n_1+1]_{q_1}^3}\right. \\
& + \frac{q_1^{n_1}[n_1-2]_{q_1}\left(3+2q_1+q_1^2\right)}{[n_1+1]_{q_1}^3} + \frac{\alpha_1 q_1^{n_1-1}\left[(q_1-1)[n_1-1]_{q_1}+3+2q_1+q_1^2\right]}{[n_1+1]_{q_1}^3} \\
& \left. \frac{q_1\left(1+2q_1+3q_1^2\right)[n_1]_{q_1}[n_1-1]_{q_1}}{[4]_{q_1}[n_1+1]_{q_1}^3} + \frac{(\alpha_1-1)q_1^{n_1-1}[2]_{q_1}\left(1+2q_1+3q_1^2\right)}{[4]_{q_1}[n_1+1]_{q_1}^3}\right\} x^2 \\
& + \left[\frac{[n_1-1]_{q_1}}{[n_1+1]_{q_1}^3} + \frac{\left(2+5q_1+3q_1^2\right)[n_1]_{q_1}}{[4]_{q_1}[n_1+1]_{q_1}^3} + \frac{q_1^{n_1-1}}{[n_1+1]_{q_1}^3}\right. \\
& \left. + \frac{(1-\alpha_1)q_1^{n_1-1}[2]_{q_1}[3]_{q_1}\left(3+2q_1+q_1^2\right)}{[4]_{q_1}[n_1+1]_{q_1}^3}\right] x + \frac{1}{[4]_{q_1}[n_1+1]_{q_1}^3} \\
:=\ & \varphi\left(q_1, q_1^{n_1\pm\cdot}, [n_1\pm\cdot]_{q_1}, \alpha_1; x\right);
\end{aligned} \tag{21}
$$

$$
K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(e_{03}; x, y) = \varphi\left(q_2, q_2^{n_2\pm\cdot}, [n_2\pm\cdot]_{q_2}, \alpha_2; y\right); \tag{22}
$$

$$
\begin{aligned}
& K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(e_{40};x,y) \\
= \quad & \Bigg[\frac{q_1^6[n_1-1]_{q_1}[n_1-2]_{q_1}[n_1-3]_{q_1}[n_1-4]_{q_1}}{[n_1+1]_{q_1}^4} + \frac{\alpha_1[3]_{q_1}q_1^{n_1+2}[n_1-2]_{q_1}[n_1-3]_{q_1}[n_1]_{q_1}}{[n_1+1]_{q_1}^4} \\
+ \quad & \frac{q_1^{n_1+5}[n_1-2]_{q_1}[n_1-3]_{q_1}[n_1-4]_{q_1}}{[n_1+1]_{q_1}^4} + (1-\alpha_1)\frac{10q_1^{n_1+7}[n_1-2]_{q_1}[n_1-3]_{q_1}[n_1-4]_{q_1}}{[2]_{q_1}[3]_{q_1}[4]_{q_1}[n_1+1]_{q_1}^4} \Bigg] x^4 \\
+ \quad & \Bigg\{ \frac{q_1^3[n_1-1]_{q_1}[n_1-2]_{q_1}[n_1-3]_{q_1}\left[1+2q_1+3q_1^2+4q_1^3+[5]_{q_1}\left(3+2q_1+q_1^2\right)\right]}{[5]_{q_1}[n_1+1]_{q_1}^4} \\
+ \quad & \frac{q_1^{n_1}[n_1-1]_{q_1}[n_1-2]_{q_1}\left[1+2q_1+3q_1^2+4q_1^3+[5]_{q_1}\left(4+3q_1+2q_1^2+q_1^3\right)\right]}{[5]_{q_1}[n_1+4]_{q_1}^4} \\
+ \quad & \frac{\alpha_1 q_1^{n_1}[n_1-1]_{q_1}[n_1-2]_{q_1}\left(-1+2q_1+8q_1^2+12q_1^3+15q_1^4+13q_1^5+7q_1^6+3q_1^7+q_1^8\right)}{[5]_{q_1}[n_1+1]_{q_1}^4} \\
+ \quad & \frac{(\alpha_1-1)q_1^{n}[2]_{q_1}[n_1-2]_{q_1}\left[[5]_{q_1}\left(4+3q_1+2q_1^2+q_1^3\right)+1+2q_1+3q_1^2+4q_1^3\right]}{[5]_{q_1}[n_1+1]_{q_1}^4} \Bigg\} x^3 \\
+ \quad & \Bigg\{ \frac{q_1[n_1-1]_{q_1}[n_1-2]_{q_1}\left(5+11q_1+16q_1^2+21q_1^3+17q_1^4+4q_1^5+q_1^6\right)}{[5]_{q_1}[n_1+1]_{q_1}^4} \\
+ \quad & \frac{q_1^{n_1-1}[n_1-1]_{q_1}\left(9+22q_1+35q_1^2+44q_1^3+36q_1^4+23q_1^5+11q_1^6+4q_1^7+q_1^8\right)}{[5]_{q_1}[n_1+1]_{q_1}^4} \\
- \quad & \frac{\alpha_1 q_1^{n_1-1}[n_1-1]_{q_1}\left(4+6q_1+9q_1^2+11q_1^3+7q_1^4+8q_1^5+6q_1^6+3q_1^7+q_1^8\right)}{[5]_{q_1}[n_1+1]_{q_1}^4} \\
+ \quad & \frac{(\alpha_1-1)q_1^{n_1-1}\left(10+26q_1+44q_1^2+50q_1^3+36q_1^4+23q_1^5+11q_1^6+4q_1^7+q_1^8\right)}{[5]_{q_1}[n_1+1]_{q_1}^4} \\
+ \quad & \frac{q_1[2]_{q_1}\left(1+3q_1+6q_1^2\right)[n_1-1]_{q_1}}{[5]_{q_1}[n_1+1]_{q_1}^4} \Bigg\} x^2 + \Bigg[\frac{\left(2+5q_1+11q_1^2+11q_1^3+q_1^4\right)[n_1+1]_{q_1}}{[5]_{q_1}[n_1+1]_{q}^4} \\
+ \quad & \frac{q_1^{n_1-1}\left(2+3q_1+4q_1^2+5q_1^3+q_1^4\right)}{[5]_{q_1}[n_1+1]_{q_1}^4} + \frac{2+7q_1+6q_1^2}{[5]_{q_1}[n_1+1]_{q_1}^4} \\
+ \quad & \frac{(1-\alpha_1)q_1^{n_1-1}\left(6+20q_1+35q_1^2+39q_1^3+29q_1^4+15q_1^5+5q_1^6+q_1^7\right)}{[5]_{q_1}[n_1+1]_{q_1}^4} \Bigg] x \\
+ \quad & \frac{1}{[5]_{q_1}[n_1+1]_{q_1}^4} := \psi\left(q_1, q_1^{n_1\pm\cdot}, [n_1\pm\cdot]_{q_1}, \alpha_1; x\right);
\end{aligned} \tag{23}
$$

$$
K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(e_{04};x,y) = \psi\left(q_2, q_2^{n_2\pm\cdot}, [n_2\pm\cdot]_{q_2}, \alpha_2; y\right), \tag{24}
$$

where $\cdot \in \mathbb{N}$.

Proof. Using q-Jackson integrals, we get

$$
\begin{aligned}
\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} t^3 d_q t \quad &= \quad (1-q)\sum_{j=0}^{\infty}\left(\frac{[k+1]_q^4}{[n+1]_q^4} - \frac{[k]_q^4}{[n+1]_q^4}\right)q^{4j} \\
&= \quad \frac{q^k\left[1+(1+3q)[k]_q+\left(1+2q+3q^2\right)[k]_q^2+[4]_q[k]_q^3\right]}{[4]_q[n+1]_q^4}.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
& K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(e_{30};x,y) \\
= \quad & [n_1+1]_{q_1}[n_2+1]_{q_2}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} p_{n_1,q_1,k_1}^{(\alpha_1)}(x)p_{n_2,q_2,k_2}^{(\alpha_2)}(y)q_1^{-k_1}q_2^{-k_2} \\
& \times \int_{\frac{[k_1]_{q_1}}{[n_1+1]_{q_1}}}^{\frac{[k_1+1]_{q_1}}{[n_1+1]_{q_1}}} \int_{\frac{[k_2]_{q_2}}{[n_2+1]_{q_2}}}^{\frac{[k_2+1]_{q_2}}{[n_2+1]_{q_2}}} t^3 d_{q_1}t d_{q_2}s \\
= \quad & [n_1+1]_{q_1}\sum_{k_1=0}^{n_1} p_{n_1,q_1,k_1}^{(\alpha_1)}(x)q_1^{-k_1}
\end{aligned}
$$

$$
\begin{aligned}
&\times\frac{q_1^{k_1}\left[1+(1+3q_1)[k_1]_{q_1}+\left(1+2q_1+3q_1^2\right)[k_1]_{q_1}^2+[4]_{q_1}[k_1]_{q_1}^3\right]}{[4]_{q_1}[n_1+1]_{q_1}^4}\\
=\;&\frac{[n_1]_{q_1}^3}{[n_1+1]_{q_1}^3}T_{n_1,q_1,\alpha_1}\left(t^3;x\right)+\frac{\left(1+2q_1+3q_1^2\right)[n_1]_{q_1}^2}{[4]_{q_1}[n_1+1]_{q_1}^3}T_{n_1,q_1,\alpha_1}\left(t^2;x\right)\\
&+\frac{(1+3q_1)[n_1]_{q_1}}{[4]_{q_1}[n_1+1]_{q_1}^3}T_{n_1,q_1,\alpha_1}(t;x)+\frac{1}{[4]_{q_1}[n_1+1]_{q_1}^3}.
\end{aligned}
$$

Then, Equation (21) can be obtained by Lemma 1, Lemma 4, and some computations. In addition, we can get Equation (22). Similarly, we have

$$
\begin{aligned}
&\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} t^4 d_qt\\
=\;&\frac{q^k\left[1+(1+4q)[k]_q+\left(1+3q+6q^2\right)[k]_q^2+\left(1+2q+3q^2+4q^3\right)[k]_q^3+[5]_q[k]_q^4\right]}{[5]_q[n+1]_q^5}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(e_{40};x,y)\\
=\;&\frac{[n_1]_{q_1}^4}{[n_1+1]_{q_1}^4}T_{n_1,q_1,\alpha_1}\left(t^4;x\right)+\frac{\left(1+2q_1+3q_1^2+4q_1^3\right)[n_1]_{q_1}^3}{[5]_{q_1}[n_1+1]_{q_1}^4}T_{n_1,q_1,\alpha_1}\left(t^3;x\right)\\
&+\frac{\left(1+3q_1+6q_1^2\right)[n_1]_{q_1}}{[5]_{q_1}[n_1+1]_{q_1}^4}T_{n_1,q_1,\alpha_1}\left(t^2;x\right)+\frac{(1+4q_1)[n_1]_{q_1}}{[5]_{q_1}[n_1+1]_{q_1}^4}T_{n_1,q_1,\alpha_1}(t;x)\\
&+\frac{1}{[5]_{q_1}[n_1+1]_{q_1}^4},
\end{aligned}
$$

using Lemmas 1 and 4, we can obtain Equations (23) and (24). □

Corollary 2. *For fixed real α_1,α_2 in $[0,1]$, from Corollary 1, Lemma 5, and, by some computations, we can get*

$$
\begin{aligned}
K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}\left((t-x)^2;x,y\right)\le\frac{C_1}{[n_1+1]_{q_1}};\ K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}\left((s-y)^2;x,y\right)\le\frac{C_2}{[n_2+1]_{q_2}};\\
K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}\left((t-x)^4;x,y\right)\le\frac{C_3}{[n_1+1]_{q_1}^2};\ K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}\left((s-y)^4;x,y\right)\le\frac{C_4}{[n_2+1]_{q_2}^2},
\end{aligned}
$$

where C_i, $(i=1,\ldots,4)$ are some positive constants.

4. Approximation Properties for $K_{n_1,n_2,q_1,q_2}^{\alpha_1,\alpha_2}(f)$

Let $I=[0,1]$, $I^2=I\times I$, $C\left(I^2\right)$ be the space of all real valued continuous functions on I^2 with the norm $||f||=\sup_{(x,y)\in I^2}|f(x,y)|$. For $f\in C\left(I^2\right)$, $\delta_1,\delta_2>0$, the complete modulus of continuous for the bivariate case is as below:

$$
\omega(f;\delta_1,\delta_2)=\sup\left\{|f(t,s)-f(x,y)|:(t,s),(x,y)\in I^2,|t-x|\le\delta_1,|s-y|\le\delta_2\right\}.
$$

Furthermore, $\omega(f;\delta_1,\delta_2)$ satisfies the following features:

$$
\begin{aligned}
&(i)\ \omega(f;\delta_1,\delta_2)\to 0,\ if\ \delta_1,\delta_2\to 0;\\
&(ii)\ |f(t,s)-f(x,y)|\le\omega(f;\delta_1,\delta_2)\left(1+\frac{|t-x|}{\delta_1}\right)\left(1+\frac{|s-y|}{\delta_2}\right).
\end{aligned}
$$

With respect to x and y, the partial modules of continuity are given by

$$\begin{aligned}\omega^{(1)}(f;\delta_1) &= \sup\{|f(x_1,y)-f(x_2,y)| : y\in I, |x_1-x_2|\le\delta_1\},\\ \omega^{(2)}(f;\delta_2) &= \sup\{|f(x,y_1)-f(x,y_2)| : x\in I, |y_1-y_2|\le\delta_2\}.\end{aligned}$$

For more information about these definitions, see [17]. Let $C^2\left(I^2\right)$ be the space of all functions $f\in C\left(I^2\right)$ such that $\frac{\partial^i f}{\partial x^i}, \frac{\partial^i f}{\partial y^i}$ for $i=1,2$ belong to $C\left(I^2\right)$. The norm on the space $C^2\left(I^2\right)$ is as below: $||f||_{C^2(I^2)} = ||f|| + \sum_{i=1}^2\left(\left|\left|\frac{\partial^i f}{\partial x^i}\right|\right| + \left|\left|\frac{\partial^i f}{\partial y^i}\right|\right|\right)$. For $f\in C\left(I^2\right)$, $\delta>0$, the Peetre's K-functional is defined as

$$K(f;\delta) = \inf_{g\in C^2(I^2)}\{||f-g||+\delta||g||\}.$$

We have

$$K(f;\delta)\le C\left[\omega_2\left(f;\sqrt{\delta}\right)+\min(1,\delta)||f||_{C(I^2)}\right],$$

where C is a constant and independent of δ and f, $\omega_2\left(f;\sqrt{\delta}\right)$ is the second modulus of continuity of bivariate function f.

Now, the estimate of the rate of convergence of $K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f)$ is obtained.

Theorem 1. *For $f\in C(I^2)$, the following inequality is given*

$$\left|K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f;x,y)-f(x,y)\right|\le 4\omega\left(f;\sqrt{\delta_{n_1}^{(\alpha_1)}(x)},\sqrt{\delta_{n_2}^{(\alpha_2)}(y)}\right),$$

where $\delta_{n_1}^{(\alpha_1)}(x)$ and $\delta_{n_2}^{(\alpha_2)}(y)$ are as in Equations (12) and (13).

Proof. By the linearity of $K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f)$, using Corollary 1 and the above property (ii), we have

$$\begin{aligned}&\left|K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f;x,y)-f(x,y)\right|\\ \le\ & K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(|f(t,s)-f(x,y)|;x,y)\\ \le\ & \omega\left(f;\sqrt{\delta_{n_1}^{(\alpha_1)}(x)},\sqrt{\delta_{n_2}^{(\alpha_2)}(y)}\right)\left[K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(1;x,y)+\frac{K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(|t-x|;x,y)}{\sqrt{\delta_{n_1}^{(\alpha_1)}(x)}}\right]\\ &\times\left[K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(1;x,y)+\frac{K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(|s-y|;x,y)}{\sqrt{\delta_{n_2}^{(\alpha_2)}(y)}}\right].\end{aligned}$$

Then, from the linear property of $K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f)$ and Cauchy–Schwarz inequality, one can obtain Theorem 1 by the fact that

$$\begin{aligned}K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(|t-x|;x,y) &\le \sqrt{K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}\left((t-x)^2;x,y\right)}\sqrt{K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(1;x,y)}\\ &= \sqrt{\delta_{n_1}^{(\alpha_1)}(x)},\\ K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(|s-y|;x,y) &\le \sqrt{K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}\left((s-y)^2;x,y\right)}\sqrt{K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(1;x,y)}\\ &= \sqrt{\delta n_2{}^{(\alpha_2)}(y)}.\end{aligned}$$

Theorem 1 is proved. □

Theorem 2. *For $f \in C(I^2)$, the following inequality is obtained*

$$\left|K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f;x,y)-f(x,y)\right| \leq 2\left[\omega^{(1)}\left(f;\sqrt{\delta_{n_1}^{(\alpha_1)}(x)}\right)+\omega^{(2)}\left(f;\sqrt{\delta_{n_2}^{(\alpha_2)}(y)}\right)\right],$$

where $\delta_{n_1}^{(\alpha_1)}(x)$ and $\delta_{n_2}^{(\alpha_2)}(y)$ are as in Equations (12) and (13).

Proof. With the help of partial moduli of continuity above and Cauchy–Schwarz inequality, we get

$$\begin{aligned}
& \left|K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f;x,y)-f(x,y)\right| \\
\leq\; & K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(|f(t,s)-f(x,y)|;x,y) \\
\leq\; & K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(|f(t,y)-f(x,y)|;x,y)+K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(|f(t,s)-f(t,y)|;x,y) \\
\leq\; & K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}\left(\omega^{(1)}(f;|t-x|);x,y\right)+K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}\left(\omega^{(2)}(f;|s-y|;x,y)\right) \\
\leq\; & \omega^{(1)}\left(f;\sqrt{\delta_{n_1}^{(\alpha_1)}(x)}\right)\left(1+\sqrt{\frac{K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}((t-x)^2;x,y)}{\delta_{n_1}^{(\alpha_1)}(x)}}\right) \\
& +\omega^{(2)}\left(f;\sqrt{\delta_{n_2}^{(\alpha_2)}(y)}\right)\left(1+\sqrt{\frac{K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}((s-y)^2;x,y)}{\delta_{n_2}^{(\alpha_2)}(y)}}\right).
\end{aligned}$$

Theorem 2 is proved. □

Theorem 3. *For $f \in C(I^2)$, the following inequality is derived:*

$$\begin{aligned}
& \left|K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f;x,y)-f(x,y)\right| \\
\leq\; & C\left[\omega_2\left(f;\frac{1}{2}\sqrt{\delta_{n_1}^{(\alpha_1)}(x)+\delta_{n_2}^{(\alpha_2)}(y)+\frac{\left(1-[2]_{q_1}q_1^{n_1}x\right)^2}{[2]_{q_1}^2[n_1+1]_{q_1}^2}+\frac{\left(1-[2]_{q_2}q_2^{n_2}y\right)^2}{[2]_{q_2}^2[n_2+1]_{q_2}^2}}\right)\right. \\
& \left.+\min\left\{1,\delta_{n_1}^{(\alpha_1)}(x)+\delta_{n_2}^{(\alpha_2)}(y)+\frac{\left(1-[2]_{q_1}q_1^{n_1}x\right)^2}{[2]_{q_1}^2[n_1+1]_{q_1}^2}+\frac{\left(1-[2]_{q_2}q_2^{n_2}y\right)^2}{[2]_{q_2}^2[n_2+1]_{q_2}^2}\right\}||f||_{C(I^2)}\right] \\
& +\omega\left(f;\left|\frac{1-[2]_{q_1}q_1^{n_1}x}{[2]_{q_1}[n_1+1]_{q_1}}\right|,\left|\frac{1-[2]_{q_2}q_2^{n_2}y}{[2]_{q_2}[n_2+1]_{q_2}}\right|\right),
\end{aligned}$$

where C is a positive constant, $\delta_{n_1}^{(\alpha_1)}(x)$, and $\delta_{n_2}^{\alpha_2}(y)$ are defined in Equations (12) and (13).

Proof. For $(x,y) \in I^2$, the auxiliary operators are as follows:

$$\begin{aligned}
& \widetilde{K}_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f;x,y) \\
=\; & K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f;x,y)-f\left(\tfrac{[2]_{q_1}[n_1]_{q_1}x+1}{[2]_{q_1}[n_1+1]_{q_1}},\tfrac{[2]_{q_2}[n_2]_{q_2}y+1}{[2]_{q_2}[n_2+1]_{q_2}}\right)+f(x,y).
\end{aligned} \tag{25}$$

Then, we get

$$\widetilde{K}_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(t-x;x,y)=\widetilde{K}_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(s-y;x,y)=0 \tag{26}$$

by using Lemma 2. Let $g \in C^2(I^2)$. By Taylor's expansion,

$$g(t,s)-g(x,y)$$

$$
\begin{aligned}
&= g(t,y)-g(x,y)+g(t,s)-g(t,y)\\
&= \frac{\partial g(x,y)}{\partial x}(t-x)+\int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du+\frac{\partial g(x,y)}{\partial y}(s-y)+\int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv.
\end{aligned}
$$

Applying $\widetilde{K}_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}$ on both sides of the above equation and using Equation (25), we get

$$
\begin{aligned}
&\left|\widetilde{K}_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(g;x,y)-g(x,y)\right|\\
\leq\ & \left|\widetilde{K}_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}\left(\int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du;x,y\right)\right|\\
&+\left|\widetilde{K}_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}\left(\int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv;x,y\right)\right|\\
\leq\ & K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}\left(\left|\int_x^t |t-u|\left|\frac{\partial^2 g(u,y)}{\partial u^2}\right|du\right|;x,y\right)\\
&+\left|\int_x^{\frac{[2]_{q_1}[n_1]_{q_1}x+1}{[2]_{q_1}[n_1+1]_{q_1}}}\left|\frac{[2]_{q_1}[n_1]_{q_1}x+1}{[2]_{q_1}[n_1+1]_{q_1}}-u\right|\left|\frac{\partial^2 g(u,y)}{\partial u^2}\right|du\right|\\
&+K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}\left(\left|\int_y^s |s-v|\left|\frac{\partial^2 g(x,v)}{\partial v^2}\right|dv\right|;x,y\right)\\
&+\left|\int_y^{\frac{[2]_{q_2}[n_2]_{q_2}y+1}{[2]_{q_2}[n_2+1]_{q_2}}}\left|\frac{[2]_{q_2}[n_2]_{q_2}y+1}{[2]_{q_2}[n_2+1]_{q_2}}-v\right|\left|\frac{\partial^2 g(x,v)}{\partial v^2}\right|dv\right|\\
\leq\ & \left[K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}\left((t-x)^2;x,y\right)+\left(\frac{[2]_{q_1}[n_1]_{q_1}x+1}{[2]_{q_1}[n_1+1]_{q_1}}-x\right)^2\right]||g||_{C^2(I^2)}\\
&+\left[K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}\left((s-y)^2;x,y\right)+\left(\frac{[2]_{q_2}[n_2]_{q_2}y+1}{[2]_{q_2}[n_2+1]_{q_2}}-y\right)^2\right]||g||_{C^2(I^2)}\\
\leq\ & \left[\delta_{n_1}^{(\alpha_1)}(x)+\delta_{n_2}^{(\alpha_2)}(y)+\frac{(1-[2]_{q_1}q_1^{n_1}x)^2}{[2]_{q_1}^2[n_1+1]_{q_1}^2}+\frac{(1-[2]_{q_2}q_2^{n_2}y)^2}{[2]_{q_2}^2[n_2+1]_{q_2}^2}\right]||g||_{C^2(I^2)}.
\end{aligned}
$$

On the one hand, by Equations (5) and (25), and Lemma 2, we obtain

$$
\begin{aligned}
&\left|\widetilde{K}_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f;x,y)\right|\\
\leq\ & \left|K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f;x,y)\right|+\left|f\left(\frac{[2]_{q_1}[n_1]_{q_1}x+1}{[2]_{q_1}[n_1+1]_{q_1}},\frac{[2]_{q_2}[n_2]_{q_2}y+1}{[2]_{q_2}[n_2+1]_{q_2}}\right)\right|+|f(x,y)| \qquad (27)\\
\leq\ & 3||f||_{C(I^2)}.
\end{aligned}
$$

Now, Equation (25) and Inequality (28) imply

$$
\begin{aligned}
&\left|K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f;x,y)-f(x,y)\right|\\
=\ & \left|\widetilde{K}_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f;x,y)-f(x,y)+f\left(\frac{[2]_{q_1}[n_1]_{q_1}x+1}{[2]_{q_1}[n_1+1]_{q_1}},\frac{[2]_{q_2}[n_2]_{q_2}y+1}{[2]_{q_2}[n_2+1]_{q_2}}\right)-f(x,y)\right|\\
\leq\ & \left|\widetilde{K}_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f-g;x,y)\right|+\left|\widetilde{K}_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(g;x,y)-g(x,y)\right|+|g(x,y)-f(x,y)|\\
&+\left|f\left(\frac{[2]_{q_1}[n_1]_{q_1}x+1}{[2]_{q_1}[n_1+1]_{q_1}},\frac{[2]_{q_2}[n_2]_{q_2}y+1}{[2]_{q_2}[n_2+1]_{q_2}}\right)-f(x,y)\right|\\
\leq\ & 4||f-g||_{C(I^2)}+\left|\widetilde{K}_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(g;x,y)-g(x,y)\right|\\
&+\left|f\left(\frac{[2]_{q_1}[n_1]_{q_1}x+1}{[2]_{q_1}[n_1+1]_{q_1}},\frac{[2]_{q_2}[n_2]_{q_2}y+1}{[2]_{q_2}[n_2+1]_{q_2}}\right)-f(x,y)\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left\{4||f-g||_{C(I^2)}+\left[\delta_{n_1}^{(\alpha_1)}(x)+\delta_{n_2}^{(\alpha_2)}(y)+\frac{\left(1-[2]_{q_1}q_1^{n_1}x\right)^2}{[2]_{q_1}^2[n_1+1]_{q_1}^2}\right.\right.\\
&\quad\left.\left.+\frac{\left(1-[2]_{q_2}q_2^{n_2}y\right)^2}{[2]_{q_2}^2[n_2+1]_{q_2}^2}\right]||g||_{C^2(I^2)}\right\}+\omega\left(f;\left|\frac{1-[2]_{q_1}q_1^{n_1}x}{[2]_{q_1}[n_1+1]_{q_{n_1}}}\right|,\left|\frac{1-[2]_{q_2}q_2^{n_2}y}{[2]_{q_2}[n_2+1]_{q_2}}\right|\right)\\
&\leq 4K\left(f;\frac{1}{4}\left[\delta_{n_1}^{(\alpha_1)}(x)+\delta_{n_2}^{(\alpha_2)}(y)+\frac{\left(1-[2]_{q_1}q_1^{n_1}x\right)^2}{[2]_{q_1}^2[n_1+1]_{q_1}^2}+\frac{\left(1-[2]_{q_2}q_2^{n_2}y\right)^2}{[2]_{q_2}^2[n_2+1]_{q_2}^2}\right]\right)\\
&\quad+\omega\left(f;\left|\frac{1-[2]_{q_1}q_1^{n_1}x}{[2]_{q_1}[n_1+1]_{q_1}}\right|,\left|\frac{1-[2]_{q_2}q_2^{n_2}y}{[2]_{q_2}[n_2+1]_{q_2}}\right|\right).
\end{aligned}
$$

Finally, by the relationship between Peetre's K-functional and second modulus of continuity, we have

$$
\begin{aligned}
&\left|K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f;x,y)-f(x,y)\right|\\
&\leq C\left[\omega_2\left(f;\frac{1}{2}\sqrt{\delta_{n_1}^{(\alpha_1)}(x)+\delta_{n_2}^{(\alpha_2)}(y)+\frac{\left(1-[2]_{q_1}q_1^{n_1}x\right)^2}{[2]_{q_1}^2[n_1+1]_{q_1}^2}+\frac{\left(1-[2]_{q_2}q_2^{n_2}y\right)^2}{[2]_{q_2}^2[n_2+1]_{q_2}^2}}\right)\right.\\
&\quad\left.+\min\left\{1,\delta_{n_1}^{(\alpha_1)}(x)+\delta_{n_2}^{(\alpha_2)}(y)+\frac{\left(1-[2]_{q_1}q_1^{n_1}x\right)^2}{[2]_{q_1}^2[n_1+1]_{q_1}^2}+\frac{\left(1-[2]_{q_2}q_{n_2}^{n_2}y\right)^2}{[2]_{q_2}^2[n_2+1]_{q_2}^2}\right\}||f||_{C(I^2)}\right]\\
&\quad+\omega\left(f;\left|\frac{1-[2]_{q_1}q_1^{n_1}x}{[2]_{q_1}[n_1+1]_{q_1}}\right|,\left|\frac{1-[2]_{q_2}q_2^{n_2}y}{[2]_{q_2}[n_2+1]_{q_2}}\right|\right),
\end{aligned}
$$

where C is a positive constant, $\delta_{n_1}^{(\alpha_1)}(x)$ and $\delta_{n_2}^{\alpha_2}(y)$ are defined in Equations (12) and (13). Theorem 3 is proved. □

Finally, we derive the rate of convergence of $K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f;x,y)$ via functions of Lipschitz class $Lip_M(\zeta,\eta)$ if

$$|f(t,s)-f(x,y)|\leq M|t-x|^{\zeta}|s-y|^{\eta},\ \ (t,s),\ (x,y)\in I^2;\ \zeta,\ \eta\in(0,1].$$

Theorem 4. *Letting $f\in Lip_M(\zeta,\eta)$, the following inequality holds:*

$$\left|K_{n_1,n_2,q_{n_1},q_{n_2}}^{(\alpha_1,\alpha_2)}(f;x,y)-f(x,y)\right|\leq M\left[\delta_{n_1}^{(\alpha_1)}(x)\right]^{\frac{\zeta}{2}}\left[\delta_{n_2}^{(\alpha_2)}(y)\right]^{\frac{\eta}{2}},$$

where M is a positive constant, $\delta_{n_1}^{(\alpha_1)}(x)$, and $\delta_{n_2}^{\alpha_2}(y)$ are as in (12) and (13).

Proof. Since $K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f)$ are positive linear operators and $f\in Lip_M(\zeta,\eta)$, then we have

$$
\begin{aligned}
&\left|K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f;x,y)-f(x,y)\right|\\
&\leq K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}\left(|f(t,s)-f(x,y)|;x,y\right)\\
&\leq MK_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}\left(|t-x|^{\zeta};x,y\right)K_{n_1,n_2,q_{n_1},q_{n_2}}^{(\alpha_1,\alpha_2)}\left(|s-y|^{\eta};x,y\right),
\end{aligned}
$$

with the help of the Hölder's inequality, respectively, we get

$$
\begin{aligned}
&\left|K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f;x,y)-f(x,y)\right|\\
&\leq M\left[K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}\left((t-x)^2;x,y\right)\right]^{\frac{\zeta}{2}}\left[K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(1;x,y)\right]^{\frac{2-\zeta}{2}}
\end{aligned}
$$

$$
\begin{aligned}
& \times \left[K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}\left((s-y)^2;x,y\right)\right]^{\frac{\eta}{2}} \left[K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(1;x,y)\right]^{\frac{2-\eta}{2}} \\
= \quad & M\left[\delta_{n_1}^{(\alpha_1)}(x)\right]^{\frac{\zeta}{2}}\left[\delta_{n_2}^{(\alpha_2)}(y)\right]^{\frac{\eta}{2}},
\end{aligned}
$$

where M is a positive constant, $\delta_{n_1}^{(\alpha_1)}(x)$, and $\delta_{n_2}^{\alpha_2}(y)$ are as in Equations (12) and (13). □

5. Approximation Properties for $UK_{n_1,n_2,q_1,q_2}^{\alpha_1,\alpha_2}(f)$

Let X and Y be compact real intervals, we give the following definitions, which can be referred to [18–20].

(i) f is called B-continuous function in $(x_0,y_0)\in X\times Y$ if

$$\lim_{(x,y)\to(x_0,y_0)} \triangle f((x,y),(x_0,y_0)) = 0,$$

where $\triangle f((x,y),(x_0,y_0)) = f(x,y) - f(x_0,y) - f(x,y_0) + f(x_0,y_0)$.

(ii) f is B-differentiable function in $(x_0,y_0)\in X\times Y$ and denoted by $D_B f(x_0,y_0)$ if it exists, and the following limit is finite:

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\triangle f((x,y),(x_0,y_0))}{(x-x_0)(y-y_0)}.$$

(iii) f is B-bounded on $X\times Y$ if there exists $k>0$ such that $|\triangle f((x,y),(t,s))|\le K$ for any (x,y), $(t,s)\in X\times Y$.

(iv) $B(X\times Y)$: the space of all bounded functions on $X\times Y$.

(v) $C(X\times Y)$: the space of all continuous functions on $X\times Y$.

(vi) $B_b(X\times Y) = \{f : X\times Y \to \mathbb{R} | f \textit{ is } B\textit{-bounded on } X\times Y\}$ with the norm $||f||_B = \sup_{(x,y),(t,s)\in X\times Y}|\triangle f((x,y),(t,s))|$.

(vii) $C_b(X\times Y) = \{f : X\times Y \to \mathbb{R} | f \textit{ is } B\textit{-continuous on } X\times Y\}$.

(viii) $D_b(X\times Y) = \{f : X\times Y \to \mathbb{R} | f \textit{ is } B\textit{-differentiable on } X\times Y\}$.

(ix) For $f\in B_b(X\times Y)$, $\delta_1,\delta_2\ge 0$,

$$\omega_{mixed}(f;\delta_1,\delta_2) = \sup\{|\triangle f((x,y),(t,s))| : |x-t|\le\delta_1, |y-s|\le\delta_2\}$$

is called the mixed modulus of smoothness.

(x) Let $L : C_b(X\times Y)\to B(X\times Y)$ be linear positive operator, for any $f\in C_b(X\times Y)$, $(x,y)\in X\times Y$,

$$UL\left(f(t,s);x,y\right) = L\left(f(t,y)+f(x,s)-f(t,s);x,y\right),$$

is called the GBS operator.

In order to estimate the rate of convergence of $UK_{n_1,n_2,q_{n_1},q_{n_2}}^{(\alpha_1,\alpha_2)}(f;x,y)$, we give the following known results.

Theorem 5. *(See [21]) Let $L : C_b(X\times Y)\to B(X\times Y)$ be a linear positive operator and $UL : C_b(X\times Y)\to B(X\times Y)$ the associated GBS operator. Then, for any $f\in C_b(X\times Y)$, any $(x,y)\in(X\times Y)$ and any $\delta_1,\delta_2>0$, we have*

$$
\begin{aligned}
& |UL\left(f(t,s);x,y\right) - f(x,y)| \\
\le \quad & |f(x,y)||1-L\left(e_{00};x,y\right)| + \left[L\left(e_{00};x,y\right) + \delta_1^{-1}\sqrt{L\left((t-x)^2;x,y\right)}\right.
\end{aligned}
$$

$$+\delta_2^{-1}\sqrt{L\left((s-y)^2;x,y\right)}+\delta_1^{-1}\delta_2^{-1}\sqrt{L\left((t-x)^2(s-y)^2;x,y\right)}\Big]\,\omega_{mixed}(f;\delta_1,\delta_2).$$

Theorem 6. *(See [22]) Let* $L : C_b(X\times Y)\to B(X\times Y)$ *be a linear positive operator and* $UL : C_b(X\times Y)\to B(X\times Y)$ *the associated GBS operator. Then, for any* $f\in D_b(X\times Y)$ *with* $D_Bf\in B(X\times Y)$*, any* $(x,y)\in(X\times Y)$ *and any* $\delta_1,\delta_2>0$*, we have*

$$\begin{aligned}
&|UL\left(f(t,s);x,y\right)-f(x,y)|\\
\le\ & |f(x,y)|\,|1-L\left(e_{00};x,y\right)|+3||D_Bf||_\infty\sqrt{L\left((t-x)^2(s-y)^2;x,y\right)}\\
&+\left[\sqrt{L\left((t-x)^2(s-y)^2;x,y\right)}+\delta_1^{-1}\sqrt{L\left((t-x)^4(s-y)^2;x,y\right)}\right.\\
&\left.+\delta_2^{-1}\sqrt{L\left((t-x)^2(s-y)^4;x,y\right)}+\delta_1^{-1}\delta_2^{-1}L\left((t-x)^2(s-y)^2;x,y\right)\right]\\
&\times\omega_{mixed}\left(D_Bf;\delta_1,\delta_2\right).
\end{aligned}$$

Now, the rate of convergence of $UK^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}(f)$ to $f\in C_b(I^2)$ is given.

Theorem 7. *For* $f\in C_b(I^2)$*, the following inequality holds:*

$$\left|UK^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}(f;x,y)-f(x,y)\right|\le M_1\omega_{mixed}\left(f;\frac{1}{\sqrt{m+1}},\frac{1}{\sqrt{n+1}}\right),$$

where M_1 *is a positive constant.*

Proof. Using Corollary 2, we have

$$\begin{aligned}
&K^{\alpha_1,\alpha_2}_{n_1,n_2,q_1,q_2}\left((t-x)^2(s-y)^2;x,y\right)\\
=\ &K^{\alpha_1,\alpha_2}_{n_1,n_2,q_1,q_2}\left((t-x)^2;x,y\right)K^{\alpha_1,\alpha_2}_{n_1,n_2,q_1,q_2}\left((s-y)^2;x,y\right)\\
=\ &\frac{C_1C_2}{[n_1+1]_{q_1}[n_2+1]_{q_2}},
\end{aligned}$$

applying Theorem 5, we get

$$\begin{aligned}
&\left|UK^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}(f;x,y)-f(x,y)\right|\\
\le\ &\left[1+\frac{1}{\delta_1}\sqrt{\frac{C_1}{[n_1+1]_{q_1}}}+\frac{1}{\delta_2}\sqrt{\frac{C_2}{[n_2+1]_{q_2}}}+\frac{1}{\delta_1\delta_2}\sqrt{\frac{C_1C_2}{[n_1+1]_{q_1}[n_2+1]_{q_2}}}\right]\omega_{mixed}(f;\delta_1,\delta_2).
\end{aligned}$$

Therefore, Theorem 7 can be derived by choosing $\delta_1=\frac{1}{\sqrt{[n_1+1]_{q_1}}}$ and $\delta_2=\frac{1}{\sqrt{[n_2+1]_{q_2}}}$. □

Finally, the rate of convergence of $UK^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}(f)$ to $f\in D_b(I^2)$ is given.

Theorem 8. *Let* $f\in D_b(I^2)$*,* $D_Bf\in B(I^2)$*,* $(x,y)\in I^2$ *and* $n_1,n_2>1$*, the following inequality holds:*

$$\begin{aligned}
&\left|UK^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}(f;x,y)-f(x,y)\right|\\
\le\ &\frac{M_2}{\sqrt{[n_1+1]_{q_1}[n_2+1]_{q_2}}}\left[||D_Bf||_\infty+\omega_{mixed}\left(D_Bf;\frac{1}{\sqrt{[n_1+1]_{q_1}}},\frac{1}{\sqrt{[n_2+1]_{q_2}}}\right)\right],
\end{aligned}$$

where M_2 *is a positive constant.*

Proof. For $(x, y), (t, s) \in I^2$ and $i, j \in \{1, 2\}$, we have

$$K^{\alpha_1,\alpha_2}_{n_1,n_2,q_1,q_2}\left((t-x)^{2i}(s-y)^{2j}; x, y\right) = K^{\alpha_1,\alpha_2}_{n_1,n_2,q_1,q_2}\left((t-x)^{2i}; x, y\right) K^{\alpha_1,\alpha_2}_{n_1,n_2,q_1,q_2}\left((s-y)^{2j}; x, y\right).$$

Then, using Theorem 6 and Corollary 2, we have

$$\begin{aligned} & \left|UK^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}(f; x, y) - f(x, y)\right| \\ \leq\ & 3\,||D_B f||_\infty \sqrt{\frac{C_1C_2}{[n_1+1]_{q_1}[n_2+1]_{q_2}}} + \left(\sqrt{\frac{C_1C_2}{[n_1+1]_{q_1}[n_2+1]_{q_2}}}\right. \\ & \left. + \frac{1}{\delta_1[n_1+1]_{q_1}}\sqrt{\frac{C_2C_3}{[n_2+1]_{q_2}}} + \frac{1}{\delta_2[n_2+1]_{q_2}}\sqrt{\frac{C_1C_4}{[n_1+1]_{q_1}}} + \frac{C_1C_2}{\delta_1\delta_2[n_1+1]_{q_1}[n_2+1]_{q_2}}\right) \\ & \times \omega_{mixed}\left(D_B f; \delta_1, \delta_2\right). \end{aligned}$$

Hence, taking $\delta_1 = \frac{1}{\sqrt{[n_1+1]_{q_1}}}$, $\delta_2 = \frac{1}{\sqrt{[n_2+1]_{q_2}}}$, we obtain the desired result of Theorem 8. □

6. Conclusions

In the present paper, a family of bivariate α, q-Bernstein–Kantorovich operators $K^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}(f; x, y)$ and a family of GBS operators of bivariate α, q-Bernstein–Kantorovich type $UK^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}(f(t,s); x, y)$ are introduced, the degree of approximation for $K^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}(f; x, y)$ is investigated by using the definitions of partial moduli of continuity and K-functional, and the rate of convergence of $UK^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}(f(t,s); x, y)$ for B-continuous and B-differentiable functions is also estimated.

Author Contributions: All authors contributed equally to this work.

Funding: This work is supported by the National Natural Science Foundation of China (Grant Nos. 11601266, 11626031), the Project for High-level Talent Innovation and Entrepreneurship of Quanzhou (Grant No. 2018C087R), the Program for New Century Excellent Talents in Fujian Province University and Sponsoring Agreement for Overseas Studies in Fujian Province, the Key Natural Science Research Project in Universities of Anhui Province (Grant No. KJ2019A0572), the Philosophy and Social Sciences General Planning Project of Anhui Province of China (Grant No. AHSKYG2017D153) and the Natural Science Foundation of Anhui Province of China (Grant No. 1908085QA29).

Acknowledgments: We thank the Fujian Provincial Key Laboratory of Data-Intensive Computing, the Fujian University Laboratory of Intelligent Computing and Information Processing, and the Fujian Provincial Big Data Research Institute of Intelligent Manufacturing of China.

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

References

1. Chen, X.; Tan, J.; Liu, Z.; Xie, J. Approximation of functions by a new family of generalized Bernstein operators. *J. Math. Anal. Appl.* **2017**, *450*, 244–261. [CrossRef]
2. Mohiuddine, S.A.; Acar, T.; Alotaibi, A. Construction of a new family of Bernstein–Kantorovich operators. *Math. Methods Appl. Sci.* **2017**, *40*, 7749–7759. [CrossRef]
3. Cai, Q.-B.; Xu, X.-W. Shape-preserving properties of a new family of generalized Bernstein operators. *J. Inequalities Appl.* **2018**, *241*. [CrossRef] [PubMed]
4. Nowak, G. Approximation properties for generalized q–Bernstein polynomials. *J. Math. Anal. Appl.* **2009**, *350*, 50–55. [CrossRef]
5. Il'inskii, A. Convergence of generalized Bernstein polynomial. *J. Approx. Theory* **2002**, *116*, 100–112. [CrossRef]
6. Wang, H.; Meng, F. The rate of convergence of q–Bernstein polynomials. *J. Approx. Theory* **2005**, *136*, 151–158. [CrossRef]

7. Gupta, V.; Finta, Z. On certain q-Durrmeyer type operators. *Appl. Math. Comput.* **2009**, *209*, 415–420. [CrossRef]
8. Gupta, V. Some approximation properties of q-Durrmeyer operators. *Appl. Math. Comput.* **2008**, *197*, 172–178. [CrossRef]
9. Gupta, V.; Wang, H. The rate of convergence of q-Durrmeyer operators for $0 < q < 1$. *Math. Methods Appl. Sci.* **2008**, *31*, 1946–1955.
10. Dalmanoglu, Ö.; Dogru, O. On statistical approximation properties of Kantorovich type q–Bernstein operators. *Math. Comput. Model.* **2010**, *52*, 760–771. [CrossRef]
11. Rahman, S.; Mursaleen, M.; Acu, A.M. Approximation properties of λ-Bernstein–Kantorovich operators with shifted knots. *Math. Methods Appl. Sci.* **2019**, *42*, 4042–4053. [CrossRef]
12. Mursaleen, M.; Ansari, K.J.; Khan, A. Approximation by a Kantorovich type q–Bernstein-Stancu operators. *Complex Anal. Oper. Theory* **2017**, *11*, 85–107. [CrossRef]
13. Mursaleen, M.; Khan, F.; Khan, A. Approximation properties for King's type modified q-Bernstein–Kantorovich operators. *Math. Methods Appl. Sci.* **2015**, *38*, 5242–5252. [CrossRef]
14. Mursaleen, M.; Khan, F.; Khan, A. Approximation properties for modified q-Bernstein–Kantorovich operators. *Numer. Funct. Anal. Optim.* **2015**, *36*, 1178–1197. [CrossRef]
15. Gasper, G.; Rahman, M. *Encyclopedia of Mathematics and Its Applications*; Basic Hypergeometric Series; Cambridge University Press: Cambridge, UK, 1990; p. 35.
16. Kac, V.G.; Cheung, P. *Quantum Calculus*; Springer: New York, NY, USA, 2002.
17. Anastassiou, G.A.; Gal, S.G. *Approximation Theory: Moduli of Continuity and Global Smoothness Preservation*; Birkhauser: Boston, MA, USA, 2000.
18. Bögel, K. Mehrdimensionale Differentiation von Funktionen mehrerer Ver änderlicher. *J. Reine Angew. Math.* **1934**, *170*, 197–217.
19. Bögel, K. Über die mehrdimensionale differentiation. *J. Reine Angew. Math.* **1935**, *173*, 5–29.
20. Cai, Q.B.; Zhou, G. Blending type approximation by GBS operators of bivariate tensor product of λ-Bernstein–Kantorovich type. *J. Inequal. Appl.* **2018**, *2018*, 268. [CrossRef] [PubMed]
21. Badea, C.; Cottin, C. Korovkin-type theorems for generalized boolean sum operators. In *Colloquia Mathematica Societatis János Bolyai*; Approximation Theory: Kecskemét, Hunagary, 1990; Volume 58, pp. 51–68.
22. Pop, O.T. Approximation of B-differentiable functions by GBS operators. *Anal. Univ. Oradea Fasc. Mat.* **2007**, *14*, 15–31.