The k -Rainbow Domination Number of $C_n \square C_m$

Hong Gao ^{1,*}, Kun Li ¹ and Yuansheng Yang ²

¹ College of Science, Dalian Maritime University, Dalian 116026, China; likun_dlm@dlmu.edu.cn

² School of Computer Science and Technology, Dalian University of Technology, Dalian 116024, China; yangys@dlut.edu.cn

* Correspondence: gaohong@dlmu.edu.cn

Received: 10 October 2019; Accepted: 21 November 2019; Published: 1 December 2019

Abstract: Given a graph G and a set of k colors, assign an arbitrary subset of these colors to each vertex of G . If each vertex to which the empty set is assigned has all k colors in its neighborhood, then the assignment is called a k -rainbow dominating function (k RDF) of G . The minimum sum of numbers of assigned colors over all vertices of G is called the k -rainbow domination number of graph G , denoted by $\gamma_{rk}(G)$. In this paper, we focus on the study of the k -rainbow domination number of the Cartesian product of cycles, $C_n \square C_m$. For $k \geq 8$, based on the results of J. Amjadi et al. (2017), $\gamma_{rk}(C_n \square C_m) = mn$. For $(4 \leq k \leq 7)$, we give a proof for the new lower bound of $\gamma_{r4}(C_n \square C_3)$. We construct some novel and recursive k RDFs which are good enough and upon these functions we get sharp upper bounds of $\gamma_{rk}(C_n \square C_m)$. Therefore, we obtain the following results: (1) $\gamma_{r4}(C_n \square C_3) = 2n$; (2) $\gamma_{rk}(C_n \square C_m) = \frac{kmn}{8}$ for $n \equiv 0 \pmod{4}, m \equiv 0 \pmod{4}$ ($4 \leq k \leq 7$); (3) for $n \not\equiv 0 \pmod{4}$ or $m \not\equiv 0 \pmod{4}$, $\frac{mn}{2} \leq \gamma_{r4}(C_n \square C_m) \leq \frac{mn}{2} + m + \frac{n}{2} - 1$ and $\frac{kmn}{8} \leq \gamma_{rk}(C_n \square C_m) \leq k \frac{mn+n}{8} + m$ for $5 \leq k \leq 7$. We also discuss Vizing's conjecture on the k -rainbow domination number of $C_n \square C_m$.

Keywords: rainbow domination; k -rainbow domination; cartesian product graph; cycle

1. Introduction

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E . The open neighborhood $N(v)$ of a vertex v consists of the vertices adjacent to v . The degree of a vertex $v \in V$ is $\deg_G(v) = |N(v)|$. The minimum and maximum degree of a graph G are denoted by $\delta(G)$ and $\Delta(G)$ respectively.

The k -rainbow domination problem is to determine the k -rainbow domination number of G . This problem can be described as the following: k -rainbow domination represents a situation in which there are k types of guards, and it is required that each location (vertex) which is not occupied by a guard has all types of guards in its neighborhood. The k -rainbow domination has many practical applications, such as in information transfer or people allocation between company departments, network security, channel assignment, logistics scheduling, storage hierarchy optimization, and so on. Therefore, it has been extensively studied [1,2].

Let G be a graph, and f be a mapping from $V(G)$ to the power set of $\{1, 2, \dots, k\}$, i.e., $f : V(G) \rightarrow \mathcal{P}(\{1, 2, \dots, k\})$. If $\bigcup_{u \in N(v)} f(u) = \{1, 2, \dots, k\}$ for each vertex $v \in V(G)$ with $f(v) = \emptyset$, then f is called a k -rainbow dominating function (k RDF) of G . The weight of f is $w(f) = \sum_{v \in V(G)} |f(v)|$, $|f(v)|$ denotes the number of elements in $f(v)$. The minimum weight of a k RDF of G is called the k -rainbow domination number of G , denoted by $\gamma_{rk}(G)$.

$G \square H$, the Cartesian product of graphs G and H , is the graph with vertex set $V(G) \times V(H)$, where two vertices are adjacent if and only if they are equal in one coordinate and adjacent in the other. Let $G = C_n \square C_m$ with $V(G) = \{v_{i,j} | 0 \leq i \leq n-1, 0 \leq j \leq m-1\}$ and $E(G) = \{e_{i,j} | e_{i,j} = (v_{i,j}, v_{i+1,j}), 0 \leq i \leq n-1, 0 \leq j \leq m-1\} \cup \{e'_{i,j} | e'_{i,j} = (v_{i,j}, v_{i,j+1}), 0 \leq i \leq n-1, 0 \leq j \leq m-1\}$, where indices i and j are read modulo n and m respectively. Figure 1 shows the graph of $C_n \square C_m$.

The problem of domination on Cartesian product graphs was first initiated by Vizing [3]. Since then, various domination numbers of $G \square H$ are extensively studied [4–6].

Figure 1. Graph $C_n \square C_m$.

The concept of k -rainbow domination is introduced by Brešar et al. [1], and they determine the exact values of 2-rainbow domination numbers of paths, cycles, Suns, etc. [7]. Since then, many scholars have begun to pay attention to and studied this parameter. There are many results on 2-rainbow domination. Stepień et al. present the exact values of $\gamma_{r2}(C_n \square C_3)$ [8], $\gamma_{r2}(C_n \square C_5)$ [9], and $\gamma_{r2}(C_m \square C_n)$ [10] for $m \equiv 0 \pmod{6}$, $n \equiv 0 \pmod{3}$. Shao Zehui et al. [11] determined the exact values of $\gamma_{r2}(C_4 \square C_n)$ and $\gamma_{r2}(C_8 \square C_n)$. Shao Zehui et al. [12] studied the 2-rainbow domination number of the generalized Petersen graphs $P(n, k)$ and prove $\gamma_{r2}(P(n, k)) = n$ for $n \leq 12$, $\gamma_{r2}(P(n, 1)) = n$ for $n \geq 5$, and $\gamma_{r2}(P(2k + 2, k)) = 2k + 2$ for $k \geq 2$. Wang Yueli et al. [13] propose a tight upper bound for $\gamma_{r2}(P(n, k))$ when $n \geq 4k + 1$. Liu Jiajie et al. [14] determine $\gamma_{r2}(S(n, m))$, $\gamma_{r2}(S^+(n, m))$, and $\gamma_{r2}(S^{++}(n, m))$, where $S(n, m)$, $S^+(n, m)$, and $S^{++}(n, m)$ are Sierpiński graphs and extended Sierpiński graphs.

The problem of k -rainbow domination will be more complex with k becoming bigger. The relative studies on k -rainbow domination for $k \geq 3$ are not as numerous as 2-rainbow domination. Michitaka Furuya et al. [15] prove that $\gamma_{r3}(G) \leq \frac{5n}{6}$ for every connected graph G with $\delta(G) \geq 2$ and $|V(G)| \geq 8$. Shao Zehui et al. [16] investigate the 3-rainbow domination number of cycles, paths and the generalized Petersen graphs. They determine the exact values of $\gamma_{r3}(P(n, 1))$, and present the upper bounds of $\gamma_{r3}(P(n, 2))$, $\gamma_{r3}(P(n, 3))$. Gerard J. Chang et al. [17] prove the k -rainbow domination problem is NP-complete, and for a given tree T , they determine the smallest k such that $\gamma_{rk}(T) = |V(T)|$. Hao Guoliang et al. [18] study the k -rainbow domination number of directed graphs and determine the exact value of $\gamma_{rk}(D)$ in the Cartesian product of directed cycles for $k \geq 4$. Kang Qiong et al. [19] initiate the study of outer-independent k -rainbow domination and they present some bounds for the outer-independent 2-rainbow domination number. Simon Brezovnik et al. [20] present some bounds on the k -rainbow independent domination number of the lexicographic product and give the exact values of the 2-rainbow independent domination number of the lexicographic product. J. Amjadi et al. [21] show the lower bounds on the k -rainbow domination number for any graphs and they present $\gamma_{rk}(G) = n$ for $k \geq 2\Delta(G)$.

In this paper, we focus on the study of the k -rainbow domination number of $C_n \square C_m$ ($k \geq 4$). Thank to J. Amjadi et al. [21] giving the lower bounds on the k -rainbow domination number for any graphs, we get the lower bounds of $\gamma_{rk}(C_n \square C_m)$ and the exact values of $\gamma_{rk}(C_n \square C_m)$ for $k \geq 8$.

Theorem 1. ([21]) Let k be a positive integer, and let G be a graph of order n , then

$$\gamma_{rk}(G) \geq \begin{cases} \frac{kn}{\Delta(G)+k}, & k \leq \Delta(G), \\ \frac{kn}{2\Delta(G)}, & \Delta(G) < k \leq 2\Delta(G), \\ n, & k > 2\Delta(G). \end{cases}$$

Corollary 1. ([21]) Let k be a positive integer, and let G be a graph of order n . If $k \geq 2\Delta(G)$, then $\gamma_{rk}(G) = n$.

Since $\Delta(C_n \square C_m) = 4$, by Corollary 1, it has

Corollary 2. For $k \geq 8$,

$$\gamma_{rk}(C_n \square C_m) = mn.$$

In this paper, we provide a proof for the new lower bound on the 4-rainbow domination number of $C_n \square C_3$. We construct some recursive k RDFs and upon these functions we obtain sharp upper bounds on the 4-rainbow domination number of $C_n \square C_m$. We determine the exact values $\gamma_{r4}(C_n \square C_3) = 2n$ and $\gamma_{rk}(C_n \square C_m) = \frac{kmn}{8}$ for $n \equiv 0(\text{mod } 4), m \equiv 0(\text{mod } 4)$ ($4 \leq k \leq 7$). We present some bounds of $\gamma_{rk}(C_n \square C_m)$ for $n \not\equiv 0(\text{mod } 4)$ or $m \not\equiv 0(\text{mod } 4)$ ($4 \leq k \leq 7$). At last, we discuss Vizing's conjecture on the k -rainbow domination number of $C_n \square C_m$.

2. 4-Rainbow Domination Number of Graph $C_n \square C_m$

2.1. Lower Bounds on the 4-Rainbow Domination Number of Graph $C_n \square C_m$

Lemma 1. For a graph $C_n \square C_m$,

$$\gamma_{r4}(C_n \square C_m) \geq \frac{mn}{2}.$$

Proof. In $C_n \square C_m$, the order is mn , $\Delta = 4$. Since $k = 4$, then $k = \Delta$. By Theorem 1, we can obtain the lower bound of $\gamma_{r4}(C_n \square C_m)$ is $\frac{mn}{2}$. $\square$

For some special graphs, the lower bound of $\gamma_{r4}(C_n \square C_m)$ can be higher than $\frac{mn}{2}$. Next, we will prove the lower bound of $\gamma_{r4}(C_n \square C_3)$ can be improved to $2n$ instead of $\frac{3n}{2}$.

Let f be a 4RDF on $C_n \square C_3$, we denote $V^i = \{v_{i,j} | 0 \leq j \leq 2\}$, $w(f(v_{i,j})) = |f(v_{i,j})|$ and $w(f_i) = \sum_{v_{i,j} \in V^i} |f(v_{i,j})|$.

Lemma 2. For $G = C_n \square C_3$, if there exists i ($0 \leq i \leq n-1$) such that $w(f_i) = 0$, then $w(f_{i-1}) \geq 6$ or $w(f_{i+1}) \geq 6$, where indices are read modulo n .

Proof. If $w(f_i) = 0$, i.e., $w(f(v_{i,0})) = w(f(v_{i,1})) = w(f(v_{i,2})) = 0$, then by the definition of 4RDF, it follows

$$\begin{aligned} & w(f_{i-1}) + w(f_{i+1}) \\ &= w(f(v_{i-1,0})) + w(f(v_{i-1,1})) + w(f(v_{i-1,2})) + w(f(v_{i+1,0})) + w(f(v_{i+1,1})) + w(f(v_{i+1,2})) \\ &= (w(f(v_{i-1,0})) + w(f(v_{i+1,0}))) + (w(f(v_{i-1,1})) + w(f(v_{i+1,1}))) + (w(f(v_{i-1,2})) + w(f(v_{i+1,2}))) \\ &\geq 4 + 4 + 4 = 12. \end{aligned}$$

So, $w(f_{i-1}) \geq 6$ or $w(f_{i+1}) \geq 6$. $\square$

Lemma 3. For $G = C_n \square C_3$, if there exists i ($0 \leq i \leq n-1$) such that $w(f_i) = 1$, then $w(f_{i-1}) \geq 3$ or $w(f_{i+1}) \geq 3$, where indices are read modulo n .

Proof. If $w(f_i) = 1$, without loss of generality, we let $w(f(v_{i,0})) = 1$, then by the definition of 4RDF, it follows

$$\begin{aligned} & w(f_{i-1}) + w(f_{i+1}) \\ &= (w(f(v_{i-1,0})) + w(f(v_{i-1,1})) + w(f(v_{i-1,2}))) + w(f(v_{i+1,0})) + w(f(v_{i+1,1})) + w(f(v_{i+1,2})) \\ &= (w(f(v_{i-1,0})) + w(f(v_{i+1,0}))) + (w(f(v_{i-1,1})) + w(f(v_{i+1,1}))) + (w(f(v_{i-1,2})) + w(f(v_{i+1,2}))) \\ &\geq 0 + 3 + 3 = 6. \end{aligned}$$

So, $w(f_{i-1}) \geq 3$ or $w(f_{i+1}) \geq 3$. $\square$

Lemma 4. Let $G = C_n \square C_3$, then $\gamma_{r4}(G) \geq 2n$.

Proof. For $0 \leq i \leq n-1$ we divide V^i into B_s by the following steps.

Step 0. Let $s = s_1 = s_2 = s_3 = 0$ and let $D[i] = 0$ for $i = 0$ up to $n-1$.

Step 1. For every i with $w(f_i) \geq 6 \wedge D[i] = 0$ do:

$s = s + 1; D[i] = 1; B_s = V^i;$
 if $w(f_{i-1}) = 0 \wedge D[i-1] = 0$, then $D[i-1] = 1; B_s = B_s \cup V^{i-1};$
 if $w(f_{i+1}) = 0 \wedge D[i+1] = 0$, then $D[i+1] = 1; B_s = B_s \cup V^{i+1};$
 if $w(f_{i-1}) = 1 \wedge D[i-1] = 0$, then $D[i-1] = 1; B_s = B_s \cup V^{i-1};$
 if $w(f_{i+1}) = 1 \wedge D[i+1] = 0$, then $D[i+1] = 1; B_s = B_s \cup V^{i+1};$

It follows $\sum_{V^i \subseteq \bigcup_{t=1}^s B_t} w(f_i) \geq \frac{|\bigcup_{t=1}^s B_t|}{3} \times 2$. Let $s_1 = s$.

By Lemma 2, by now, for every row with $w(f_i) \geq 6$, $D[i] = 1$, and $w(f_i) = 0$, $D[i] = 1$.

Step 2. For every i with $4 \leq w(f_i) \leq 5 \wedge D[i] = 0$ do:

$s = s + 1; D[i] = 1; B_s = V^i;$
 if $w(f_{i-1}) = 1 \wedge D[i-1] = 0$, then $D[i-1] = 1; B_s = B_s \cup V^{i-1};$
 if $w(f_{i+1}) = 1 \wedge D[i+1] = 0$, then $D[i+1] = 1; B_s = B_s \cup V^{i+1};$

It has $\sum_{V^i \subseteq \bigcup_{t=s_1+1}^s B_t} w(f_i) \geq \frac{|\bigcup_{t=s_1+1}^s B_t|}{3} \times 2$. Let $s_2 = s$.

Step 3. For every i with $w(f_i) = 3 \wedge D[i] = 0$ do:

$s = s + 1; D[i] = 1; B_s = V^i;$
 if $w(f_{i+1}) = 1 \wedge D[i+1] = 0$, then $D[i+1] = 1; B_s = B_s \cup V^{i+1};$

It follows $\sum_{V^i \subseteq \bigcup_{t=s_2+1}^s B_t} w(f_i) \geq \frac{|\bigcup_{t=s_2+1}^s B_t|}{3} \times 2$. Let $s_3 = s$.

By Lemma 2 and 3, by now, $w(f_i) = 2$ for all $D[i] = 0$ ($0 \leq i \leq n-1$).

Step 4. For every i with $w(f_i) = 2 \wedge D[i] = 0$ do:

$s = s + 1; D[i] = 1; B_s = V^i.$
 It has $\sum_{V^i \subseteq \bigcup_{t=s_3+1}^s B_t} w(f_i) \geq \frac{|\bigcup_{t=s_3+1}^s B_t|}{3} \times 2$.

Thus, $\sum_{i=0}^{n-1} w(f_i) \geq 2n$, that is $\gamma_{r4}(G) \geq 2n$. $\square$

2.2. Upper Bounds on the 4-Rainbow Domination Number of Graph $C_n \square C_m$

Lemma 5. For $m = 3$, $n \geq 3$,

$$\gamma_{r4}(C_n \square C_3) \leq 2n.$$

Proof. First, we define a function g on $C_4 \square C_3$ as follows.

$$g(v_{i,j}) = \begin{cases} \emptyset, & i(\bmod 2) \neq j(\bmod 2), \\ \{1\}, & i = 0, j = 0 \vee i = 2, j = 2, \\ \{2\}, & i = 0, j = 2 \vee i = 2, j = 0, \\ \{3, 4\}, & i = 1, j = 1 \vee i = 3, j = 1. \end{cases}$$

Figure 2a shows g on $C_4 \square C_3$. For convenience, we use $0, 1, 2, 3, 4$ to encode the color sets $\emptyset, \{1\}, \{2\}, \{3\}, \{4\}$, and use $1, 2; 1, 3; \dots; 3, 4$ to encode the color sets $\{1, 2\}, \{1, 3\}, \dots, \{3, 4\}$. We use Figure 2b to show a function on a graph in the rest of this paper.

Figure 2. The g for $C_4 \square C_3$. (a) Vertex labelled with color sets. (b) Vertex labelled with codes.

Then, we construct a function f as follows.

$$f(v_{i,j}) = \begin{cases} \emptyset, & n \equiv 2 \pmod{4} \wedge (i = n-3, j = 0 \vee i = n-2, j = 1, 2 \vee i = n-1, j = 0), \\ \{1\}, & n \equiv 2 \pmod{4} \wedge i = n-3, j = 1, 2, \\ \{2\}, & n \equiv 2 \pmod{4} \wedge i = n-1, j = 1, 2, \\ \{3, 4\}, & n \equiv 2 \pmod{4} \wedge i = n-2, j = 0, \\ g(v_{i(\bmod 4), j}), & \text{otherwise.} \end{cases}$$

Figure 3 shows f on $C_8 \square C_3$, $C_9 \square C_3$, $C_{10} \square C_3$ and $C_{11} \square C_3$. One can check f is a 4RDF and its weight is $w(f) = n \times 2 = 2n$. Hence, $\gamma_{r4}(C_n \square C_3) \leq 2n$.

Figure 3. f on $C_8 \square C_3$, $C_9 \square C_3$, $C_{10} \square C_3$, $C_{11} \square C_3$.

□

For $C_n \square C_m$ ($n, m \geq 4$), by symmetry of $C_n \square C_m$, we construct 4RDFs in the following cases:

- (1) m, n are evens and $m \equiv 0 \pmod{4}$, $n \equiv 0 \pmod{4}$ (Lemma 6).
- (2) m, n are evens and $m \equiv 0 \pmod{4}$, $n \equiv 2 \pmod{4}$ (Lemma 7).
- (3) m, n are evens and $m \equiv 2 \pmod{4}$, $n \equiv 2 \pmod{4}$ (Lemma 8).
- (4) m, n are odds and $m \equiv 1 \pmod{4}$, $n \equiv 1 \pmod{4}$ (Lemma 9).
- (5) m, n are odds and $m \equiv 1 \pmod{4}$, $n \equiv 3 \pmod{4}$ (Lemma 10).
- (6) m, n are odds and $m \equiv 3 \pmod{4}$, $n \equiv 3 \pmod{4}$ (Lemma 9).
- (7) m is odd, n is even and $m \equiv 1 \pmod{2}$, $n \equiv 0 \pmod{4}$ (Lemma 11).

(8) m is odd, n is even and $m \equiv 1 \pmod{2}$, $n \equiv 2 \pmod{4}$ (Lemma 12).

Lemma 6. For $m \equiv 0 \pmod{4}$, $n \equiv 0 \pmod{4}$ ($m, n \geq 4$), $\gamma_{r4}(C_n \square C_m) \leq \frac{mn}{2}$.

Proof. We first define a 4RDF g_1 on $C_4 \square C_4$, and Figure 4 shows the function.

$$g_1(v_{i,j}) = \begin{cases} \emptyset, & i \pmod{2} \neq j \pmod{2}, \\ \{1\}, & i = j = 0, 2, \\ \{2\}, & i + j = 2 \wedge i \neq j, \\ \{3\}, & i = j = 1, 3, \\ \{4\}, & \text{otherwise.} \end{cases}$$

Figure 4. The g_1 for $C_4 \square C_4$.

Then, we construct a recursive 4RDF f on $C_n \square C_m$,

$$f(v_{i,j}) = g_1(v_{i \pmod{4}, j \pmod{4}}).$$

Figure 5 shows f on $C_8 \square C_8$. The weight $w(f) = \frac{mn}{2}$. Thus, $\gamma_{r4}(C_n \square C_m) \leq \frac{mn}{2}$.

Figure 5. f on $C_8 \square C_8$

□

Lemma 7. For $m \equiv 0 \pmod{4}$, $n \equiv 2 \pmod{4}$ ($m, n \geq 4$), $\gamma_{r4}(C_n \square C_m) \leq \frac{mn}{2} + m$.

Proof. We construct the function f as follows.

$$f(v_{i,j}) = \begin{cases} \{1\}, & i = n-1 \wedge j \equiv 0 \pmod{2}, \\ \{3, 4\}, & i = n-1 \wedge j \equiv 1 \pmod{2}, \\ g_1(v_{i \pmod{4}, j \pmod{4}}), & \text{otherwise.} \end{cases}$$

where g_1 is defined as shown in Figure 4.

Figure 6 shows f on $C_{10} \square C_8$. The weight $w(f) = \frac{m \times (n-1)}{2} + \frac{3m}{2} = \frac{mn}{2} + m$. Hence, $\gamma_{r4}(C_n \square C_m) \leq \frac{mn}{2} + m$.

Figure 6. f on $C_{10} \square C_8$.

□

Lemma 8. For $m \equiv 2 \pmod{4}$, $n \equiv 2 \pmod{4}$ ($m, n \geq 4$),

$$\gamma_{r4}(C_n \square C_m) \leq \begin{cases} \frac{mn}{2} + \frac{2n}{m}, & n \equiv 0 \pmod{m}, \\ \frac{mn}{2} + n - m - 2(m-2) \lfloor \frac{\frac{n}{m}-1}{2} \rfloor, & n \not\equiv 0 \pmod{m}. \end{cases}$$

Proof. First, we define six different functions on $C_4 \square C_4$ and Figure 7 shows the six functions.

$$\begin{aligned} g_1(v_{i,j}) &= \begin{cases} \emptyset, & i \pmod{2} \neq j \pmod{2}, \\ \{1\}, & i+j=0,2, \\ \{2\}, & i+j=2 \wedge i \neq j, \\ \{3\}, & i+j=1,3, \\ \{4\}, & i+j=4 \wedge i \neq j. \end{cases} & g_2(v_{i,j}) &= \begin{cases} \emptyset, & i \pmod{2} \neq j \pmod{2}, \\ \{1\}, & i+j=2 \wedge i \neq j, \\ \{2\}, & i+j=0,2, \\ \{3\}, & i+j=4 \wedge i \neq j, \\ \{4\}, & i+j=1,3. \end{cases} \\ g_3(v_{i,j}) &= \begin{cases} \emptyset, & i \pmod{2} \neq j \pmod{2}, \\ \{1\}, & i+j=4 \wedge i \neq j, \\ \{2\}, & i+j=1,3, \\ \{3\}, & i+j=2 \wedge i \neq j, \\ \{4\}, & i+j=0,2. \end{cases} & g_4(v_{i,j}) &= \begin{cases} \emptyset, & i \pmod{2} \neq j \pmod{2}, \\ \{1\}, & i+j=0,2, \\ \{2\}, & i+j=2 \wedge i \neq j, \\ \{3\}, & i+j=4 \wedge i \neq j, \\ \{4\}, & i+j=1,3. \end{cases} \\ g_5(v_{i,j}) &= \begin{cases} \emptyset, & i \pmod{2} \neq j \pmod{2}, \\ \{1\}, & i+j=2 \wedge i \neq j, \\ \{2\}, & i+j=0,2, \\ \{3\}, & i+j=1,3, \\ \{4\}, & i+j=4 \wedge i \neq j. \end{cases} & g_6(v_{i,j}) &= \begin{cases} \emptyset, & i \pmod{2} \neq j \pmod{2}, \\ \{1\}, & i+j=4 \wedge i \neq j, \\ \{2\}, & i+j=1,3, \\ \{3\}, & i+j=0,2, \\ \{4\}, & i+j=2 \wedge i \neq j. \end{cases} \end{aligned}$$

Figure 7. Six different functions on $C_4 \square C_4$.

Then, we design a novel partition, $\{B_{11}, B_{12}, B_{13}, B_{14}, B_{15}\}$, on $C_m \square C_m$. The blocks are defined as the following (shown in Figure 8).

$$\begin{aligned} B_{11} &= \{v_{i,j} | i+j \leq \frac{m}{2} - 2\}, & B_{12} &= \{v_{i,j} | j-i \geq \frac{m}{2}\}, \\ B_{13} &= \{v_{i,j} | i-j \geq \frac{m}{2}\}, & B_{14} &= \{v_{i,j} | i+j \geq \frac{3m}{2} - 2\}, \\ B_{15} &= V(C_m \square C_m) - B_{11} \cup B_{12} \cup B_{13} \cup B_{14}. \end{aligned}$$

Figure 8. Partition on $C_{14} \square C_{14}$.

Now, we construct g_m and g'_m on $C_m \square C_m$.

For $m \equiv 6 \pmod{8}$,

$$g_m(v_{i,j}) = \begin{cases} \{1\}, & v_{i,j} \in \{v_{\frac{m}{2}-1, m-1}, v_{m-1, \frac{m}{2}-1}\} \\ g_1(v_{i(\bmod 4), j(\bmod 4)}), & v_{i,j} \in B_{11} \cup B_{14} - \{v_{\frac{m}{2}-1, m-1}, v_{m-1, \frac{m}{2}-1}\}, \\ g_2(v_{i(\bmod 4), j(\bmod 4)}), & v_{i,j} \in B_{12} \cup B_{13} - \{v_{\frac{m}{2}-1, m-1}, v_{m-1, \frac{m}{2}-1}\}, \\ g_3(v_{i(\bmod 4), j(\bmod 4)}), & v_{i,j} \in B_{15}. \end{cases}$$

For $m \equiv 2 \pmod{8}$,

$$g'_m(v_{i,j}) = \begin{cases} \{1\}, & v_{i,j} \in \{v_{\frac{m}{2}-1, m-1}, v_{m-1, \frac{m}{2}-1}\} \\ g_4(v_{i(\bmod 4), j(\bmod 4)}), & v_{i,j} \in B_{11} \cup B_{14} - \{v_{\frac{m}{2}-1, m-1}, v_{m-1, \frac{m}{2}-1}\}, \\ g_5(v_{i(\bmod 4), j(\bmod 4)}), & v_{i,j} \in B_{12} \cup B_{13} - \{v_{\frac{m}{2}-1, m-1}, v_{m-1, \frac{m}{2}-1}\}, \\ g_6(v_{i(\bmod 4), j(\bmod 4)}), & v_{i,j} \in B_{15}. \end{cases}$$

Case 1. For $n \equiv 0 \pmod{m}$, we construct f on $C_n \square C_m$ as follows.

$$f(v_{i,j}) = \begin{cases} g_m(v_{i(\bmod m), j}), & m \equiv 6 \pmod{8}, \\ g'_m(v_{i(\bmod m), j}), & m \equiv 2 \pmod{8}. \end{cases}$$

Figure 9 shows f on $C_{42} \square C_{14}$ and $C_{54} \square C_{18}$. One can check f is a 4RDF and its weight is $w(f) = (\frac{mn}{2} + 2) \times \frac{n}{m} = \frac{mn}{2} + \frac{2n}{m}$. Hence, $\gamma_{r4}(C_n \square C_m) \leq \frac{mn}{2} + \frac{2n}{m}$ for $n \equiv 0 \pmod{m}$.

Case 2. For $n \not\equiv 0 \pmod{m}$, we design two kinds of partition, $\{B_{11}, B_{12}, B_{13}, B_{14}, B_{15}\}$ and $\{B_{21}, B_{22}, B_{23}, B_{24}, B_{25}\}$. The former is defined as shown in Figure 8, and the later is defined on the lines from $nn+1$ to $n-1$ as shown in Figure 10, where $nn = 2m \times \lfloor \frac{\frac{n}{m}-1}{2} \rfloor - 1$.

Figure 9. Function f on $C_{42} \square C_{14}$ and $C_{54} \square C_{18}$.

$$\begin{aligned}
B_{21} &= \{v_{i,j} | j+i \leq \frac{m}{2} + nn - 1\}, & B_{22} &= \{v_{i,j} | i-j \leq nn + 1 - \frac{m}{2}\}, \\
B_{23} &= \{v_{i,j} | i-j \geq n - \frac{m}{2}\}, & B_{24} &= \{v_{i,j} | j+i \geq \frac{m}{2} + n - 2\}, \\
B_{25} &= \{v_{i,j} | nn + 1 \leq i \leq n-1, 0 \leq j \leq m-1\} - B_{21} \cup B_{22} \cup B_{23} \cup B_{24}.
\end{aligned}$$

Figure 10. Partition on the last $[(n-1) - (nn+1)]$ lines.

We construct functions h and h' on the lines from $nn+1$ to $n-1$.

For $m \equiv 6 \pmod{8}$,

$$h(v_{i,j}) = \begin{cases} \{1\}, & v_{i,j} = v_{n-1, \frac{m}{2}-1}, \\ \{1\}, & v_{i,j} \in B_{25} \wedge j = m-1 \wedge i \equiv 0 \pmod{2}, \\ g_1(v_{(i-nn-1) \pmod{4}, j \pmod{4}}), & v_{i,j} \in B_{21} \cup B_{24} - \{v_{n-1, \frac{m}{2}-1}\}, \\ g_2(v_{(i-nn-1) \pmod{4}, j \pmod{4}}), & v_{i,j} \in B_{22} \cup B_{23} - \{v_{n-1, \frac{m}{2}-1}\}, \\ g_3(v_{(i-nn-1) \pmod{4}, j \pmod{4}}), & v_{i,j} \in B_{25} \wedge j \leq m-2, \\ \{1, 2\}, & v_{i,j} \in B_{25} \wedge j = m-1 \wedge i \equiv 1 \pmod{2}. \end{cases}$$

For $m \equiv 2 \pmod{8}$,

$$h'(v_{i,j}) = \begin{cases} \{1\}, & v_{i,j} = v_{n-1, \frac{m}{2}-1}, \\ \{1\}, & v_{i,j} \in B_{25} \wedge j = m-1 \wedge i \equiv 0 \pmod{2}, \\ g_4(v_{(i-nn-1) \pmod{4}, j \pmod{4}}), & v_{i,j} \in B_{21} \cup B_{24} - \{v_{n-1, \frac{m}{2}-1}\}, \\ g_5(v_{(i-nn-1) \pmod{4}, j \pmod{4}}), & v_{i,j} \in B_{22} \cup B_{23} - \{v_{n-1, \frac{m}{2}-1}\}, \\ g_6(v_{(i-nn-1) \pmod{4}, j \pmod{4}}), & v_{i,j} \in B_{25} \wedge j \leq m-2, \\ \{1, 2\}, & v_{i,j} \in B_{25} \wedge j = m-1 \wedge i \equiv 1 \pmod{2}. \end{cases}$$

Now, we construct f on $C_n \square C_m$.

$m \equiv 6 \pmod{8}$,

$$f(v_{i,j}) = \begin{cases} g_m(v_{i \pmod{m}, j}), & 0 \leq i \leq nn, \\ h(v_{i,j}), & nn+1 \leq i \leq n-1. \end{cases}$$

$m \equiv 2 \pmod{8}$,

$$f(v_{i,j}) = \begin{cases} g'_m(v_{i \pmod{m}, j}), & 0 \leq i \leq nn, \\ h'(v_{i,j}), & nn+1 \leq i \leq n-1. \end{cases}$$

Figure 11 shows f on $C_{46} \square C_{14}$ and $C_{58} \square C_{18}$. One can check f is a 4RDF and its weight is

$$\begin{aligned} w(f) &= \frac{mn}{2} + \lfloor \frac{\frac{n}{m}-1}{2} \rfloor \times 2 \times 2 + n - 2m \times \lfloor \frac{\frac{n}{m}-1}{2} \rfloor - m \\ &= \frac{mn}{2} + n - m + \lfloor \frac{\frac{n}{m}-1}{2} \rfloor \times (2 - m) \times 2. \end{aligned}$$

Hence, $\gamma_{r4}(C_n \square C_m) \leq \frac{mn}{2} + n - m - 2(m-2) \lfloor \frac{\frac{n}{m}-1}{2} \rfloor$ for $n \not\equiv 0 \pmod{m}$.

Lemma 9. For $m(\bmod 4) = n(\bmod 4) = 1, 3$ ($m, n \geq 4$), $\gamma_{r4}(C_n \square C_m) \leq \frac{mn+m+n-1}{2}$.

Proof. We construct f on $C_n \square C_m$ as follows.

$$f(v_{i,j}) = \begin{cases} \{3\}, & i = n-1 \wedge j \equiv 3(\bmod 4) \vee j = m-1 \wedge i \equiv 3(\bmod 4), \\ \{4\}, & i = n-1 \wedge j \equiv 1(\bmod 4) \vee j = m-1 \wedge i \equiv 1(\bmod 4), \\ g_1(v_{i(\bmod 4), j(\bmod 4)}), & \text{otherwise.} \end{cases}$$

where g_1 is defined as shown in Figure 4.

Figure 12 shows f on $C_9 \square C_9$ and $C_{11} \square C_{11}$. The weight $w(f) = \frac{(m-1)(n-1)}{2} + m + n - 1 = \frac{mn+m+n-1}{2}$. Hence, $\gamma_{r4}(C_n \square C_m) \leq \frac{mn+m+n-1}{2}$.

Figure 12. f on $C_9 \square C_9$ and $C_{11} \square C_{11}$.

□

Lemma 10. For $m \equiv 1(\bmod 4)$, $n \equiv 3(\bmod 4)$ ($m, n \geq 4$), $\gamma_{r4}(C_n \square C_m) \leq \frac{mn+m+n-5}{2}$.

Proof. We construct f on $C_n \square C_m$ as follows,

$$f(v_{i,j}) = \begin{cases} \{1\}, & i = n-2, j = m-3, \\ \{2\}, & i = n-3, j = m-2, m-3, \\ \{3\}, & i = n-1 \wedge j \equiv 3(\bmod 4) \vee j = m-1 \wedge i \equiv 3(\bmod 4), \\ \{4\}, & i = n-1 \wedge j \equiv 1(\bmod 4) \vee j = m-1 \wedge i \equiv 1(\bmod 4), \\ \emptyset, & i = n-3, j = m-1 \vee i = n-2, j = m-2 \vee i = n-1, j = m-3, m-1, \\ g_1(v_{i(\bmod 4), j(\bmod 4)}), & \text{otherwise,} \end{cases}$$

where g_1 is defined as shown in Figure 4.

Figure 13 shows f on $C_{11} \square C_9$. One can check f is a 4RDF and its weight is $w(f) = \frac{(m-3)(n-3)}{2} + \frac{m-3}{2} \times 2 + m - 3 + \frac{n-3}{2} \times 2 + n - 3 + 5 = \frac{mn+m+n-5}{2}$. Hence, $\gamma_{r4}(C_n \square C_m) \leq \frac{mn+m+n-5}{2}$. □

Lemma 11. For $m \equiv 1(\bmod 2)$, $n \equiv 0(\bmod 4)$ ($m, n \geq 4$), $\gamma_{r4}(C_n \square C_m) \leq \frac{mn}{2} + \frac{n}{2}$.

Proof. We construct f on $C_n \square C_m$ as follows,

$$f(v_{i,j}) = \begin{cases} \{3\}, & j = m-1 \wedge i \equiv 3(\bmod 4), \\ \{4\}, & j = m-1 \wedge i \equiv 1(\bmod 4), \\ g_1(v_{i(\bmod 4), j(\bmod 4)}), & \text{otherwise,} \end{cases}$$

where g_1 is defined as shown in Figure 4.

Figure 13. Function f on $C_{11} \square C_9$.

Figure 14 shows f on $C_8 \square C_9$, $C_8 \square C_{11}$. One can check f is a 4RDF and its weight is $w(f) = \frac{(m-1) \times n}{2} + n = \frac{mn}{2} + \frac{n}{2}$. Hence, $\gamma_{r4}(C_n \square C_m) \leq \frac{mn}{2} + \frac{n}{2}$.

Figure 14. f on $C_8 \square C_9$, $C_8 \square C_{11}$.

□

Lemma 12. For $m \equiv 1(\text{mod } 2)$, $n \equiv 2(\text{mod } 4)$ ($m, n \geq 4$),

$$\gamma_{r4}(C_n \square C_m) \leq \begin{cases} \frac{mn}{2} + m + \frac{n}{2} - 1, & m \equiv 1(\text{mod } 4), \\ \frac{mn}{2} + m + \frac{n}{2} - 2, & m \equiv 3(\text{mod } 4). \end{cases}$$

Proof. Case 1. For $m \equiv 1(\text{mod } 4)$, we construct f as follows,

$$f(v_{i,j}) = \begin{cases} \{1\}, & i = n-1 \wedge j \equiv 0(\text{mod } 2), \\ \{3\}, & j = m-1 \wedge i \equiv 3(\text{mod } 4), \\ \{4\}, & j = m-1 \wedge i \equiv 1(\text{mod } 4) \wedge i \leq n-2, \\ \{3, 4\}, & i = n-1 \wedge j \equiv 1(\text{mod } 2), \\ g_1(v_{i(\text{mod } 4), j(\text{mod } 4)}), & \text{otherwise,} \end{cases}$$

where g_1 is defined as shown in Figure 4.

Figure 15 shows f on $C_{10} \square C_9$. The weight $w(f) = \frac{(m-1) \times (n-1)}{2} + n + \frac{3(m-1)}{2} = \frac{mn}{2} + m + \frac{n}{2} - 1$. Hence, $\gamma_{r4}(C_n \square C_m) \leq \frac{mn}{2} + m + \frac{n}{2} - 1$ for $m \equiv 1(\text{mod } 4)$.

Case 2. For $m \equiv 3(\text{mod } 4)$, we construct f as follows,

$$f(v_{i,j}) = \begin{cases} \emptyset, & i = n-1, j = m-1, \\ \{1\}, & i = n-1 \wedge j \equiv 0(\text{mod } 2) \wedge j \neq m-1, \\ \{3\}, & j = m-1 \wedge i \equiv 3(\text{mod } 4) \wedge i \neq n-1, \\ \{4\}, & j = m-1 \wedge i \equiv 1(\text{mod } 4) \wedge i \neq n-1, \\ \{3, 4\}, & i = n-1 \wedge j \equiv 1(\text{mod } 2), \\ g_1(v_{i(\text{mod } 4), j(\text{mod } 4)}), & \text{otherwise,} \end{cases}$$

where g_1 is defined as shown in Figure 4.

Figure 15. Function f on $C_{10} \square C_9$ and $C_{10} \square C_{11}$.

Figure 15 shows f on $C_{10} \square C_{11}$. The weight $w(f) = \frac{(n-2) \times (m-1)}{2} + n - 2 + 2 \times (m-3) + 5 = \frac{mn}{2} + m + \frac{n}{2} - 2$. Hence, $\gamma_{r4}(C_n \square C_m) \leq \frac{mn}{2} + m + \frac{n}{2} - 2$ for $m \equiv 3(\text{mod } 4)$. $\square$

2.3. The Values and Bounds of $\gamma_{r4}(C_n \square C_m)$

By Lemma 4 and Lemma 5, we can get the exact values of $\gamma_{r4}(C_n \square C_3)$.

Theorem 2. For any integer $n \geq 3$, $\gamma_{r4}(C_n \square C_3) = 2n$.

By Lemma 1 and Lemma 6, we have

Theorem 3. For $m, n \geq 4$, and $m \equiv 0 \pmod{4}$, $n \equiv 0 \pmod{4}$, $\gamma_{r4}(C_n \square C_m) = \frac{mn}{2}$.

By Lemmas 1 and 7–12, we have

Theorem 4. For $m, n \geq 4$, and $m \not\equiv 0 \pmod{4}$ or $n \not\equiv 0 \pmod{4}$,

$$\frac{mn}{2} \leq \gamma_{r4}(C_n \square C_m) \leq \frac{mn}{2} + m + \frac{n}{2} - 1.$$

3. The k -Rainbow Domination Number of Graph $C_n \square C_m$ ($5 \leq k \leq 7$)

Lemma 13. For $5 \leq k \leq 7$, $\gamma_{rk}(C_n \square C_m) \geq \frac{kmn}{8}$.

Proof. Since $\Delta(C_n \square C_m) = 4$, then $k > \Delta$, by Theorem 1, $\gamma_{rk}(C_n \square C_m) \geq \frac{kmn}{8}$. $\square$

Next, we will present upper bounds on the k -rainbow domination number of graph $C_n \square C_m$ ($5 \leq k \leq 7$). Let $|V_t^{clr}|$ denotes the number of vertex containing color t ($t = 1, 2, 3, 4$). Based on 4RDFs, we can construct k RDFs for $C_n \square C_m$ ($k \geq 5$). The main idea is: (1) Find $|V_{t_0}^{clr}|$, where $|V_{t_0}^{clr}| = \min\{|V_1^{clr}|, |V_2^{clr}|, |V_3^{clr}|, |V_4^{clr}|\}$. (2) Replace color t_0 with colors $t_0, 5, 6, \dots, k$. Then, we can get upper bounds of $\gamma_{rk}C_n \square C_m$ ($5 \leq k \leq 7$).

Lemma 14. For $n \geq 3$, $5 \leq k \leq 7$,

$$\gamma_{rk}(C_n \square C_3) \leq \begin{cases} \min\{\frac{kn}{2}, 3n\}, & n \equiv 0 \pmod{4}, \\ \min\{\frac{k(n-1)}{2} + 4, 3n\}, & n \equiv 1, 3 \pmod{4}, \\ \min\{\frac{k(n-2)}{2} + 8, 3n\}, & n \equiv 2 \pmod{4}. \end{cases}$$

Proof. By the 4RDF f we construct for $C_n \square C_3$ in Lemma 5 (see Figure 3), we can count:

$$\begin{aligned} n \equiv 0 \pmod{4}, & |V_1^{clr}| = |V_2^{clr}| = |V_3^{clr}| = |V_4^{clr}| = \frac{n}{2}, \\ n \equiv 1 \pmod{4}, & |V_1^{clr}| = |V_2^{clr}| = \frac{n+1}{2}, |V_3^{clr}| = |V_4^{clr}| = \frac{n-1}{2}, \\ n \equiv 2 \pmod{4}, & |V_1^{clr}| = |V_2^{clr}| = \frac{n+2}{2}, |V_3^{clr}| = |V_4^{clr}| = \frac{n-2}{2}, \\ n \equiv 3 \pmod{4}, & |V_1^{clr}| = |V_2^{clr}| = \frac{n+1}{2}, |V_3^{clr}| = |V_4^{clr}| = \frac{n-1}{2}. \end{aligned}$$

Then,

$$|V_{t_0}^{clr}| = \begin{cases} \frac{n}{2}, & n \equiv 0 \pmod{4}, \\ \frac{n-1}{2}, & n \equiv 1, 3 \pmod{4}, \\ \frac{n-2}{2}, & n \equiv 2 \pmod{4}. \end{cases}$$

Thus,

$$\begin{aligned} \gamma_{rk}(C_n \square C_3) & \leq w(f) + (k-4)|V_{t_0}^{clr}| \\ & \leq 2n + (k-4)|V_{t_0}^{clr}| \\ & = \begin{cases} \min\{\frac{kn}{2}, 3n\}, & n \equiv 0 \pmod{4}, \\ \min\{\frac{k(n-1)}{2} + 4, 3n\}, & n \equiv 1, 3 \pmod{4}, \\ \min\{\frac{k(n-2)}{2} + 8, 3n\}, & n \equiv 2 \pmod{4}. \end{cases} \end{aligned}$$

$\square$

Lemma 15. For $m, n \geq 4$, $m \equiv 0 \pmod{4}$, $n \equiv 0 \pmod{4}$ and $5 \leq k \leq 7$, $\gamma_{rk}(C_n \square C_m) \leq \frac{kmn}{8}$.

Proof. By the 4RDF f in Lemma 6 (see Figure 5), $|V_{t_0}^{clr}| = |V_1^{clr}| = |V_2^{clr}| = |V_3^{clr}| = |V_4^{clr}| = \frac{mn}{8}$. Thus, $\gamma_{rk}(C_n \square C_m) \leq w(f) + (k-4)|V_{t_0}^{clr}| \leq \frac{mn}{2} + (k-4)\frac{mn}{8} = \frac{kmn}{8}$. $\square$

Lemma 16. For $m, n \geq 4$, $m \not\equiv 0 \pmod{4}$ or $n \not\equiv 0 \pmod{4}$ and $5 \leq k \leq 7$, $\gamma_{rk}(C_n \square C_m) \leq k\frac{mn+n}{8} + m$.

Proof. (1) For $m \pmod{4} = 0$, $n \pmod{4} = 2$, by f in Lemma 7 (see Figure 6), $|V_{t_0}^{clr}| = \frac{mn}{8}$, thus $\gamma_{rk}(C_n \square C_m) \leq w(f) + (k-4)|V_{t_0}^{clr}| \leq \frac{mn}{2} + m + (k-4) \times \frac{mn}{8} = \frac{kmn}{8} + m$.

(2) For $m \pmod{4} = 2$, $n \pmod{4} = 2$, by f in Lemma 8 (see Figures 9 and 11).

For $n \equiv 0 \pmod{m}$, $|V_{t_0}^{clr}| = \frac{mn-4}{8}$. Thus

$$\gamma_{rk}(C_n \square C_m) \leq w(f) + (k-4)|V_{t_0}^{clr}| \leq \frac{mn}{2} + \frac{2n}{m} + (k-4) \times \frac{mn-4}{8} = k\frac{mn-4}{8} + \frac{2n}{m} + 2.$$

For $n \not\equiv 0 \pmod{m}$, $|V_{t_0}^{clr}| \leq \frac{mn+4}{8}$. Thus

$$\begin{aligned} \gamma_{rk}(C_n \square C_m) &\leq w(f) + (k-4)|V_{t_0}^{clr}| \\ &\leq \frac{mn}{2} + n - m - 2(m-2)\lfloor \frac{\frac{n}{m}-1}{2} \rfloor + (k-4) \times \frac{mn+4}{8} \\ &= k\frac{mn+4}{8} - 2(m-2)\lfloor \frac{\frac{n}{m}-1}{2} \rfloor + n - m - 2. \end{aligned}$$

(3) For $m \pmod{4} = n \pmod{4} = 1, 3$, by f in Lemma 9 (see Figure 12).

For $m \pmod{4} = n \pmod{4} = 1$, $|V_{t_0}^{clr}| = \frac{mn+m+n-3}{8}$, $t_0 = 2, 3, 4$.

Thus,

$$\begin{aligned} \gamma_{rk}(C_n \square C_m) &\leq w(f) + (k-4)|V_{t_0}^{clr}| \\ &\leq \frac{mn+m+n-1}{2} + (k-4) \times \frac{mn+m+n-3}{8} \\ &= \frac{k(mn+m+n-3)}{8} + 1. \end{aligned}$$

For $m \pmod{4} = n \pmod{4} = 3$, $|V_{t_0}^{clr}| = \frac{mn+m+n-7}{8}$, $t_0 = 3$.

Thus,

$$\begin{aligned} \gamma_{rk}(C_n \square C_m) &\leq w(f) + (k-4)|V_{t_0}^{clr}| \\ &\leq \frac{mn+m+n-1}{2} + (k-4) \times \frac{mn+m+n-7}{8} \\ &= \frac{k(mn+m+n-7)}{8} + 3. \end{aligned}$$

(4) For $m \pmod{4} = 1$, $n \pmod{4} = 3$, by f in Lemma 10 (see Figure 13), $|V_{t_0}^{clr}| = \frac{mn+m+n-7}{8}$, thus $\gamma_{rk}(C_n \square C_m) \leq w(f) + (k-4)|V_{t_0}^{clr}| \leq \frac{mn+m+n-5}{2} + (k-4) \times \frac{mn+m+n-7}{8} = \frac{k(mn+m+n-7)}{8} + 1$.

(5) For $m \pmod{2} = 1$, $n \pmod{4} = 0$, by f in Lemma 11 (see Figure 14), $|V_{t_0}^{clr}| = \frac{mn+n}{8}$, $\gamma_{rk}(C_n \square C_m) \leq \frac{mn}{2} + (k-4)|V_{t_0}^{clr}| = \frac{mn}{2} + (k-4) \times \frac{mn+n}{8} = \frac{k(mn+n)}{8}$.

(6) For $m \pmod{2} = 1$, $n \pmod{4} = 2$, by f in Lemma 12 (see Figure 15).

$$|V_{t_0}^{clr}| = \begin{cases} \frac{mn+n-4}{8} & m \equiv 1 \pmod{4} \\ \frac{mn+n+6}{8} & m \equiv 3 \pmod{4}. \end{cases}$$

Thus,

$$\begin{aligned}\gamma_{rk}(C_n \square C_m) &\leq w(f) + (k-4)|V_{t_0}^{clr}| \\ &= \begin{cases} \frac{mn}{2} + m + \frac{n}{2} - 1 + (k-4)\frac{mn+n-4}{8}, & m \equiv 1 \pmod{4}, \\ \frac{mn}{2} + m + \frac{n}{2} - 2 + (k-4)\frac{mn+n+6}{8}, & m \equiv 3 \pmod{4}, \end{cases} \\ &= \begin{cases} k\frac{mn+n-4}{8} + m + 1, & m \equiv 1 \pmod{4}, \\ k\frac{mn+n+6}{8} + m - 3, & m \equiv 3 \pmod{4}. \end{cases}\end{aligned}$$

To sum up, for $m \not\equiv 0 \pmod{4}$ or $n \not\equiv 0 \pmod{4}$ and $5 \leq k \leq 7$, $\gamma_{rk}(C_n \square C_m) \leq k\frac{mn+n}{8} + m$. $\square$

By Lemma 13 and Lemma 15, we have the following Theorem.

Theorem 5. For $m, n \geq 4$, $m \equiv 0 \pmod{4}$, $n \equiv 0 \pmod{4}$ and $5 \leq k \leq 7$,

$$\gamma_{rk}(C_n \square C_m) = \frac{kmn}{8}.$$

By Lemma 13 and Lemma 14, we have

Theorem 6. For $n \geq 3$, $5 \leq k \leq 7$,

$$\frac{3kn}{8} \leq \gamma_{rk}(C_n \square C_3) \leq \begin{cases} \min\{\frac{kn}{2}, 3n\}, & n \equiv 0 \pmod{4}, \\ \min\{\frac{k(n-1)}{2} + 4, 3n\}, & n \equiv 1, 3 \pmod{4}, \\ \min\{\frac{k(n-2)}{2} + 8, 3n\}, & n \equiv 2 \pmod{4}. \end{cases}$$

By Lemma 13 and Lemma 16, we can get

Theorem 7. For $m, n \geq 4$, $m \not\equiv 0 \pmod{4}$ or $n \not\equiv 0 \pmod{4}$, and $5 \leq k \leq 7$,

$$\frac{kmn}{8} \leq \gamma_{rk}(C_n \square C_m) \leq k\frac{mn+n}{8} + m.$$

4. Discussion on Vizing's Conjecture

Vizing's conjecture [3] concerns a relation between the domination number and the cartesian product of graphs. It states $\gamma(G \square H) \geq \gamma(G)\gamma(H)$, where $\gamma(G)$ denotes the domination number of G . In this section, we check Vizing's generalized conjecture for the k -rainbow domination number of $C_n \square C_m$ (shown in Table 1). From Table 1, one can see $\gamma_{rk}(C_m \square C_n) \geq \gamma_{rk}(C_m)\gamma_{rk}(C_n)$ is not always true.

Table 1. Known results of k -rainbow domination number of C_n and $C_n \square C_m$

$\gamma_{rk}(C_n)$	$\gamma_{rk}(C_m \square C_n)$	Check Vizing's Conjecture
$k \geq 5$ ([21]) $\gamma_{rk}(C_n) = n$	$k \geq 8$ by Corollary 2 $\gamma_{rk}(C_n \square C_m) = mn$	$\gamma_{rk}(C_m \square C_n) = \gamma_{rk}(C_m) \gamma_{rk}(C_n)$
	$5 \leq k \leq 7$, by Theorem 5 $n \equiv 0(\text{mod } 4), m \equiv 0(\text{mod } 4)$ $\gamma_{rk}(C_n \square C_m) = \frac{kmn}{8}$	$\gamma_{rk}(C_m \square C_n) < \gamma_{rk}(C_m) \gamma_{rk}(C_n)$
	$5 \leq k \leq 7$, by Theorem 7 $n \not\equiv 0(\text{mod } 4)$ or $m \not\equiv 0(\text{mod } 4)$ $\gamma_{rk}(C_n \square C_m) \leq k \frac{mn+n}{8} + m$	$m, n \geq 15$ $\gamma_{rk}(C_m \square C_n) \leq \gamma_{rk}(C_m) \gamma_{rk}(C_n)$
$k = 4$ ([21]) $\gamma_{r4}(C_n) = n$ $\gamma_{r4}(C_3) = 3$	by Theorem 2 $\gamma_{r4}(C_n \square C_3) = 2n$ by Theorem 3 $n \equiv 0(\text{mod } 4), m \equiv 0(\text{mod } 4)$ $\gamma_{r4}(C_n \square C_m) = \frac{mn}{2}$	$\gamma_{r4}(C_n \square C_3) < \gamma_{r4}(C_n) \gamma_{r4}(C_3)$ $\gamma_{r4}(C_n \square C_m) < \gamma_{r4}(C_n) \gamma_{r4}(C_m)$
	by Theorem 4 $n \not\equiv 0(\text{mod } 4)$ or $m \not\equiv 0(\text{mod } 4)$ $\gamma_{r4}(C_n \square C_m) \leq \frac{mn}{2} + m + \frac{n}{2} - 1$	$\gamma_{r4}(C_m \square C_n) < \gamma_{r4}(C_m) \gamma_{r4}(C_n)$
$k = 3$ ([16]) $\gamma_{r3}(C_n) = \lceil \frac{3n}{4} \rceil$	$\gamma_{r3}(C_n \square C_m) \leq \gamma_{r4}(C_n \square C_m)$ by Theorem 4 $\gamma_{r3}(C_m \square C_n) \leq \frac{mn}{2} + m + \frac{n}{2} - 1$	$m, n \geq 24$ $\gamma_{r3}(C_m \square C_n) < \gamma_{r3}(C_m) \gamma_{r3}(C_n)$
$k = 2$ ([7]) $\gamma_{r2}(C_n) = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$ $\gamma_{r2}(C_3) = 2$		
$\gamma_{r2}(C_n) = \begin{cases} \frac{n}{2}, & n \equiv 0, 4, 8(\text{mod } 12), \\ \frac{n}{2} + 1, & n \equiv 2, 6, 10(\text{mod } 12), \\ \frac{n-1}{2} + 1, & n \equiv 1, 3, 5, 7, 9, 11(\text{mod } 12). \end{cases}$	$([8]) \gamma_{r2}(C_3 \square C_n) = \begin{cases} n, & n \equiv 0(\text{mod } 6), \\ n + 1, & n \equiv 1, 2, 3, 5(\text{mod } 6), \\ n + 2, & n \equiv 4(\text{mod } 6). \end{cases}$	$\begin{cases} n \equiv 0, 1, 3, 5, 7, 9, 10, 11(\text{mod } 12) \\ \gamma_{r2}(C_3 \square C_n) = \gamma_{r2}(C_3) \gamma_{r2}(C_n), \\ n \equiv 2, 6(\text{mod } 12) \\ \gamma_{r2}(C_3 \square C_n) > \gamma_{r2}(C_3) \gamma_{r2}(C_n), \\ n \equiv 4, 8(\text{mod } 12) \\ \gamma_{r2}(C_3 \square C_n) < \gamma_{r2}(C_3) \gamma_{r2}(C_n). \end{cases}$
$\gamma_{r2}(C_4) = 2$	$([11]) \gamma_{r2}(C_4 \square C_n) = \begin{cases} \lfloor \frac{3n}{2} \rfloor, & n \equiv 0(\text{mod } 8), \\ \lfloor \frac{3n}{2} \rfloor + 1, & n \equiv 2, 4, 5(\text{mod } 8), \\ \lfloor \frac{3n}{2} \rfloor + 2, & n \equiv 1, 3, 6, 7(\text{mod } 8). \end{cases}$	$\gamma_{r2}(C_4 \square C_n) > \gamma_{r2}(C_4) \gamma_{r2}(C_n)$
$\gamma_{r2}(C_5) = 3$	$([9]) \gamma_{r2}(C_5 \square C_n) = 2n$	$\gamma_{r2}(C_5 \square C_n) > \gamma_{r2}(C_5) \gamma_{r2}(C_n)$
$\gamma_{r2}(C_8) = 4$	$([11]) \gamma_{r2}(C_8 \square C_n) = 3n$ $([8]) n \equiv 0(\text{mod } 6), m \equiv 0(\text{mod } 3)$ $\gamma_{r2}(C_n \square C_m) = \frac{mn}{3}$	$\gamma_{r2}(C_8 \square C_n) > \gamma_{r2}(C_8) \gamma_{r2}(C_n)$ $m, n \geq 13$ $\gamma_{r2}(C_n \square C_m) \geq \gamma_{r2}(C_n) \gamma_{r2}(C_m)$

5. Conclusions

In this paper, we study the k -rainbow domination number of $C_n \square C_m$ ($4 \leq k \leq 7$). We provide a proof for the new lower bound on the 4-rainbow domination number of $C_n \square C_3$. We construct some novel and recursive 4RDFs, and upon these functions we obtain some sharp upper bounds on the 4-rainbow domination number of $C_n \square C_m$. Therefore, for $k = 4$, we determine $\gamma_{r4}(C_n \square C_3) = 2n$; for $4 \leq k \leq 7$, we determine $\gamma_{rk}(C_n \square C_m) = \frac{kmn}{8}$ for $n \equiv 0(\text{mod } 4), m \equiv 0(\text{mod } 4)$; we present the bounds of $\gamma_{rk}(C_n \square C_m)$ for $n \not\equiv 0(\text{mod } 4)$ or $m \not\equiv 0(\text{mod } 4)$. Finally, we discuss Vizing's generalized conjecture for the k -rainbow domination number of $C_n \square C_m$.

Author Contributions: H.G. contributes for supervision, methodology, validation, project administration and formal analysing. K.L. contributes for resource, some computations and wrote the initial draft of the paper. Y.Y. wrote the final draft.

Funding: This work is supported by the Fundamental Research Funds for the Central University, Grand No. is 3132019323.

Acknowledgments: Authors gratefully acknowledge the helpful comments and suggestions of the reviewers, which have improved the presentation. This work is supported by the Fundamental Research Funds for the Central University, Grand No. is 3132019323.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Brešar, B.; Henning, M.A.; Rall, D.F. Rainbow domination in graphs. *Taiwan J. Math.* **2008**, *12*, 213–225. [[CrossRef](#)]
2. Dehghardi, N.; Sheikholeslami, S.M.; Volkmann, L. The k -rainbow bondage number of a graph. *Discret. Appl. Math.* **2014**, *174*, 133–139.
3. Vizing, V.G. Some unsolved problems in graph theory. *Uspehi Mater. Nauk.* **1968**, *23*, 117–134.
4. Chen, Q.; Zhao, C.Y.; Zhao, M. Dominator colorings of certain cartesian products of paths and cycles. *Graph Combinator.* **2017**, *33*, 73–83.
5. Li, Z.P.; Shao, Z.H.; Xu, J. Weak $\{2\}$ -domination number of Cartesian products of cycles. *J. Comb. Optim.* **2018**, *35*, 75–85.
6. Ye, A.S.; Miao, F.; Shao, Z.H.; Liu, J.B.; Žerovnik, J.; Repolusk, P. More Results on the Domination Number of Cartesian Product of Two Directed Cycles. *Mathematics* **2019**, *7*, 210. [[CrossRef](#)]
7. Brešar, B.; Šumenjak, T.K. On the 2-rainbow domination in graphs. *Discret. Appl. Math.* **2007**, *155*, 2397–2400.
8. Stepień, Z.; Zwierchowski, M. 2-rainbow domination number of cartesian products: $C_n \square C_3$ and $C_n \square C_5$. *J. Comb. Optim.* **2014**, *28*, 748–755. [[CrossRef](#)]
9. Stepień, Z.; Szymaszkiewicz, A.; Szymaszkiewicz, L.; Zwierchowski, M. 2-Rainbow domination number of $C_n \square C_5$. *Discret. Appl. Math.* **2014**, *170*, 113–116. [[CrossRef](#)]
10. Stepień, Z.; Szymaszkiewicz, L.; Zwierchowski, M. The Cartesian product of cycles with small 2-rainbow domination number. *J. Comb. Optim.* **2015**, *30*, 668–674. [[CrossRef](#)]
11. Shao, Z.H.; Li, Z.P.; Erveš, R.; Žerovnik, J. The 2-Rainbow domination numbers of $C_4 \square C_n$ and $C_8 \square C_n$. *Natl. Acad. Sci. Lett.* **2019**, *2*, 1–8.
12. Shao, Z.H.; Jiang, H.Q.; Wu, P.; Wang, S.H.; Žerovnik, J.; Zhang, X.S.; Liu, J.B. On 2-rainbow domination of generalized Petersen graphs. *Discret. Appl. Math.* **2019**, *257*, 370–384. [[CrossRef](#)]
13. Wang, Y.L.; Wu, K.H. A tight upper bound for 2-rainbow domination in generalized Petersen graphs. *Discret. Appl. Math.* **2013**, *161*, 2178–2188. [[CrossRef](#)]
14. Liu, J.J.; Chang, S.C.; Lin, C.J. The 2-rainbow domination of Sierpiński graphs and extended Sierpiński graphs. *Theor. Comput. Syst.* **2017**, *61*, 893–906. [[CrossRef](#)]
15. Furuya, M.; Koyanagi, M.; Yokota, M. Upper bound on 3-rainbow domination in graphs with minimum degree 2. *Discret. Optim.* **2018**, *29*, 45–76. [[CrossRef](#)]
16. Shao, Z.H.; Liang, M.L.; Yin, C.; Xu, X.D.; Pavlič, P.; Žerovnik, J. On rainbow domination numbers of graphs. *Inform. Sci.* **2014**, *254*, 225–234. [[CrossRef](#)]
17. Chang, G.J.; Wu, J.J.; Zhu, X.D. Rainbow domination on trees. *Discret. Appl. Math.* **2010**, *158*, 8–12. [[CrossRef](#)]
18. Hao, G.L.; Qian, J.G. On the rainbow domination number of digraphs. *Graph Combinator.* **2016**, *32*, 1903–1913. [[CrossRef](#)]
19. Kang, Q.; Samodivkin, V.; Shao, Z.H.; Sheikholeslami, S.M.; Soroudi, M. Outer-independent k -rainbow domination. *J. Taibah Univ. Sci.* **2019**, *1*, 883–891. [[CrossRef](#)]
20. Brezovnik, S.; Šumenjak, T.K. Complexity of k -rainbow independent domination and some results on the lexicographic product of graphs. *Appl. Math. Comput.* **2019**, *349*, 214–220. [[CrossRef](#)]
21. Amjadi, J.; Dehghardi, N.; Furuya, M.; Sheikholeslam, S.M. A sufficient condition for large rainbow domination number. *Int. J. Comput. Math. Comput. Syst. Theory* **2017**, *2*, 53–65. [[CrossRef](#)]

© 2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).