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On Jacobi-Type Vector Fields on Riemannian Manifolds

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Abstract: In this article, we study Jacobi-type vector fields on Riemannian manifolds. A Killing vector field is a Jacobi-type vector field while the converse is not true, leading to a natural question of finding conditions under which a Jacobi-type vector field is Killing. In this article, we first prove that every Jacobi-type vector field on a compact Riemannian manifold is Killing. Then, we find several necessary and sufficient conditions for a Jacobi-type vector field to be a Killing vector field on non-compact Riemannian manifolds. Further, we derive some characterizations of Euclidean spaces in terms of Jacobi-type vector fields. Finally, we provide examples of Jacobi-type vector fields on non-compact Riemannian manifolds, which are non-Killing.

Keywords: Jacobi-type vector fields; Killing vector fields; conformal vector fields; Euclidean space

MSC: 53C20; 53B21

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1. Introduction

Throughout this article, we assume that manifolds are connected and differentiable. There are several important types of smooth vector fields on an n-dimensional Riemannian manifold (M, g), whose existence influences the geometry of the Riemannian manifold M. A smooth vector field ξ on M is called a Killing vector field if its local flow consists of local isometries of the Riemannian manifold M. The geometry of Riemannian manifolds with Killing vector fields has been studied quite extensively (cf., e.g., [1–6]). The presence of a non-zero Killing vector field on a compact Riemannian manifold constrains its geometry, as well as topology; for instance, it does not allow the Riemannian manifold to have negative Ricci curvature, and on a Riemannian manifold of positive curvature, its fundamental group contains a cyclic subgroup with a constant index depending only on n (cf. [1,2]).

In Riemannian geometry, Jacobi vector fields are vector fields along a geodesic defined by the Jacobi equation that arise naturally in the study of the exponential map. More precisely, a vector field J along a geodesic γ in a Riemannian manifold M is called a Jacobi vector field if it satisfies the Jacobi equation (cf. [7]):

$$\frac{D^2}{dt^2} J(t) + R(J(t), \dot{\gamma}(t))\dot{\gamma}(t) = 0,$$

where D denotes the covariant derivative with respect to the Levi–Civita connection ∇, R is the Riemann curvature tensor of M, $\dot{\gamma}(t)$ is the tangent vector field, and t is the parameter of the geodesic. Clearly, the Jacobi equation is a linear, second order ordinary differential equation; in particular, the values of J and $\frac{D}{dt}J(t)$ at one point of γ uniquely determine the Jacobi vector field. Further, a Killing vector field ξ on a Riemannian manifold (M, g) is a Jacobi vector field along each geodesic, since it satisfies the differential equation: $\ddot{\gamma} + R(\xi, \dot{\gamma})\dot{\gamma} = 0$. Furthermore, it follows from the Jacobi equation that Jacobi vector fields on a Euclidean space are simply those vector fields that are linear in t.

As a natural extension of Jacobi vector fields, one of the authors introduced in [8] the notion of Jacobi-type vector fields as follows. A vector field η on a Riemannian manifold M is called a Jacobi-type vector field if it satisfies the following Jacobi-type equation:

$$\nabla_X \nabla_X \eta - \nabla_{\nabla_X X} \eta + R(\eta, X)X = 0, \quad X \in \mathfrak{X}(M),$$

where $\mathfrak{X}(M)$ denotes the Lie algebra of smooth vector fields on M. Obviously, every Jacobi-type vector field is a Jacobi vector field along each geodesic on M.

Since each Killing vector field is a Jacobi-type vector field (see [8]), a natural question is the following:

Question 1: "For a given Riemannian manifold M, under which topological or geometric conditions is every Jacobi-type vector on M Killing?"

One objective of this article is to study this question. In Section 3, we prove that if a Riemannian manifold M is compact, then every Jacobi-type vector field on M is Killing. In contrast, not every Jacobi-type vector field on a non-compact Riemannian manifold is Killing (see the examples in Section 6). Therefore, the second interesting question is

Question 2: "Under what conditions is a Jacobi-type vector field on a non-compact Riemannian manifold a Killing vector field?"

In Section 4, we obtain three necessary and sufficient conditions for a Jacobi-type vector field on a non-compact Riemannian manifold to be Killing (see Theorems 2–4). In Section 5, we prove two characterizations of Euclidean spaces using Jacobi-type vector fields (see Theorems 6 and 7). In the last section, we provide some explicit examples of non-Killing Jacobi-type vector fields.

2. Preliminaries

First, we recall the following result from [8].

Proposition 1. *A Killing vector field on a Riemannian manifold is a Jacobi-type vector field.*

Although each Killing vector field on a Riemannian manifold is a Jacobi-type vector field, there do exist Jacobi-type vector fields that are non-Killing. For instance, let us consider the Euclidean space $(\mathbb{R}^n, g)$ with the canonical Euclidean metric $g = \sum_{i=1}^n dx^i \otimes dx^i$. Then, it is easy to verify that the position vector field ψ of $\mathbb{R}^n$:

$$\psi = \sum x^i \frac{\partial}{\partial x^i}$$

is of the Jacobi type and it satisfies $\left(\mathcal{L}_\psi g\right)(X, Y) = 2g(X, Y)$, where $\mathcal{L}$ denotes the Lie derivative. Hence, ψ is a non-Killing vector field.

We need the following lemma from [8].

Lemma 1. *If η is a Jacobi-type vector field on a Riemannian manifold M, then we have the following equation:*

$$\nabla_X \nabla_Y \eta - \nabla_{\nabla_X Y} \eta + R(\eta, X)Y = 0, \quad X, Y \in \mathfrak{X}(M).$$

For a given Jacobi-type vector field η on a Riemannian manifold M, let us denote by ω the one-form dual to η. Furthermore, we define a symmetric tensor field B of type $(1,1)$ and a skew-symmetric tensor field φ of type $(1,1)$ respectively by:

$$\left(\mathcal{L}_\eta g\right)(X,Y) = 2g(BX,Y) \quad \text{and} \quad d\omega(X,Y) = 2g(\varphi X, Y)$$

for $X, Y \in \mathfrak{X}(M)$. By applying Koszul's formula, we find:

$$\nabla_X \eta = BX + \varphi X, \quad X \in \mathfrak{X}(M). \tag{1}$$

Combining this with Lemma 1 yields:

$$(\nabla_X B)Y + (\nabla_X \varphi)Y + R(\eta, X)Y = 0, \tag{2}$$

where $(\nabla_X A)\,Y = \nabla_X(AY) - A\nabla_X Y$ for a tensor field A of type $(1,1)$. If we define a smooth function h on M by $h = \operatorname{Tr} B$, then for a local orthonormal frame $\{e_1, .., e_n\}$ on M, by choosing $Y = e_i$ in Equation (2) and by taking the inner product with e_i, we find:

$$\sum_{i=1}^{n} g\left((\nabla_X B)\,e_i, e_i\right) = 0,$$

where we have used the skew-symmetry of the tensor φ. Hence, the above equation gives $Xh = 0$ for any $X \in \mathfrak{X}(M)$. Thus, h is a constant function. Consequently, we have the following.

Lemma 2. *Let η be a Jacobi-type vector field on a Riemannian manifold (M, g). If B is the symmetric operator associated with η defined by $\left(\mathcal{L}_\eta g\right)(X,Y) = 2g(BX,Y)$, then $\operatorname{Tr} B$ is a constant.*

3. Jacobi-Type Vector Fields on Compact Riemannian Manifolds

For Question 1, we prove the following.

Theorem 1. *Every Jacobi-type vector field on a compact Riemannian manifold is a Killing vector field.*

Proof. Let η be a Jacobi-type vector field on an n-dimensional compact Riemannian manifold (M, g). Consider the Ricci operator Q defined by:

$$g(QX, Y) = Ric(X, Y), \quad X, Y \in \mathfrak{X}(M),$$

where Ric is the Ricci tensor. Then, for a local orthonormal frame $\{e_1, .., e_n\}$ on M, we have:

$$QX = \sum_{i=1}^{n} R(X, e_i)e_i, \quad X \in \mathfrak{X}(M)$$

and consequently, Equation (2) gives:

$$\sum_{i=1}^{n} (\nabla_{e_i} B)\,e_i + \sum_{i=1}^{n} (\nabla_{e_i} \varphi)\,e_i + Q(\xi) = 0. \tag{3}$$

Furthermore, using Equation (1), we get:

$$R(X,Y)\eta = (\nabla_X B)\,Y + (\nabla_X \varphi)\,Y - (\nabla_Y B)\,X - (\nabla_Y \varphi)\,X,$$

which yields:

$$Ric(Y,\eta) = g\left(Y, \sum_{i=1}^{n}(\nabla_{e_i}B)e_i\right) - g\left(Y, \sum_{i=1}^{n}(\nabla_{e_i}\varphi)e_i\right),$$

where we have applied Lemma 2 and the facts that B is symmetric and φ is skew-symmetric. The above equation implies:

$$Q(\eta) = \sum_{i=1}^{n}(\nabla_{e_i}B)\,e_i - \sum_{i=1}^{n}(\nabla_{e_i}\varphi)\,e_i,$$

which together with Equation (3) gives:

$$\sum_{i=1}^{n}(\nabla_{e_i}B)\,e_i = 0 \quad \text{and} \quad \sum_{i=1}^{n}(\nabla_{e_i}\varphi)e_i + Q(\eta) = 0. \tag{4}$$

Since B is a symmetric operator, we can choose a local orthonormal frame $\{e_1,\ldots,e_n\}$ on M that diagonalizes B, and as φ is skew-symmetric, we have:

$$\sum_{i=1}^{n} g(Be_i, \varphi e_i) = 0. \tag{5}$$

Recall that the divergence of a vector field X on M is given by (see, e.g., [9]):

$$\operatorname{div} X = \sum_{i=1}^{n}\langle \nabla_{e_i}X, e_i\rangle\,. \tag{6}$$

Now, by using Equations (1), (5), and (6), we see that the divergence of the vector field $B\eta$ satisfies:

$$\operatorname{div}(B\eta) = \|B\|^2,$$

where $\|B\|^2$ denotes the squared norm of B. Thus, after integrating the above equation over the compact M, we get $B = 0$. Consequently, Equation (1) confirms that η is a Killing vector field. □

Remark 1. *Let M be a compact real hypersurface of a Kähler manifold with a unit normal vector field N. In view of Theorem 1, we observe that the assumption "the characteristic vector field $\xi = -JN$ is of the Jacobi type" in the results of [10,11] is redundant.*

4. Jacobi-Type Vector Fields on Non-Compact Riemannian Manifolds

On a compact Riemannian manifold, the notions of Jacobi-type vector fields and Killing vector fields are equivalent according to Theorem 1, yet on non-compact Riemannian manifolds, they are different in general (see the examples in Section 6). Therefore, it is an interesting question to seek some conditions under which a Jacobi-type vector field is a Killing vector field on a non-compact Riemannian manifold.

Note that if η is a Killing vector field on an n-dimensional Riemannian manifold M, then $B = 0$ in Equation (1). Thus, we have $\varphi\eta = \nabla_\eta \eta$. Hence, we obtain:

$$\operatorname{div}(\varphi\eta) = -\|\varphi\|^2 - g\left(\eta, \sum_{i=1}^{n}(\nabla_{e_i}\varphi)e_i\right).$$

Using Equation (4) in the above equation shows that, for a Killing vector field η, we have:

$$\operatorname{div}(\varphi\eta) = Ric\,(\eta,\eta) - \|\varphi\|^2\,.$$

Moreover, if we define a smooth function f on M by $f = \frac{1}{2}\|\eta\|^2$, we get $\nabla f = -\varphi\eta$, and thus, for a Killing vector field η, the Laplacian Δf is given by:

$$\Delta f = \|\varphi\|^2 - Ric(\eta, \eta). \tag{7}$$

A natural question is the following:

Question 3: "Does the function $f = \frac{1}{2}\|\eta\|^2$ for a Jacobi-type vector field η on a Riemannian manifold satisfying (7) make η a Killing vector field?"

The next theorem provides an answer to this question.

Theorem 2. *Let η be a Jacobi-type vector field on a Riemannian manifold M. Then, η is a Killing vector field if and only if the function $f = \frac{1}{2}\|\eta\|^2$ satisfies:*

$$\Delta f \leq \|\varphi\|^2 - Ric(\eta, \eta).$$

Proof. Let η be a Jacobi-type vector field on an n-dimensional Riemannian manifold M. Then, using Equation (1), the gradient ∇f of $f = \frac{1}{2}\|\eta\|^2$ is given by:

$$\nabla f = B\eta - \varphi\eta. \tag{8}$$

Now, using Equations (1) and (4), we compute:

$$\operatorname{div}(B\eta) = \|B\|^2 \quad \text{and} \quad \operatorname{div}(\varphi\eta) = -\|\varphi\|^2 - g\left(\eta, \sum_{i=1}^{n}\left(\nabla_{e_i}\varphi\right)e_i\right). \tag{9}$$

Thus, by using Equation (8), we conclude:

$$\Delta f = \|B\|^2 + \|\varphi\|^2 + g\left(\eta, \sum_{i=1}^{n}(\nabla_{e_i}\varphi)e_i\right). \tag{10}$$

Applying Equation (2) and Lemma 2, we find:

$$Ric(\eta, \eta) = -g\left(\eta, \sum_{i=1}^{n}(\nabla_{e_i}\varphi)e_i\right), \tag{11}$$

which together with Equation (10) yields:

$$\Delta f = \|B\|^2 + \|\varphi\|^2 - Ric(\eta, \eta).$$

Hence, if the inequality $\Delta f \leq \|\varphi\|^2 - Ric(\eta, \eta)$ holds, then the above equation implies $B = 0$, that is η is a Killing vector field.

The converse is trivial as a Killing vector field is a Jacobi vector field and the function f satisfies Equation (7). $\square$

Recall that the flow $\{\psi_t\}$ of a vector field $X \in \mathfrak{X}(M)$ on a Riemannian manifold M is called a geodesic flow, if for each point $p \in M$, the curve $\sigma(t) = \psi_t(p)$ is a geodesic on M passing through the point p. As the local flow of a Killing vector field on a Riemannian manifold M consists of isometries of M, it follows that a local flow of a Killing vector field is a geodesic flow, but the converse is not true. For example, the Reeb vector field ζ of a proper trans-Sasakian manifold has as the local flow a geodesic flow, yet ζ is not a Killing vector field (cf. [12]).

In the next theorem, we provide a very simple characterization for a Killing vector field to have constant length via a Jacobi-type vector field on a Riemannian manifold.

Theorem 3. *Let η be a Jacobi-type vector field on a Riemannian manifold M with the flow of η a geodesic flow. Then, η is a Killing vector field of constant length if and only if the Ricci curvature $Ric\,(\eta,\eta)$ satisfies:*

$$Ric\,(\eta,\eta) \geq \|\varphi\|^2 .$$

Proof. Let η be a Jacobi-type vector field on an n-dimensional Riemannian manifold M. Since the local flow of η is a geodesic flow, Equation (1) implies:

$$B\eta + \varphi\eta = 0. \tag{12}$$

Now, using Equation (9), we conclude:

$$\|B\|^2 - \|\varphi\|^2 - g\left(\eta, \sum_{i=1}^{n} (\nabla_{e_i}\varphi)\, e_i\right) = 0,$$

which upon using Equation (11) gives:

$$Ric\,(\eta,\eta) = \|\varphi\|^2 - \|B\|^2 .$$

Using the inequality $Ric\,(\eta,\eta) \geq \|\varphi\|^2$ in the above equation, we get $B = 0$, that is η is a Killing vector field. Moreover, Equation (12) gives $\varphi\eta = 0$, and consequently, Equation (8) implies $\nabla f = 0$, that is η has constant length.

Conversely, if η is a Killing vector field of constant length, then using $B = 0$ and Equation (1) in $X(||\eta||^2) = 0$ gives $g(X, \varphi\eta) = 0$, $X \in \mathfrak{X}(M)$. This gives $\varphi\eta = 0$, which together with Equation (1) confirms $\nabla_\eta \eta = 0$, that is the local flow of η is a geodesic flow. As f is a constant, Equation (7) implies the equality $Ric\,(\eta,\eta) = \|\varphi\|^2$. $\square$

Recall that a smooth function f on a Riemannian manifold M is said to be harmonic if $\Delta f = 0$ and superharmonic if $\Delta f \leq 0$. The Hessian operator A_f of a smooth function f is a symmetric operator defined by:

$$A_f X = \nabla_X \nabla f, \quad X \in \mathfrak{X}(M),$$

and the Hessian of f, denoted by $\mathrm{Hess}(f)$, is given by:

$$\mathrm{Hess}(f)(X,Y) = g(A_f X, Y), \quad X, Y \in \mathfrak{X}(M).$$

Now, we prove the following characterization of a Killing vector field using a Jacobi-type vector field on a Riemannian manifold.

Theorem 4. *A Jacobi-type vector field η on a Riemannian manifold M is a Killing vector field of constant length if and only if the function $f = \frac{1}{2}\|\eta\|^2$ is superharmonic and the Hessian of f satisfies $\mathrm{Hess}(f)(\eta,\eta) \leq 0$.*

Proof. Let η be a Jacobi-type vector field on a Riemannian manifold M. Suppose the function $f = \frac{1}{2}\|\eta\|^2$ satisfies:

$$\mathrm{Hess}(f)(\eta,\eta) \leq 0 \quad \text{and} \quad \Delta f \leq 0. \tag{13}$$

Using Equation (1), we have:

$$\nabla_\eta \eta = B\eta + \varphi\eta. \tag{14}$$

After combining (14) with Equation (8), we get:

$$2B\eta = \nabla f + \nabla_\eta \eta, \quad 2\varphi\eta = \nabla_\eta \eta - \nabla f. \tag{15}$$

Now, by taking the inner product in Equation (8) with η, we get $\eta(f) = g(B\eta, \eta)$, which gives:

$$\eta\eta(f) = g((\nabla_\eta B)\,\eta, \eta) + 2g\,(B\eta, \nabla_\eta \eta)\,. \tag{16}$$

Furthermore, the first equation in Equation (15) implies:

$$2g(B\eta, \nabla_\eta \eta) = \nabla_\eta \eta(f) + \|\nabla_\eta \eta\|^2\,.$$

Using the above equation in Equation (16) gives:

$$\mathrm{Hess}(f)(\eta, \eta) = g((\nabla_\eta B)\,\eta, \eta) + \|\nabla_\eta \eta\|^2\,. \tag{17}$$

Note that Equation (2) implies $(\nabla_\eta B)\,\eta = -\,(\nabla_\eta \varphi)\,\eta$, and as φ is skew-symmetric, we obtain $g((\nabla_\eta \varphi)\,\eta, \eta) = 0$. Consequently, the above equation implies $g((\nabla_\eta B)\,\eta, \eta) = 0$. Thus, Equation (17) reduces to:

$$\mathrm{Hess}(f)(\eta, \eta) = \|\nabla_\eta \eta\|^2$$

and using the condition in Equation (13) forces the above equation to yield $\nabla_\eta \eta = 0$. Consequently, the first equation in Equation (15) gives $\nabla f = 2B\eta$, and on account of Equation (9), we conclude that $\Delta f = 2\,\|B\|^2$.

Now, using the fact that the function f is superharmonic, we conclude $B = 0$. Hence, η is a Killing vector field. Moreover, using $\nabla_\eta \eta = 0$ and $B = 0$ in Equation (15), we find $\nabla f = 0$ on the connected M, which proves that f is a constant. Thus, η is a Killing vector field of constant length.

Conversely, if η is a Killing vector field of constant length, then obviously, η is a Jacobi-type vector field that satisfies $\mathrm{Hess}(f)(\eta, \eta) = 0$ and $\Delta f = 0$. □

5. Jacobi-Type Vector Fields on Euclidean Spaces

A vector field X on a Riemannian manifold (M, g) is called conformal if it satisfies (cf. e.g., [7,13]):

$$\mathcal{L}_X g = 2\rho g \tag{18}$$

for some smooth function $\rho : M \to \mathbf{R}$. The conformal vector field X is called non-trivial if the function ρ in (18) is a non-zero function. Further, a conformal vector field X is called a gradient conformal vector field if X is the gradient of some smooth function. Non-Killing conformal vector fields have been used, e.g., in [2,3,5,14–18] to characterize spheres among compact Riemannian manifolds.

We already known from Section 2 that the position vector field ξ of the Euclidean n-space $\mathbb{R}^n$ is a Jacobi-type vector field satisfying $\mathcal{L}_\xi g = 2g$. Hence, ξ is conformal. In fact, it is also a gradient conformal vector field with $\xi = \nabla f$ with $f = \frac{1}{2}\,\|\xi\|^2$. Furthermore, it is known that if ψ denotes the position vector field on the complex Euclidean n-space $\mathbb{C}^n$, then $\zeta = \psi + J\psi$ is of the Jacobi type, which is a non-gradient conformal vector field on $\mathbb{C}^n$, where J denotes the complex structure on $\mathbb{C}^n$.

From these properties of the vector fields ζ, we ask the next question.

Question 4: *"Is a Jacobi-type vector field on a complete Riemannian manifold that is also a conformal vector field characterized as a Euclidean space?"*

The main purpose of this section is to study this question. First, we show that a complete Riemannian manifold admits a Jacobi-type vector field that is also a non-trivial gradient conformal vector field if and only if it is isometric to the Euclidean space $\mathbb{R}^n$. Then, we prove that a complete Riemannian manifold admits a Jacobi-type vector field that is also a conformal vector field (not necessarily a gradient conformal vector field) that annihilates the operator φ if and only if it is isometric to the Euclidean space $\mathbb{R}^n$.

To prove these results mentioned above, we need the following result from [19] (cf. Theorem 1).

Theorem 5. *Let M be a complete Riemannian manifold. If there exists a smooth function $f : M \to R$ satisfying $\operatorname{Hess}(f) = cg$ for some non-zero constant c, then M is isometric to $\mathbb{R}^n$.*

Now, we prove the following result, which is an easy application of Theorem 5.

Theorem 6. *Let M be a complete Riemannian manifold. Then, M admits a Jacobi-type vector field that is also a non-trivial gradient conformal vector field if and only if M is isometric to a Euclidean space.*

Proof. Clearly, if M is isometric to the Euclidean n-space $\mathbb{R}^n$, then the position vector field ξ is a Jacobi-type vector field, which is also a non-trivial gradient conformal vector field.

Conversely, suppose that the complete Riemannian manifold M admits a Jacobi-type vector field η that is also a non-trivial gradient conformal vector field. Then, as η is closed, we have that $\varphi = 0$ and $B = \rho I$ in Equation (1) and that the smooth function ρ is a constant by Lemma 2. Moreover, ξ being a gradient conformal vector field, there is a smooth function $f : M \to R$ that satisfies $\eta = \nabla f$, and consequently, Equation (1) takes the form:

$$\nabla_X \nabla f = \rho X, \quad X \in \mathfrak{X}(M),$$

where the constant $\rho \neq 0$ is guaranteed by the fact that η is a non-trivial gradient conformal vector field. The above equation implies that $\operatorname{Hess}(f) = \rho g$ with non-zero constant ρ. Consequently, by Theorem 5, we conclude that M is isometric to a Euclidean space. □

Finally, we prove the following.

Theorem 7. *Let M be a complete Riemannian manifold. Then, M admits a Jacobi-type vector field η, which is also a non-trivial conformal vector field that annihilates the operator φ (associated with η) if and only if M is isometric to a Euclidean space.*

Proof. Clearly, if M is isometric to the Euclidean n-space $\mathbb{R}^n$, then its position vector field ξ is a Jacobi-type vector field with $\varphi = 0$, which is also a non-trivial conformal vector field.

Conversely, if the complete Riemannian manifold (M, g) admits a Jacobi-type vector field η that is also a non-trivial conformal vector field with $\varphi(\eta) = 0$, then as η is a conformal vector field, we have $B = \rho I$ in Equation (1), which thus takes the form:

$$\nabla_X \eta = \rho X + \varphi X, \quad X \in \mathfrak{X}(M) \tag{19}$$

and the smooth function ρ is a constant by Lemma 2.

Define a smooth function $f : M \to R$ by $f = \frac{1}{2}\|\xi\|^2$, whose gradient is easily found using Equation (19), as:

$$\nabla f = \rho\eta - \varphi(\eta) = \rho\eta.$$

Then, after taking the covariant derivative in the above equation with respect to $X \in \mathfrak{X}(M)$ and using Equation (19), we conclude that:

$$\nabla_X \nabla f = \rho^2 X + \rho\varphi X.$$

Thus, we get:

$$\operatorname{Hess}(f)(X, X) = \rho^2 g(X, X).$$

Now, using polarization in the above equation, we get $\operatorname{Hess}(f) = \rho^2 g$. Note that the constant ρ has to be non-zero as the vector field η is a non-trivial conformal vector field. Hence, by Theorem 5, we conclude that M is isometric to a Euclidean space. □

Remark 2. *It was proven in [20] that a complete Kähler n-manifold* (M, J, g) *is isometric to a complex Euclidean n-space* $\mathbf{C}^n$ *if and only if* (M, J, g) *admits a "special kind" of non-trivial Jacobi-type vector field.*

6. Examples of Non-Killing Jacobi-Type Vector Fields

In this section, we provide some examples of Jacobi-type vector fields that are non-trivial conformal vector fields.

Example 1. *Let* $x^1, \cdots, x^n$ *be Euclidean coordinates of the Euclidean n-space* $(\mathbb{R}^n, \langle\ ,\ \rangle)$. *Consider the vector field:*

$$\xi = \psi - \left\langle \psi, \frac{\partial}{\partial x^i} \right\rangle \frac{\partial}{\partial x^j} + \left\langle \psi, \frac{\partial}{\partial x^j} \right\rangle \frac{\partial}{\partial x^i},$$

where ψ *is the position vector field of* $\mathbb{R}^n$ *and* i, j *are two fixed indices with* $i \neq j$. *If we denote by* ∇ *the covariant derivative operator with respect to the Euclidean connection on* $(\mathbb{R}^n, \langle\ ,\ \rangle)$, *then it is easy to verify that:*

$$\nabla_X \xi = X + \varphi(X), \quad X \in \mathfrak{X}(\mathbb{R}^n),$$

where:

$$\varphi(X) = -(Xx^i)\frac{\partial}{\partial x^j} + (Xx^j)\frac{\partial}{\partial x^i},$$

is skew symmetric. Hence:

$$\mathcal{L}_\xi \langle\ ,\ \rangle = 2\langle\ ,\ \rangle,$$

that is, ξ *is a conformal vector field, which is non-closed. Moreover, we have:*

$$\nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi = -\mathrm{Hess}(x^i)(X, X)\frac{\partial}{\partial x^j} + \mathrm{Hess}(x^j)(X, X)\frac{\partial}{\partial x^i},$$

where $\mathrm{Hess}(f)$ *is the Hessian of* f. *However, the Hessians* $\mathrm{Hess}(x^i)$ *and* $\mathrm{Hess}(x^j)$ *of the coordinate functions* x^i *and* x^j *are zero. Therefore, the above equation confirms that* ξ *is a Jacobi-type vector field on* $(\mathbb{R}^n, \langle\ ,\ \rangle)$. *Therefore,* ξ *is a Jacobi-type vector field, which is a non-trivial conformal vector field. Hence,* ξ *is a non-Killing vector field on* $\mathbb{R}^n$.

Example 2. *Let* $M'(\varphi', \xi', \eta', g')$ *be a* $(2n+1)$*-dimensional Sasakian manifold (cf. [21]). Then:*

$$\nabla'_X \xi' = -\varphi' X, \quad (\nabla' \varphi')(X, Y) = g'(X, Y)\xi' - \eta'(Y)X, \ X, Y \in \mathfrak{X}(M'), \tag{20}$$

where ∇' *denotes the covariant derivative operator with respect to the Riemannian connection on* M'. *Using the above equation, we conclude:*

$$R'(X, Y)\xi' = \eta'(Y)X - \eta'(X)Y, \quad X, Y \in \mathfrak{X}(M'),$$

which upon taking the inner product with $Z \in \mathfrak{X}(M')$ *gives:*

$$R''(X, Y; \xi', Z) = \eta'(Y)g'(X, Z) - \eta'(X)g'(Y, Z),$$

that is,

$$R'(\xi', Z)X = g'(X, Z)\xi' - \eta'(X)Z, \quad X, Z \in \mathfrak{X}(M'). \tag{21}$$

Now, let $M = (0, \infty) \times_t M'$ *be the warped product with the warping function the coordinate function* t *on the open interval* $(0, \infty)$ *and with the warped product metric* $g = dt^2 + t^2 g'$. *We shall show that the vector field* $\xi \in \mathfrak{X}(M)$ *defined by:*

$$\xi = t\frac{\partial}{\partial t} - \xi'$$

is a Jacobi-type vector field, as well as a non-trivial conformal vector field, which is non-Killing on M.

We denote by ∇ the covariant derivative operator with respect to the Riemannian connection on the Riemannian manifold (M, g), and let $E = h\frac{\partial}{\partial t} + V$, where $V \in \mathfrak{X}(M')$ is a vector field on M and $h : (0, \infty) \to R$ is a smooth function. Then, using Proposition 35 in [22], an easy computation gives:

$$\nabla_E \xi = \left(h\frac{\partial}{\partial t} + V\right) + \varphi'(V) + t\eta'(V)\frac{\partial}{\partial t} - \frac{h}{t}\xi' = E + \varphi(E), \tag{22}$$

where φ is a $(1, 1)$-tensor field on M defined by:

$$\varphi(E) = \varphi'(V) + t\eta'(V)\frac{\partial}{\partial t} - \frac{h}{t}\xi'.$$

It is easy to verify that φ is a skew-symmetric tensor field. Furthermore, we may compute that:

$$\nabla_E E = \left(hh' - tg'(V, V)\right)\frac{\partial}{\partial t} + \nabla'_V V + \frac{2h}{t}V. \tag{23}$$

Now, using Equation (22), we conclude:

$$\begin{aligned}\nabla_E \nabla_E \xi = {} & \nabla_E E + \frac{2h}{t}\varphi'(V) + \nabla'_V \varphi'(V) + \eta'(V)V - \frac{hh'}{t}\xi' \\ & + \left(2h\eta'(V) + tV(\eta'(V))\right)\frac{\partial}{\partial t},\end{aligned} \tag{24}$$

which upon using Equations (22) and (23), gives:

$$\begin{aligned}\nabla_{\nabla_E E} \xi = {} & \nabla_E E + \frac{2h}{t}\varphi'(V) + \varphi'\left(\nabla'_V V\right) - \frac{hh'}{t}\xi' + g'(V, V)\xi' \\ & + \left(2h\eta'(V) + t\eta'\left(\nabla'_V V\right)\right)\frac{\partial}{\partial t}.\end{aligned} \tag{25}$$

Using Proposition 40 in [22] (note the difference in sign convention for the curvature tensor in our work and [22]), first we get:

$$R(\xi, E)E = -R(\xi', V)V,$$

where R is the curvature tensor field for the Riemannian manifold (M, g), and then, using (5) of Proposition 40 in [22] or by a direct calculation, we find:

$$R(\xi, E)E = -R'(\xi', V)V + g'(V, V)\xi' - \eta'(V)V. \tag{26}$$

Hence, from Equations (20), (21), and (24)–(26), we may conclude:

$$\nabla_E \nabla_E \xi - \nabla_{\nabla_E E} \xi + R(\xi, E)E = 0.$$

Hence, ξ is a Jacobi-type vector field on the Riemannian manifold M. Furthermore, using Equation (22), it is easy to verify that ξ is a non-trivial conformal vector field, which is non-Killing on the Riemannian manifold M.

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