mathematics

Article

An Inverse Problem for a Generalized Fractional Derivative with an Application in Reconstruction of Time- and Space-Dependent Sources in Fractional Diffusion and Wave Equations

Nataliia Kinash* and **Jaan Janno**

Department of Cybernetics, Tallinn University of Technology, Ehitajate tee 5, 19086 Tallinn, Estonia; jaan.janno@taltech.ee

* Correspondence: nataliia.kinash@taltech.ee

Received: 24 October 2019; Accepted: 19 November 2019; Published: 21 November 2019

Abstract: In this article, we consider two inverse problems with a generalized fractional derivative. The first problem, IP1, is to reconstruct the function u based on its value and the value of its fractional derivative in the neighborhood of the final time. We prove the uniqueness of the solution to this problem. Afterwards, we investigate the IP2, which is to reconstruct a source term in an equation that generalizes fractional diffusion and wave equations, given measurements in a neighborhood of final time. The source to be determined depends on time and all space variables. The uniqueness is proved based on the results for IP1. Finally, we derive the explicit solution formulas to the IP1 and IP2 for some particular cases of the generalized fractional derivative.

Keywords: inverse problem; source reconstruction; final overdetermination; subdiffusion; tempered subdiffusion; fractional wave equation; generalized fractional derivative; Atangana–Baleanu derivative

MSC: 35R30; 35R11

1. Introduction

Fractional derivatives are increasingly used in modeling various processes in physics, biology, economics, engineering sciences, etc. [1]. In addition to classical fractional derivatives, several generalizations have been introduced to better match the models to the reality in different situations. In this paper, we work with generalized fractional derivatives of Riemann–Liouville and Caputo type where the power-type kernel (fractional derivative case) is replaced by an arbitrary function k. Such a generalization was previously used in [2–5] and covers many specific cases that are important in applications (see Section 2.1).

Fractional derivatives of Riemann–Liouville and Caputo type are non-local: the derivative of a function $u(t)$ at $t = T$ depends on values of u at $t < T$. We consider an inverse problem (IP1) to recover a history of a function u at $0 < t < T$ by means of measurements of $u(t)$ and its generalized fractional derivative in a left neighborhood of T. To the authors' knowledge, such a problem has not yet been considered in the literature.

We use the results obtained for IP1 in order to investigate an inverse problem of reconstruction of a history of a source in a general PDE that includes as particular cases fractional diffusion and wave equations from the measurements in a left neighborhood of final time T (IP2).

Quite often in the inverse source problem, the goal is to determine a source that is either a space- or time-dependent function. The space-dependent source term is usually reconstructed based on the final time overdetermination condition [6–11]. The time-dependent source term can be recovered from additional boundary measurements [7] or from integral conditions [12,13]. In this paper [14], the source term dependent on time and part of the space variables has been determined. In this paper, we assume that the overdetermination condition is given not only at the final moment of time T, but in its neighborhood. This enables us to reconstruct the source term that depends on both time and all space variables.

In Section 2, we explain the concept of generalized fractional derivative with examples. Next, we formulate the inverse problems and give hints to their physical applications. In Section 3, we prove the uniqueness for a general class of kernels k and reduce IP1 to an integral equation that is further used to derive the solution formulas. Finally, in Section 4, we derive the solution formulas in some particular cases of k based on the expansion with the Legendre polynomials.

2. Problem Formulation

2.1. Generalized Fractional Derivatives

In this paper, $L_p(0,T)$ and $W_p^n(0,T)$ stand for real Lebesgue and Sobolev spaces.

We are solving problems with a generalized fractional derivative. This concept has been used in [2–5]. We utilize $D_a^{\{k\},n}$ as a *unified notation* that stands for the generalized fractional derivatives in Riemann–Liouville ${}^RD_a^{\{k\},n}$ and Caputo sense ${}^CD_a^{\{k\},n}$:

$$({}^RD_a^{\{k\},n}v)(t) = \frac{d^n}{dt^n}\int_a^t k(t-\tau)v(\tau)d\tau, \; ({}^CD_a^{\{k\},n}v)(t) = \int_a^t k(t-\tau)v^{(n)}(\tau)d\tau,$$
$$t > a, \; n \in \{0\} \cup \mathbb{N}, \; k \in L_{1,loc}(0,\infty).$$

The notation of generalized fractional derivative incorporates the following possibilities.

The basic case is

(k1) $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$, $\beta \in (0,1)$. Then, ${}^RD_a^{\{k\},n}$ and ${}^CD_a^{\{k\},n}$ are the Riemann–Liouville and Caputo fractional derivatives of the order $n+\beta-1$, i.e.,

$$({}^RD_a^{\{k\},n}v)(t) = ({}^RD_a^{n+\beta-1}v)(t) = \frac{d^n}{dt^n}\int_a^t \frac{(t-\tau)^{-\beta}}{\Gamma(1-\beta)}v(\tau)d\tau,$$
$$({}^CD_a^{\{k\},n}v)(t) = ({}^CD_a^{n+\beta-1}v)(t) = \int_a^t \frac{(t-\tau)^{-\beta}}{\Gamma(1-\beta)}v^{(n)}(\tau)d\tau.$$

Moreover, in case $k(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$, ${}^RD_a^{\{k\},0}$ is the Riemann–Liouville fractional integral of the order $\beta > 0$, i.e.,

$$({}^RD_a^{\{k\},0}v)(t) = (I_a^\beta v)(t) = \int_a^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}v(\tau)d\tau.$$

Often a memory is not of power-type. A direct generalization of (k1) leads to **multiterm and distributed order fractional derivatives** [15–17]. These derivatives have the following kernels:

(k2) $k(t) = \sum_{j=1}^m p_j \frac{t^{-\beta_j}}{\Gamma(1-\beta_j)}$, $\beta_j \in (0,1)$, $p_j \neq 0$, and

(k3) $k(t) = \int_0^1 p(\beta)\frac{t^{-\beta}}{\Gamma(1-\beta)}d\beta$, $p \in L_1(0,1)$, respectively.

Distributed order and multiterm derivatives enable to model accelerating and retarding sub(super) diffusion, since different powers of t dominate as $t \to 0^+$ and $t \to \infty$ in the kernel. A proper choice of p in (k3) allows modelling ultraslow diffusion [16].

The cases (k2) and (k3) can be unified to a form of Lebesque–Stiltjes integral $k(t) = \int_0^1 \frac{t^{-\beta}}{\Gamma(1-\beta)}d\mu(\beta)$, but we will treat them separately.

Tempered fractional derivatives are used to describe slow transition of anomalous diffusion to a normal one. There are two models of this type in the literature that differ in their mathematical

derivations. The corresponding kernels are:
(k4) $k(t) = \frac{e^{-\lambda t} t^{-\beta}}{\Gamma(1-\beta)} + \lambda \int_0^t \frac{e^{-\lambda\tau}\tau^{-\beta}}{\Gamma(1-\beta)} d\tau$, $0 < \beta < 1$, $\lambda > 0$ [18,19]; and
(k5) $k(t) = e^{-\lambda t} t^{\beta-1} E_{\beta,\beta}(\lambda^\beta t^\beta)$, $0 < \beta < 1$, $\lambda > 0$ [19,20].
We will call derivatives with kernels (k4) and (k5) tempered fractional derivatives of type I and II, respectively.

Removing the singularity of kernels at $t = 0$ allows to highlight memory effects better [21]. In this paper, we consider the following bounded kernels:
(k6) $k(t) = \frac{1}{1-\beta} e^{-\frac{\beta}{1-\beta} t}$, $0 < \beta < 1$ is the kernel of Caputo-Fabrizio derivative [21,22];
(k7) $k(t) = \frac{1}{1-\beta} E_\beta \left(-\frac{\beta t^\beta}{1-\beta} \right)$, $0 < \beta < 1$ is a kernel of Atangana–Baleanu fractional derivative [23,24].
Here, E_β and $E_{\beta,\beta}$ are one-parametric and two-parametric Mittag-Leffler functions, respectively, given by the formulas:

$$E_\alpha(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + 1)}, \quad \operatorname{Re} \alpha > 0,$$
$$E_{\alpha,\beta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + \beta)}, \quad \operatorname{Re} \alpha > 0, \operatorname{Re} \beta > 0.$$

2.2. Formulation of Inverse Problems

Let $0 < t_0 < T < \infty$. Our basic inverse problem consists in a reconstruction of a function in $(0, t_0)$ provided that this function and its derivative are given in (t_0, T).

IP1. Given $\varphi, g : (t_0, T) \to \mathbb{R}$, find $u : (0, T) \to \mathbb{R}$ such that

$$u|_{(t_0,T)} = \varphi \quad \text{and} \quad D_0^{\{k\},n} u|_{(t_0,T)} = g \,. \tag{1}$$

An example of IP1 is the reconstruction of physical quantities in constitutive relations involving fractional derivatives. In the Scott–Blair model of viscoelasticity, the stress is proportional to a time fractional derivative of the strain [25]. In this context, IP1 means the reconstruction of a history of the strain of a body by means of the measurement of strain and stress in a left neighborhood of a time value T. A similar meaning for IP1 can be given in the subdiffusion where the flux is proportional to a time fractional derivative of the concentration (temperature) gradient [26].

Next, we formulate IP2 that is an inverse source problem that can be reduced to IP1:

IP2. Given $\varphi, \Phi : \Omega \times (t_0, T) \to \mathbb{R}$, find $u, F : \Omega \times (0, T) \to \mathbb{R}$, such that

$$(D_0^{\{k\},n} Bu)(x,t) + D^l u(x,t) - Au(x,t) = F(x,t), \quad x \in \Omega,\ t \in (0,T) \tag{2}$$

is fulfilled and

$$u|_{\Omega\times(t_0,T)} = \varphi, \quad F|_{\Omega\times(t_0,T)} = \Phi.$$

Here, $\Omega \subseteq \mathbb{R}^N$ with some $N \in \mathbb{N}$, $D^l = \sum_{j=1}^{l} q_j \frac{\partial^j}{\partial t^j}$ with some $l \in \mathbb{N}$, $q_j \in \mathbb{R}$, and A and B are operators that act on functions depending on x. Throughout the paper, assume that A and B with their domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$ are such that $A : \mathcal{D}(A) \subseteq C(\Omega) \to C(\Omega)$, $B : \mathcal{D}(B) \subseteq C(\Omega) \to C(\Omega)$. We also assume that B is invertible.

Equation (2) generalizes the fractional wave equation ${}^C D_0^\beta u + \lambda(-\Delta)^\alpha u = F$, $\beta \in (1,2)$, $\alpha \in [0.5, 1]$, $\lambda > 0$ [13,27,28], the attenuated wave equation $\frac{\partial^2}{\partial t^2} u + \mu\, {}^R D_0^\beta u - \lambda \Delta u = F$, $\beta \in (0,1) \cup (1,2)$ [29,30] and different subdiffusion equations ${}^C D_0^{\{k\},1} u - \lambda \Delta u = F$ and $\frac{\partial}{\partial t} u - \lambda\, {}^R D_0^{\{k\},1} \Delta u = F$, where k has one of the above forms (k1)–(k7) [16–18,20,23,26,31]. In the latter equation, $B = -\lambda\Delta$ and, in order to guarantee the invertibility of B, proper boundary conditions must be specified in the domain $\mathcal{D}(B)$.

We point out that the operators A and B in (2) are not necessarily linear.

In case if $\Phi = 0$, IP2 means a reconstruction of a source that was active in the past using a measurement of the state of u in a left neighborhood of T. Such an inverse problem may occur in seismology, ground water pollution, etc.

Now, we reduce IP2 to IP1. Let (u, F) solve IP2. Then, Equation (2) restricted to $\Omega \times (t_0, T)$ has the form $(D_0^{\{k\},n} Bu)(x,t) + D^l \varphi(x,t) - A\varphi(x,t) = \Phi(x,t)$. Therefore, Bu is a solution of the following family of IP1:

$$Bu|_{\Omega\times(t_0,T)} = B\varphi \quad \text{and} \quad D_0^{\{k\},n} Bu|_{\Omega\times(t_0,T)} = g, \tag{3}$$

where

$$g(x,t) = \Phi(x,t) + A\varphi(x,t) - D^l\varphi(x,t), \quad x \in \Omega,\ t \in (t_0, T). \tag{4}$$

The solution of IP2 is expressed by means of Bu explicitly: $u = B^{-1}Bu$, $F = D_0^{\{k\},n} Bu + D^l u - Au$.

3. Results in Case of General k

3.1. Uniqueness Results

Lemma 1. *Let k be real analytic in $(0,\infty)$ and $v \in L_1(0,t_0)$. Then, $w(t) = \int_0^{t_0} k(t-\tau)v(\tau)d\tau$ is real analytic in (t_0,∞).*

Proof. The function k can be extended as a complex analytic function $k_{\mathbb{C}}$ in an open domain $D \subset \mathbb{C}$ containing the positive part of the real axis. Let us define $w_{\mathbb{C}}(z) = \int_0^{t_0} k_{\mathbb{C}}(z-\tau)v(\tau)d\tau$ for $z \in D_{t_0} = \{z : z = \xi + t_0,\ \xi \in D\}$. Using the analyticity of $k_{\mathbb{C}}$, it is possible to show that functions u and v involved in the formula $w_{\mathbb{C}}(t+is) = u(t,s) + iv(t,s)$, are continuously differentiable and satisfy Cauchy-Riemann equations in $\{(t,s) : t + is \in D_{t_0}\}$. This implies that $w_{\mathbb{C}}$ is complex analytic in D_{t_0}. On the other hand, its restriction to the subset $\{z = t + i0 : t \in (t_0,\infty)\}$ is the function w. Therefore, w is real analytic in (t_0,∞). $\square$

We will denote the Laplace transform of a function $f : (0,\infty) \to \mathbb{R}$ by

$$\widehat{f}(s) = (\mathcal{L}_{t\to s} f)(s) = \int_0^\infty e^{-st} f(t)dt.$$

The symbol $*$ will stand for the time convolution, i.e., $(f_1 * f_2)(t) = \int_0^t f_1(t-\tau)f_2(\tau)d\tau$.

We prove a uniqueness theorem for IP1.

Theorem 1. *Assume that k satisfies the following conditions:*

$$\exists \mu \in \mathbb{R} : \int_0^\infty e^{-\mu t}|k(t)|dt < \infty, \tag{5}$$

$$k \text{ is real analytic in } (0,\infty), \tag{6}$$

$$\widehat{k}(s) \text{ cannot be meromorphically extended to the whole complex plane } \mathbb{C}. \tag{7}$$

Then, the following assertions hold.

(i) *If $u \in L_1(0,T)$, $k * u \in W_1^n(0,T)$ and $u|_{(t_0,T)} = {}^R D_0^{\{k\},n} u|_{(t_0,T)} = 0$, then $u = 0$.*
(ii) *If $u \in W_1^n(0,T)$ and $u|_{(t_0,T)} = {}^C D_0^{\{k\},n} u|_{(t_0,T)} = 0$, then $u = 0$.*

Proof. (i) Let us extend $u(t)$ by zero for $t > T$ and define the function $f : (0, \infty) \to \mathbb{R}$:

$$f = {}^R D_0^{\{k\},n} u.$$

Since $u(t) = 0, t > t_0$, it holds that

$$f(t) = \frac{d^n}{dt^n} \int_0^{t_0} k(t-\tau) u(\tau) d\tau = \int_0^{t_0} k^{(n)}(t-\tau) u(\tau) d\tau, \quad t > t_0.$$

The function k is real analytic, therefore, $k^{(n)}$ is also real analytic. Hence, Lemma 1 implies that f is real analytic in (t_0, ∞). Since $f(t) = 0,\ t \in (t_0, T)$, and f is real analytic, we obtain that $f(t) = 0,\ t > t_0$.

Due to (5) the $\widehat{k}(s)$ exists and is holomorphic for $\mathrm{Re} s > \mu$. Moreover, in view the properties of f, the $\widehat{f}(s)$ also exists and is expressed by the formula

$$\widehat{f}(s) = s^n \widehat{k}(s) \widehat{u}(s) - p_0 s^{n-1} - ... - p_{n-1}, \quad p_j = \frac{d^j}{dt^j}(k * u)(t)\Big|_{t=0}, \quad \mathrm{Re} s > \mu.$$

Therefore,

$$\widehat{k}(s) = \frac{\widehat{f}(s) + p_0 s^{n-1} + ... + p_{n-1}}{s^n \widehat{u}(s)} \quad \text{for any } s \text{ such that } \mathrm{Re} s > \mu \text{ and } s^n \widehat{u}(s) \neq 0.$$

Since the values $f(t)$ and $u(t)$ vanish for $t > t_0$, $\widehat{f}$ and $\widehat{u}$ are entire functions. Thus, the function $\widehat{f}(s) + p_0 s^{n-1} + ... + p_{n-1}$ is also entire. Assume that $\widehat{u}$ does not vanish on $\mathbb{C}$. Then, by Identity theorem and the fact that $\widehat{u}$ is entire the set of zeros of $\widehat{u}$ does not contain accumulation points. This implies that the extension of $\widehat{k}$ is meromorphic on $\mathbb{C}$. This contradicts to the assumption (7) of the theorem. Therefore, the assumption $\widehat{u} \not\equiv 0$ is invalid, which implies $u = 0$ in $L_1(0, T)$.

(ii) At this part of the proof, let us use the notation $v := u^{(n)}$. Then, $v|_{(t_0,T)} = {}^R D_0^{\{k\},0} v|_{(t_0,T)} = 0$ and $v,\ k * v \in L_1(0, T)$. Therefore, by the assertion (i) of this theorem $v = 0$. Consequently, $u^{(n)} = 0$ and $u|_{(t_0,T)} = 0$ imply that $u = 0$ in $W_1^n(0, T)$. $\square$

Let us compute the Laplace transform for the kernels from Section 1 to see if they satisfy the conditions of Theorem 1.

(k1) In the basic case $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$, $\beta \in (0, 1)$, it holds $\widehat{k}(s) = \frac{1}{s^{1-\beta}}$.

(k2) Similarly for $k(t) = \sum_{j=1}^m p_j \frac{t^{-\beta_j}}{\Gamma(1-\beta_j)}$, $0 < \beta_j < 1$, $p_j \neq 0$, we have $\widehat{k}(s) = \sum_{j=1}^m p_j \frac{1}{s^{1-\beta_j}}$.

(k3) For the distributed fractional derivative $k(t) = \int_0^1 p(\beta) \frac{t^{-\beta}}{\Gamma(1-\beta)} d\beta,\ p \in L_1(0, 1)$, the Laplace transform is $\widehat{k}(s) = \int_0^1 p(\beta) \frac{1}{s^{1-\beta}} d\beta$.

(k4) For the tempered fractional derivative of type I $k(t) = \frac{e^{-\lambda t} t^{-\beta}}{\Gamma(1-\beta)} + \lambda \int_0^t \frac{e^{-\lambda \tau} \tau^{-\beta}}{\Gamma(1-\beta)} d\tau,\ 0 < \beta < 1,\ \lambda > 0$, it holds $\widehat{k}(s) = \frac{(s+\lambda)^\beta}{s}$.

(k5) For the tempered fractional derivative of type II $k(t) = e^{-\lambda t} t^{\beta-1} E_{\beta,\beta}(\lambda^\beta t^\beta)$, $0 < \beta < 1$, $\lambda > 0$, we have that $\widehat{k}(s) = \frac{1}{(s+\lambda)^\beta - \lambda^\beta}$ [19].

(k6) The kernel of Caputo-Fabrizio fractional derivative $k(t) = \frac{1}{1-\beta} e^{-\frac{\beta}{1-\beta} t}$, $0 < \beta < 1$, has a Laplace transform $\widehat{k}(s) = \frac{1}{(1-\beta)s + \beta}$.

(k7) In case of Atangana–Baleanu fractional derivative $k(t) = \frac{1}{1-\beta} E_\beta \left(-\frac{\beta t^\beta}{1-\beta} \right)$, $0 < \beta < 1$, it follows from [32] that $\widehat{k}(s) = \frac{s^{\beta-1}}{(1-\beta)s^\beta + \beta}$.

The kernels (k1)–(k7) satisfy (5),(6). Moreover, it is evident that the kernels (k1), (k2), (k4), (k5), (k7) satisfy (7), because Laplace transforms of these functions have branch points. To guarantee that (k3) also satisfies (7) we assume additionally that $p \neq 0$, $p \geq 0$. Then,

$$\lim_{\substack{\operatorname{Arg} s \to \pm\pi, \\ |s|=1}} \operatorname{Im} \widehat{k}(s) = \int_0^1 p(\beta) \sin((\beta - 1) \times (\pm\pi)) d\beta \underset{>}{<} 0.$$

This shows that $\widehat{k}(s)$ has a jump at $s = -1$, hence (7) holds.

Summing up, the solution of IP1 for a derivative containing a kernel (k1)–(k5) or (k7) is unique.

The kernel of Caputo-Fabrizio fractional derivative (k6) does not satisfy (7) because it has the meromorphic in $\mathbb{C}$ Laplace transform. IP1 with this kernel has infinitely many solutions. Any function such that $\int_0^{t_0} e^{\frac{\beta}{1-\beta}\tau} u(\tau) d\tau = 0$, $u|_{(t_0,T)} = 0$ satisfies the homogeneous IP1 in case $D_0^{\{k\},n} = {}^R D_0^{\{k\},n}$ and any function such that $\int_0^{t_0} e^{\frac{\beta}{1-\beta}\tau} u^{(n)}(\tau) d\tau = 0$, $u|_{(t_0,T)} = 0$ satisfies the homogeneous IP1 in case $D_0^{\{k\},n} = {}^C D_0^{\{k\},n}$.

Now, we proceed to IP2. We define the following set related to operators A, B and D^l:

$$\mathcal{U} = \{u : \Omega \times (0,T) \to \mathbb{R} : u(\cdot,t) \in \mathcal{D}(A) \cap \mathcal{D}(B)\ \forall t \in (0,T),$$
$$u, Au, Bu \in C(\Omega \times (0,T)) \text{ and } q_j \frac{\partial^j}{\partial t^j} u \in C(\Omega \times (0,T)),\ j = 1, \ldots, l\}.$$

From Theorem 1, we can immediately deduce a uniqueness statement for IP2.

Corollary 1. *Let k satisfy (5)–(7). Then, the following assertions hold.*

(i) *If $(u_j, F_j) \in \{u \in \mathcal{U} : (k * Bu)(x,\cdot) \in W_1^n(0,T)\ \forall x \in \Omega\} \times C(\Omega \times (0,T))$, $j = 1,2$, solve (2) with $D_0^{\{k\},n} = {}^R D_0^{\{k\},n}$ and $(u_1,F_1)|_{\Omega \times (t_0,T)} = (u_2,F_2)|_{\Omega \times (t_0,T)}$, then $(u_1,F_1) = (u_2,F_2)$.*

(ii) *If $(u_j, F_j) \in \{u \in \mathcal{U} : Bu(x,\cdot) \in W_1^n(0,T)\ \forall x \in \Omega\} \times C(\Omega \times (0,T))$, $j = 1,2$, solve (2) with $D_0^{\{k\},n} = {}^C D_0^{\{k\},n}$ and $(u_1,F_1)|_{\Omega \times (t_0,T)} = (u_2,F_2)|_{\Omega \times (t_0,T)}$, then $(u_1,F_1) = (u_2,F_2)$.*

Proof. Proof is technically the same in cases (i) and (ii). After considering the formulation of IP2 in terms of IP1 (3) and subtracting the corresponding equations for (u_1, F_1) and (u_2, F_2), we obtain that

$$(Bu_1 - Bu_2)|_{\Omega \times (t_0,T)} = 0 \quad \text{and} \quad D_0^{\{k\},n}(Bu_1 - Bu_2)|_{\Omega \times (t_0,T)} = 0.$$

Then, it follows from Theorem 1 that $(Bu_1 - Bu_2)|_{\Omega \times (0,T)} = 0$ and, consequently, since the operator B is invertible it holds $u_1(x,t) = u_2(x,t)$, $(x,t) \in \Omega \times (0,T)$. Finally, the Equation (2) implies $F_1(x,t) = F_2(x,t)$, $(x,t) \in \Omega \times (0,T)$. □

3.2. Reduction to Integral Equations

In this subsection, we reduce IP1 to integral equations. Let us assume that k satisfies (6).

Firstly, we consider the case $D_0^{\{k\},n} = {}^R D_0^{\{k\},n}$. Assume that $u \in L_1(0,T)$ solves IP1 and $k * u \in W_1^n(0,T)$. Then,

$$\int_0^t k(t-\tau) u(\tau) d\tau = \int_0^{t_0} k(t-\tau) u(\tau) d\tau + \int_{t_0}^t k(t-\tau) \varphi(\tau) d\tau \tag{8}$$

for $t \in (t_0, T)$, where the left hand side belongs to $W_1^n(t_0,T)$ and the first addend in the right-hand side belongs to $C^\infty(t_0,T]$. Thus, the data φ necessarily satisfies $\int_{t_0}^t k(t-\tau)\varphi(\tau)d\tau \in W_1^n(t_0+\delta, T)$,

$\forall\delta \in (t_0, T)$. Applying $\frac{d^n}{dt^n}$ to (8), using the second condition in (1) and rearranging the terms, we obtain the following integral equation of the first kind for $u|_{(0,t_0)}$:

$$\int_0^{t_0} k^{(n)}(t-\tau)u(\tau)d\tau = f(t),\ \ t \in (t_0, T), \quad \text{where} \quad f = g - {}^R D_{t_0}^{\{k\},n}\varphi. \tag{9}$$

Secondly, let us consider the case $D_0^{\{k\},n} = {}^C D_0^{\{k\},n}$, $n \geq 1$. If $u \in W_1^n(0,T)$ solves IP1, then $u^{(n)}|_{(0,t_0)}$ is a solution of the integral equation

$$\int_0^{t_0} k(t-\tau)u^{(n)}(\tau)d\tau = f(t),\ \ t \in (t_0, T), \quad \text{where} \quad f = g - {}^C D_{t_0}^{\{k\},n}\varphi. \tag{10}$$

Since $\lim\limits_{\tau\to t_0^-} u^{(j)}(\tau) = \lim\limits_{\tau\to t_0^+} \varphi^{(j)}(\tau)$, $j = 0, \dots, n-1$, the function $u|_{(0,t_0)}$ is obtained from $u^{(n)}|_{(0,t_0)}$ by the integration:

$$u(t) = \int_{t_0}^{t} \frac{(t-\tau)^{n-1}}{(n-1)!} u^{(n)}(\tau)d\tau + \sum_{j=0}^{n-1} \lim_{\tau\to t_0^+} \varphi^{(j)}(\tau)\frac{(t-t_0)^j}{j!}, \quad t \in (0, t_0).$$

Due to Lemma 1, the integral operators involved in (9),(10) map $L_1(0, t_0)$ into the space of functions that are real analytic in $t > t_0$. This means that IP1 is severely ill-posed and necessarily, f is real analytic in (t_0, T). In the next section, we will derive solution formulas for IP1 that contain the quantities

$$f^{(m)}(t_1), \quad m \in \{0\} \cup \mathbb{N},$$

where t_1 is an arbitrary point in (t_0, T).

4. Solution Formulas in Particular Cases of k

4.1. A Basic Theorem

Theorem 2. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, $t_1 > t_0 > 0$ and $f \in C^\infty(t_0, \infty)$. Let us introduce the following family of sums that depend on a variable $t \in (0, t_0)$ and parameters α, f, t_1, t_0:*

$$V_N(\alpha, f, t_1, t_0)(t) = (t_1 - t)^{-\alpha-2} \sum_{n=0}^{N} A_n P_n\left(\frac{2t_1(t_1-t_0)}{t_0(t_1-t)} - \frac{2t_1 - t_0}{t_0}\right).$$

Here, $N \in \{0\} \cup \mathbb{N} \cup \{\infty\}$, P_n are normalized in $L_2(-1,1)$ Legendre polynomials

$$P_n(t) = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} c_{n,l} t^{n-2l}, \quad \text{where } c_{n,l} = \sqrt{\frac{2n+1}{2}}\frac{1}{2^n}(-1)^l \binom{n}{l}\binom{2n-2l}{n},$$

and

$$\begin{aligned} A_n = A_n(\alpha, f, t_1, t_0) = &\sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} c_{n,l} \sum_{m=0}^{n-2l} \binom{n-2l}{m}\left(\frac{t_0 - 2t_1}{t_0}\right)^{n-2l-m} \\ &\times \left(\frac{2t_1(t_1-t_0)}{t_0}\right)^m \Gamma(\alpha - m + 1) f^{(m)}(t_1). \end{aligned}$$

Assume that $v \in L_2(0, t_0)$ and f is given by $f(t) = \int_0^{t_0} \frac{(t-\tau)^\alpha}{\Gamma(\alpha+1)} v(\tau)d\tau$, $t > t_0$. Then, the series $V_\infty(\alpha, f, t_1, t_0)(t)$ converges almost everywhere in $(0, t_0)$ and

$$v(t) = V_\infty(\alpha, f, t_1, t_0)(t), \quad a.e.\ t \in (0, t_0). \tag{11}$$

Moreover, $V_N(\alpha, f, t_1, t_0) \to v$ *in* $L_2(0, t_0)$ *as* $N \to \infty$*. If in addition,* $v \in BV[0, t_0]$*, then* $V_\infty(\alpha, f, t_1, t_0)(t)$ *converges pointwise in* $(0, t_0)$ *and the estimate is valid:*

$$|v(t) - V_N(\alpha, f, t_1, t_0)(t)| \le \frac{c(t)}{N}, \quad t \in (0, t_0),$$

where $c(t)$ *is a positive constant depending on* t*.*

Proof. For $t_1 > t_0$ we have

$$\frac{1}{\Gamma(\alpha - n + 1)} \int_0^{t_0} (t_1 - \tau)^{\alpha - n} v(\tau) d\tau = f^{(n)}(t_1), \quad n \in \{0\} \cup \mathbb{N}. \tag{12}$$

The substitution $s = \frac{1}{t_1 - \tau}$ under the integral takes (12) to the form

$$\int_{\frac{1}{t_1}}^{\frac{1}{t_1 - t_0}} s^n w(s) ds = \Gamma(\alpha - n + 1) f^{(n)}(t_1), \quad n \in \{0\} \cup \mathbb{N}, \tag{13}$$

where $w(s) = s^{-\alpha-2} v\left(t_1 - \frac{1}{s}\right)$.

We would like to expand our function into series by means of orthonormal Legendre polynomials; thus, we apply a linear substitution that takes us from $[\frac{1}{t_1}, \frac{1}{t_1 - t_0}]$ to the interval $[-1, 1]$, where such an expansion can be applied:

$$\tilde{s} = as + b, \text{ where } a = \frac{2t_1(t_1 - t_0)}{t_0}, \quad b = -\frac{2t_1 - t_0}{t_0}.$$

We also denote $\tilde{w}(\tilde{s}) = w(s)$. Since the performed changes of variables under the integrals are diffeomorphic, $v \in L_2(0, t_0)$ implies $w \in L_2(\frac{1}{t_1}, \frac{1}{t_1 - t_0})$ and $\tilde{w} \in L_2(-1, 1)$ (cf. [33] Section 16.4). Similarly, $v \in BV[0, t_0]$ implies $\tilde{w} \in BV[-1, 1]$.

Since $\tilde{w} \in L_2(-1, 1)$, it can be expanded into the Fourier-Legendre series. It follows from (13) that for $n \in \{0\} \cup \mathbb{N}$

$$\int_{-1}^{1} \frac{1}{a^{n+1}} (\tilde{s} - b)^n \tilde{w}(\tilde{s}) d\tilde{s} = \Gamma(\alpha - n + 1) f^{(n)}(t_1)$$

and, therefore,

$$\int_{-1}^{1} \tilde{s}^n \tilde{w}(\tilde{s}) d\tilde{s} = \int_{-1}^{1} ((\tilde{s} - b) + b)^n \tilde{w}(\tilde{s}) d\tilde{s}$$
$$= \sum_{m=0}^{n} \binom{n}{m} b^{n-m} \int_{-1}^{1} (\tilde{s} - b)^m \tilde{w}(\tilde{s}) d\tilde{s} = \sum_{m=0}^{n} \binom{n}{m} b^{n-m} a^m \Gamma(\alpha - m + 1) f^{(m)}(t_1).$$

It implies that for the normalized Legendre polynomials

$$\int_{-1}^{1} P_n(\tilde{s}) \tilde{w}(\tilde{s}) d\tilde{s} = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} c_{n,l} \int_{-1}^{1} \tilde{s}^{n-2l} \tilde{w}(\tilde{s}) d\tilde{s} = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} c_{n,l} \sum_{m=0}^{n-2l} \binom{n-2l}{m}$$
$$\times b^{n-2l-m} a^m \Gamma(\alpha - m + 1) f^{(m)}(t_1) = A_n.$$

Then, $\tilde{w}(\tilde{s}) = \sum_{n=0}^{\infty} A_n P_n(\tilde{s})$. This series converges in $L_2(-1, 1)$ and for almost every $\tilde{s} \in (-1, 1)$ [34].

For $\tilde{w} \in BV[-1,1]$, the series for $\tilde{w}$ is convergent pointwise for $\tilde{s} \in (-1,1)$ and according to Theorem 1 [35]

$$|\tilde{w}(\tilde{s}) - \sum_{n=0}^{N} A_n P_n(\tilde{s})| \leq \frac{c_1(\tilde{s})}{N}, \quad \tilde{s} \in (-1,1),$$

where $c_1(\tilde{s})$ is a positive constant.

Since the change of variables $\tilde{s} = \frac{a}{t_1 - t} + b$, $t \in [0, t_0]$, is diffeomorphic and $v(t) = (t_1 - t)^{-\alpha-2} \tilde{w}(\frac{a}{t_1-t} + b)$, all assertions of the theorem follow from the proved properties of the series $\tilde{w}(\tilde{s}) = \sum_{n=0}^{\infty} A_n P_n(\tilde{s})$. $\square$

Remark 1. *It follows from* (11) *that for f of form $f(t) = \int_0^{t_0} \frac{(t-\tau)^\alpha}{\Gamma(\alpha+1)} v(\tau) d\tau$, $t > t_0$, where $v \in L_2(0, t_0)$, the sum of series $V_\infty(\alpha, f, t_1, t_0)(t)$ is independent of $t_1 > t_0$. The partial sums $V_N(\alpha, f, t_1, t_0)(t)$, $N < \infty$, however, still may depend on t_1 in case of such f. For example, if $v = 1$, then $V_0(\alpha, f, t_1, t_0)(t) = \frac{\sqrt{0.5}}{\alpha+1} (t_1 - t)^{-\alpha-2} \left[t_1^{\alpha+1} - (t_1 - t_0)^{\alpha+1} \right]$.*

4.2. Solution Formulas in Case of Usual Fractional Derivatives

In this subsection, we consider the case $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$, $\beta \in (0,1)$, $n \geq 1$. Then, ${}^R D_0^{\{k\},n}$ and ${}^C D_0^{\{k\},n}$ are the Riemann–Liouville and Caputo fractional derivatives of the order $n + \beta - 1$, respectively.

Theorem 3. *Let $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$, $0 < \beta < 1$. Then, the following assertions hold.*

(i) *If $u \in L_2(0,T)$, $k * u \in W_1^n(0,T)$ and u solves IP1 with $D_0^{\{k\},n} = {}^R D_0^{\{k\},n}$, then*

$$u(t) = \mathcal{F}_{R,t_1}^{\beta,n}(g - {}^R D_{t_0}^{\{k\},n} \varphi)(t), \quad a.e.\ t \in (0, t_0), \tag{14}$$

where the operator $\mathcal{F}_{R,t_1}^{\beta,n}$ is given by the rule

$$\mathcal{F}_{R,t_1}^{\beta,n}(f)(t) = V_\infty(-n - \beta, f, t_1, t_0)(t). \tag{15}$$

(ii) *If $u \in W_2^n(0,T)$, $n \geq 1$, solves IP1 with $D_0^{\{k\},n} = {}^C D_0^{\{k\},n}$, then*

$$u(t) = \mathcal{F}_{C,t_1}^{\beta,n}(\varphi; g - {}^C D_{t_0}^{\{k\},n} \varphi)(t), \quad t \in (0, t_0), \tag{16}$$

where

$$\mathcal{F}_{C,t_1}^{\beta,n}(\varphi; f)(t) = \sum_{j=0}^{n-1} \lim_{\tau \to t_0^+} \varphi^{(j)}(\tau) \frac{(t - t_0)^j}{j!} + \int_{t_0}^{t} \frac{(t - \tau)^{n-1}}{\Gamma(n)} V_\infty(-\beta, f, t_1, t_0)(\tau) d\tau. \tag{17}$$

The Formulas (14), (16) *are valid for any $t_1 \in (t_0, T)$.*

Proof. (i) Firstly, we represent the IP1 in form (9) with $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$. That is identical to $f(t) = \int_0^{t_0} \frac{(t-\tau)^\alpha}{\Gamma(\alpha+1)} v(\tau) d\tau$ with $\alpha = -n - \beta$, $v(t) = u(t)$ and $f(t) = g(t) - {}^R D_{t_0} \varphi(t)$ and Theorem 2 implies (14).

(ii) Similarly to the previous case we start from representing the problem in a form (10) with $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$. This gives us the relation $f(t) = \int_0^{t_0} \frac{(t-\tau)^\alpha}{\Gamma(\alpha+1)} v(\tau) d\tau$ with $\alpha = -\beta$, $v(t) = u^{(n)}(t)$ and $f(t) = g(t) - {}^C D_{t_0}^{\beta,n} \varphi^{(n)}(t)$. By applying Theorem 2 to it, we obtain

$$u^{(n)}(t) = V_\infty(-\beta, f, t_1, t_0)(t), \ \ a.e.\ t \in (0, t_0), \quad f = g - {}^C D_{t_0}^{\{k\},n} u.$$

Since the condition $u|_{(t_0,T)} = \varphi$ implies $u^{(j)}(t_0) = \lim_{\tau\to t_0^+} \varphi^{(j)}(\tau)$, $\quad j = 0\ldots n-1$, the solution Formula (16) is valid. □

Remark 2. *Let us consider the approximations of the exact solutions defined by* $u_{N,t_1}(t) = V_N(-n-\beta, f, t_1, t_0)(t)$, $t \in (0,t_0)$, $N < \infty$, *in case (i) and* $u_{N,t_1}(t) = \sum_{j=0}^{n-1} \lim_{\tau\to t_0^+} \varphi^{(j)}(\tau)\frac{(t-t_0)^j}{j!} + \int_{t_0}^{t} \frac{(t-\tau)^{n-1}}{\Gamma(n)} V_N(-\beta, f, t_1, t_0)(\tau)d\tau$, $t \in (0,t_0)$, $N < \infty$, *in case (ii). Then, Theorem 2 can be used to compare* u_{N,t_1} *with* u *in the process* $N \to \infty$. *In case (i),* $u_{N,t_1}|_{(0,t_0)} \to u|_{(0,t_0)}$ *in* $L_2(0,t_0)$ *and* $u_{N,t_1}(t) \to u(t)$ *a.e.* $t \in (0,t_0)$. *Similarly, in case (ii),* $u_{N,t_1}|_{(0,t_0)} \to u|_{(0,t_0)}$ *in* $W_2^n(0,t_0)$ *and* $u_{N,t_1}^{(n)}(t) \to u^{(n)}(t)$ *a.e.* $t \in (0,t_0)$. *If in addition to the assumptions of (i),* $u|_{(0,t_0)} \in BV[0,t_0]$ *holds, then* $|u_{N,t_1}(t) - u(t)|$ *is of the order* $1/N$ *for every* $t \in (0,t_0)$. *Similarly, if in addition to the assumptions of (ii),* $u^{(n)}|_{(0,t_0)} \in BV[0,t_0]$ *is valid, then* $|u_{N,t_1}^{(n)}(t) - u^{(n)}(t)|$ *is of the order* $1/N$ *for every* $t \in (0,t_0)$.

Corollary 2. *Let* $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$, $0 < \beta < 1$. *Then, the following assertions hold.*

(i) *If* $(u,F) \in \{u \in \mathcal{U} : (k*Bu)(x,\cdot) \in W_1^n(0,T)\,\forall x \in \Omega\} \times C(\Omega\times(0,T))$ *solves* IP2 *with* $D_0^{\{k\},n} = {}^RD_0^{\{k\},n}$, *then*

$$u(x,t) = \left[B^{-1}\mathcal{F}_{R,t_1}^{\beta,n}(g(x,\cdot) - {}^RD_{t_0}^{\{k\},n}\varphi(x,\cdot))\right](t), \ \text{a.e. } (x,t) \in \Omega\times(0,t_0).$$

(ii) *If* $(u,F) \in \{u \in \mathcal{U} : Bu(x,\cdot) \in W_2^n(0,T)\,\forall x \in \Omega\} \times C(\Omega\times(0,T))$, $n \geq 1$, *solves* IP2 *with* $D_0^{\{k\},n} = {}^CD_0^{\{k\},n}$, *then*

$$u(x,t) = \left[B^{-1}\mathcal{F}_{C,t_1}^{\beta,n}(\varphi(x,\cdot); g(x,\cdot) - {}^RD_{t_0}^{\{k\},n}\varphi(x,\cdot))\right](t), \ (x,t) \in \Omega\times(0,t_0).$$

In both cases g *is given by* (4), t_1 *is an arbitrary number in* (t_0,T) *and* $F|_{\Omega\times(0,t_0)} = \left[D_0^{\{k\},n}Bu + D^l u - Au\right]\Big|_{\Omega\times(0,t_0)}$.

Proof. The proof follows from Theorem 3 and the relations, (3), (4), that describe the transition from IP2 to IP1. □

4.3. Solution Formulas in Case of Tempered and Atangana–Baleanu Derivatives

In this subsection, we derive the solution formulas for particular subcases of the generalized fractional derivative of the order $n = 1$. They are based on solution formulas derived for the usual fractional derivative and involve the operators $\mathcal{F}_{R,t_1}^{\beta,1}$, $\mathcal{F}_{C,t_1}^{\beta,1}$. Again, we assume that t_1 is an arbitrary number in the interval (t_0,T).

Firstly, let us consider the tempered fractional derivatives of type I.

Theorem 4. *Let* $k(t) = \frac{e^{-\lambda t}t^{-\beta}}{\Gamma(1-\beta)} + \lambda\int_0^t \frac{e^{-\lambda\tau}\tau^{-\beta}}{\Gamma(1-\beta)}d\tau$, $0 < \beta < 1$, $\lambda > 0$. *Then, the following assertions hold.*

(i) *If* $u \in L_2(0,T)$, $k*u \in W_1^1(0,T)$ *and* u *solves* IP1 *with* $D_0^{\{k\},1} = {}^RD_0^{\{k\},1}$, *then*

$$u(t) = e^{-\lambda t}\mathcal{F}_{R,t_1}^{\beta,1}(e^{\lambda t}g - e^{\lambda t}\,{}^RD_{t_0}^{\{k\},1}\varphi)(t), \quad \text{a.e. } t \in (0,t_0). \tag{18}$$

(ii) *If* $u \in W_2^1(0,T)$ *solves* IP1 *with* $D_0^{\{k\},1} = {}^CD_0^{\{k\},1}$, *then*

$$u(t) = \lim_{\tau\to t_0^+}\varphi(\tau) - \int_t^{t_0} e^{-\lambda\tau}\mathcal{F}_{R,t_1}^{\beta,1}\left(e^{-\lambda\tau}(g - {}^RD_{t_0}^{\{k\},1}\varphi)'\right)(\tau)d\tau,$$
$$t \in (0,t_0). \tag{19}$$

Proof. Before starting the proof, let us point out that $k'(t) = \frac{e^{-\lambda t} t^{-1-\beta}}{\Gamma(-\beta)}$. Hence, for $t \in (t_0, T)$ and $v \in L_1(0, t_0)$:

$$\int_0^{t_0} k'(t-\tau)v(\tau)d\tau = e^{-\lambda t} \int_0^{t_0} \frac{(t-\tau)^{-1-\beta}}{\Gamma(-\beta)} e^{\lambda\tau} v(\tau)d\tau. \tag{20}$$

(i) Firstly, the IP1 can be rewritten by means of (9), and then Formula (20) leads us to the equation with the unknown term $e^{\lambda t}u(t)$

$$\int_0^{t_0} \frac{(t-\tau)^{-1-\beta}}{\Gamma(-\beta)} e^{\lambda\tau} u(\tau)d\tau = e^{\lambda t} g(t) - e^{\lambda t R} D_{t_0}^{\{k\},1} \varphi(t), \quad t \in (t_0, T).$$

Thus, by applying Theorem 2 and using the notation (15), we obtain (18).

(ii) Let us write IP1 in the form (10), differentiate it and obtain for $t \in (t_0, T)$

$$\int_0^{t_0} k'(t-\tau)u'(\tau)d\tau = \frac{d}{dt}(g(t) - {}^C D_{t_0}^{\{k\},1} \varphi(t)).$$

Then, due to (20) we have $\int_0^{t_0} \frac{(t-\tau)^{-1-\beta}}{\Gamma(-\beta)} e^{\lambda\tau} u'(\tau)d\tau = e^{\lambda t} \frac{d}{dt}(g(t) - {}^C D_{t_0}^{\{k\},1} \varphi(t))$ and similarly to (i) we deduce Formula (19) using the notation (17). □

To handle IP1 for derivatives that contain Mittag-Leffler functions, we need the following lemma.

Lemma 2. *Let $0 < \beta < 1$, $\lambda \in \mathbb{R}$ and $f \in W_1^1(0, T)$. Then, the function $p(t) = \int_0^t (t-\tau)^{\beta-1} E_{\beta,\beta}(\lambda(t-\tau)^\beta) f(\tau)d\tau$ is a solution of the equation ${}^C D_0^\beta p(t) - \lambda p(t) = f(t)$, $t \in (0, T)$, and the function $q(t) = \int_0^t E_\beta(\lambda(t-\tau)^\beta) f(\tau)d\tau$ is a solution of the equation ${}^C D_0^\beta q(t) - \lambda q(t) = I_0^{1-\beta} f(t)$, $t \in (0, T)$.*

Proof. The proof of the first assertion can be found e.g., in [32], p. 174, and the second assertion follows from the first one because $[t^{\beta-1} E_{\beta,\beta}(\lambda t^\beta)] * I_0^{1-\beta} f = E_\beta(\lambda t^\beta) * f$ [6]. □

Next, we consider the case of a tempered fractional derivative of type II.

Theorem 5. *Let $k(t) = e^{-\lambda t} t^{\beta-1} E_{\beta,\beta}(\lambda^\beta t^\beta)$, $\frac{1}{2} < \beta < 1$, $\lambda > 0$. Then, the following assertions are valid:*

(i) *If $u \in W_1^1(0, T)$ and u solves IP1 with $D_0^{\{k\},1} = {}^R D_0^{\{k\},1}$, then*

$$u(t) = \int_{t_0}^t e^{-\lambda\tau} ({}^R D_0^\beta - \lambda^\beta \mathcal{I}) \mathcal{F}_{R,t_1}^{\beta,1} \left(e^{\lambda\tau}(\varphi' + \lambda^\beta g) - {}^R D_{t_0}^\beta e^{\lambda\tau} g \right)(\tau)d\tau + \lim_{\tau \to t_0} \varphi(\tau), \quad t \in (0, t_0). \tag{21}$$

(ii) *If $u \in W_1^2(0, T)$ and u solves IP1 with $D_0^{\{k\},1} = {}^C D_0^{\{k\},1}$, then*

$$u(t) = \int_{t_0}^t e^{-\lambda\tau} ({}^C D_0^\beta - \lambda^\beta \mathcal{I}) \mathcal{F}_{C,t_1}^{\beta,1} \left(e^{\lambda\tau} g; e^{\lambda\tau}(\varphi' + \lambda^\beta g) - {}^C D_{t_0}^\beta e^{\lambda\tau} g \right)(\tau)d\tau + \lim_{\tau \to t_0} \varphi(\tau), \quad t \in (0, t_0). \tag{22}$$

Here, $\mathcal{I}$ is the unity operator.

Proof. Firstly, we prove (ii). Let us define the function w as

$$w(t) = e^{\lambda t}\, {}^C D^{\{k\},1} u(t) = \int_0^t (t-\tau)^{\beta-1} E_{\beta,\beta}(\lambda^\beta (t-\tau)^\beta)(e^{\lambda\tau} u'(\tau))d\tau.$$

Due to Lemma 2, this function solves the equation

$$ {}^{C}D_0^{\beta} w(t) - \lambda^{\beta} w(t) = e^{\lambda t} u'(t), \quad t \in (0,T). \tag{23} $$

Therefore, ${}^{C}D_0^{\beta} w = e^{\lambda t} u' + \lambda^{\beta} w$ and, in view of the condition (1), we have the IP1 with usual fractional derivative:

$$ w|_{(t_0,T)} = e^{\lambda t} g, \quad {}^{C}D_0^{\beta} w|_{(t_0,T)} = e^{\lambda t}\varphi' + \lambda^{\beta} e^{\lambda t} g. \tag{24} $$

In order to apply Theorem 3 (ii) to this problem, we must verify that $w \in W_2^1(0,T)$ is valid. Let us compute:

$$ w'(t) = t^{\beta-1} E_{\beta,\beta}(\lambda^{\beta} t^{\beta}) u'(0) + [t^{\beta-1} E_{\beta,\beta}(\lambda^{\beta} t^{\beta})] * (e^{\lambda t} u')'(t). $$

Due to the assumptions $\frac{1}{2} < \beta < 1$ and $u \in W_1^2(0,T)$ we have $t^{\beta-1}E_{\beta,\beta}(\lambda^{\beta} t^{\beta}) \in L_2(0,T)$ and $(e^{\lambda t} u')' \in L_1(0,T)$. Using the Young's theorem for convolutions, we deduce $w' \in L_2(0,T)$. Thus, $w \in W_2^1(0,T)$.

By applying Theorem 3 (ii) to (24), we obtain

$$ w(t) = \mathcal{F}_{C,t_1}^{\beta,1}(e^{\lambda t} g; e^{\lambda t}\varphi' + \lambda^{\beta} e^{\lambda t} g - {}^{C}D_{t_0}^{\beta} e^{\lambda t} g)(t), \quad t \in (0, t_0). $$

Since by (23), $u' = e^{-\lambda t}({}^{C}D_0^{\beta} - \lambda^{\beta}\mathcal{I})w$, this implies Formula (22).

Secondly we prove (i). Let us define $w(t) = e^{\lambda t R} D_0^{\{k\},1} u(t)$. Then, $w(t) = (\frac{d}{dt} - \lambda) z(t)$, where

$$ z(t) = \int_0^t (t-\tau)^{\beta-1} E_{\beta,\beta}(\lambda^{\beta}(t-\tau)^{\beta})(e^{\lambda\tau} u(\tau)) d\tau. $$

By Lemma 2, z solves the equation

$$ {}^{C}D_0^{\beta} z(t) - \lambda^{\beta} z(t) = e^{\lambda t} u(t), \quad t \in (0,T). \tag{25} $$

Let us differentiate Equation (25) to derive the equation for w:

$$ {}^{R}D_0^{\beta}(z' - \lambda z)(t) + {}^{R}D_0^{\beta}(\lambda z)(t) - \lambda^{\beta} z'(t) = \lambda e^{\lambda t} u(t) + e^{\lambda t} u'(t), \quad a.e. \ t \in (0,T). $$

That is

$$ {}^{R}D_0^{\beta} w(t) - \lambda^{\beta} w(t) + \lambda({}^{R}D_0^{\beta}(z)(t) - \lambda^{\beta} z(t)) = \lambda e^{\lambda t} u(t) + e^{\lambda t} u'(t), \quad a.e. \ t \in (0,T). $$

Since $z(0) = 0$, we have that ${}^{R}D_0^{\beta} z = {}^{C}D_0^{\beta} z$ and using (25) again, we obtain

$$ {}^{R}D_0^{\beta} w(t) = \lambda^{\beta} w(t) + e^{\lambda t} u'(t), \quad a.e. \ t \in (0,T). \tag{26} $$

Based on (1),(26), we formulate IP1 for w:

$$ w|_{(t_0,T)} = e^{\lambda t} g, \quad {}^{R}D_0^{\beta} w|_{(t_0,T)} = e^{\lambda \tau}(\varphi' + \lambda^{\beta} g). \tag{27} $$

To apply Theorem 3 (i), we should prove that $w \in L_2(0,T)$, and $\left(\frac{t^{-\beta}}{\Gamma(1-\beta)}\right) * w = I_0^{1-\beta} w \in W_1^1(0,T)$, that is ${}^R D_0^\beta w \in L_1(0,T)$. Let us investigate

$$w(t) = \left(\frac{d}{dt} - \lambda\right)\left(t^{\beta-1} E_{\beta,\beta}(\lambda^\beta t^\beta)\right) * (e^{\lambda t} u(t)) = u(0)\left(t^{\beta-1} E_{\beta,\beta}(\lambda^\beta t^\beta)\right)$$
$$+ \left(t^{\beta-1} E_{\beta,\beta}(\lambda^\beta t^\beta)\right) * ((e^{\lambda t} u(t))' - \lambda e^{\lambda t} u(t)), \quad t \in (0,T).$$

Since $t^{\beta-1} E_{\beta,\beta}(\lambda^\beta t^\beta) \in L_2(0,T)$ for $\beta \in (1/2, 1)$ and $e^{\lambda t} u(t) \in W_1^1(0,T)$, we obtain that $\left(t^{\beta-1} E_{\beta,\beta}(\lambda^\beta t^\beta)\right) * ((e^{\lambda t} u(t))' - \lambda e^{\lambda t} u(t)) \in L_2(0,T)$; thus, $w \in L_2(0,T)$. Due to the (26) ${}^R D_0^\beta w \in L_1(0,T)$, because $w \in L_2(0,T)$ and $u \in W_1^1(0,T)$.

That enables us to apply Theorem 3 (i) to (27):

$$w(t) = \mathcal{F}_{R,t_1}^{\beta,1}\left(e^{\lambda t}(\varphi' + \lambda^\beta g) - {}^R D_{t_0}^\beta e^{\lambda t} g\right)(t), \quad a.e. \ t \in (0, t_0).$$

This in view of (26) implies Formula (21). □

Remark 3. *It is possible to extend the range of β to $0 < \beta < 1$ in Theorem 5 assuming more regularity of u and the conditions $u(0) = 0$ and $u'(0) = 0$ in cases (i) and (ii), respectively.*

Finally, we consider the case of Atangana–Baleanu fractional derivative.

Theorem 6. *Let $k(t) = \frac{1}{1-\beta} E_\beta\left(-\frac{\beta t^\beta}{1-\beta}\right)$, $0 < \beta < 1$. Then, the following assertions hold:*

(i) *If $u \in W_1^1(0,T)$ and u solves IP1 with $D_0^{\{k\},1} = {}^R D_0^{\{k\},1}$, then*

$$u(t) = \left(\frac{1-\beta}{\beta}\, {}^R D_0^\beta + \mathcal{I}\right) \mathcal{F}_{R,t_1}^{\beta,1}\left(\beta g - {}^R D_{t_0}^\beta(\varphi - (1-\beta) g)\right)(t),$$
$$a.e. \ t \in (0, t_0). \tag{28}$$

(ii) *If $u \in W_1^2(0,T)$ and u solves IP1 with $D_0^{\{k\},1} = {}^C D_0^{\{k\},1}$, then*

$$u(t) = \left(\frac{1-\beta}{\beta}\, {}^C D_0^\beta + \mathcal{I}\right) \mathcal{F}_{C,t_1}^{\beta,1}\left(\varphi - (1-\beta) g; \beta g - {}^C D_{t_0}^\beta(\varphi - (1-\beta) g)\right)(t),$$
$$t \in (0, t_0). \tag{29}$$

Proof. (ii) Let us denote $w = (1-\beta) {}^C D_0^{\{k\},1} u$. For this particular kernel type the relation holds:

$$w(t) = \int_0^t E_\beta\Big(-\frac{\beta(t-\tau)^\beta}{1-\beta}\Big) u'(\tau) d\tau.$$

By Lemma 2 and the identity $I_0^{1-\beta} u' = {}^C D_0^\beta u$, w solves the equation

$${}^C D_0^\beta w(t) + \frac{\beta}{1-\beta} w(t) = {}^C D_0^\beta u(t), \quad t \in (0,T). \tag{30}$$

Since the relation (1) is valid, $w|_{(t_0,T)} = (1-\beta) g$. It follows from (30) that ${}^C D_0^\beta (u - w) = \frac{\beta}{1-\beta} w$. Thus, we have the IP1 with usual fractional derivative

$$(u - w)|_{(t_0,T)} = \varphi - (1-\beta) g, \quad {}^C D_0^\beta (u - w)|_{(t_0,T)} = \beta g.$$

To apply Theorem 3 (ii), we have to show that $u - w \in W_2^1(0,T)$. Since $E'_\beta = \frac{1}{\beta}E_{\beta,\beta}$ and $E_\beta(0) = 1$, we obtain $(u-w)' = -\frac{1}{1-\beta}[t^{\beta-1}E_{\beta,\beta}(-\frac{\beta t^\beta}{1-\beta})] * u'$. Due to the assumptions of (ii), this belongs to $L_2(0,T)$, hence $u - w \in W_2^1(0,T)$. According to Theorem 3 (ii)

$$(u-w)|_{(0,t_0)} = \mathcal{F}_{C,t_1}^{\beta,1}\left(\varphi - (1-\beta)g; \beta g - {}^C D_{t_0}^\beta(\varphi - (1-\beta)g)\right). \tag{31}$$

The relation (30) implies $w = \frac{1-\beta}{\beta}\,{}^C D_0^\beta(u-w)$. Therefore,

$$w|_{(0,t_0)} = \frac{1-\beta}{\beta}\,{}^C D_0^\beta \mathcal{F}_{C,t_1}^{\beta,1}\left(\varphi - (1-\beta)g; \beta g - {}^C D_{t_0}^\beta(\varphi - (1-\beta)g)\right).$$

Hence, from (31), we obtain (29).
(i) Let us denote $w = (1-\beta)^R D_0^{\{k\},1} u$. Then

$$w = \frac{d}{dt}z, \quad \text{where} \quad z(t) = \int_0^t E_\beta\Big(-\frac{\beta(t-\tau)^\beta}{1-\beta}\Big)u(\tau)d\tau.$$

The function z solves the equation

$${}^C D_0^\beta z(t) + \frac{\beta}{1-\beta}z(t) = I_0^{1-\beta}u(t), \quad t \in (0,T). \tag{32}$$

Next, we differentiate Equation (32) and obtain

$${}^R D_0^\beta w(t) + \frac{\beta}{1-\beta}w(t) = {}^R D_0^\beta u(t), \quad a.e.\ t \in (0,T). \tag{33}$$

Therefore, ${}^R D_0^\beta (u-w)(t) = \frac{\beta}{1-\beta}w(t)$ that leads us to the IP1 with a usual fractional derivative

$$(u-w)|_{(t_0,T)} = \varphi - (1-\beta)g, \quad {}^R D_0^\beta(u-w)|_{(t_0,T)} = \beta g.$$

Now, we have to show that $u - w \in L_2(0,T)$ and ${}^R D_0^\beta(u-w)(t) \in L_1(0,T)$. Firstly,

$$w(t) = \frac{d}{dt}\left(E_\beta\Big(-\frac{\beta}{1-\beta}t^\beta\Big)*u(t)\right) = u(0)E_\beta\Big(-\frac{\beta}{1-\beta}t^\beta\Big) + E_\beta\Big(-\frac{\beta}{1-\beta}t^\beta\Big)*u'(t)$$

Since $E_\beta\left(-\frac{\beta}{1-\beta}t^\beta\right) \in L_2(0,T)$ for any $\beta \in (0,1)$, we obtain that $w \in L_2(0,T)$. Due to the Sobolev embedding Theorem $u \in W_1^1(0,T) \subset L_2(0,T)$. Thus, $u - w \in L_2(0,T)$. Secondly, ${}^R D_0^\beta(u-w)(t) = \frac{\beta}{1-\beta}w(t) \in L_2(0,T)$.
We continue the proof by applying Theorem 3 (i) to the IP1 for $u - w$:

$$(u-w)|_{(0,t_0)} = \mathcal{F}_{R,t_1}^{\beta,1}\left(\beta g - {}^R D_{t_0}^\beta(\varphi - (1-\beta)g)\right).$$

It follows from (33) that $w = \frac{1-\beta}{\beta}{}^R D_0^\beta(u-w)$; thus, the Formula (28) holds. □

Similarly to Corollary 2, formulas of solutions of IP2 can be derived in cases of tempered and Atangana–Baleanu derivatives.

5. Conclusions

In this paper, two inverse problems were considered . The goal of IP1 was to reconstruct the history of a function based on its value and the value of its generalized fractional derivative on a final time subinterval. Afterwards, the obtained results were applied to IP2 that includes reconstruction of a source term in a fractional PDE based on the final time subinterval measurements. Defining the overdetermination condition on a final time subinterval, not pointwise, enabled us to treat the problem of the reconstruction of a source term (IP2) in a different manner than usual.

In this article, we have proved the uniqueness of the solution to IP1 and IP2 in case the derivative contains general kernel k and derived the solution formulas for some particular cases of k. Namely, these are the cases of usual fractional derivative, tempered, and Atangana–Baleanu fractional derivatives.

In the case of multiterm and distributed fractional derivatives, solution formulas cannot be derived by means of the method presented in this paper. The problem of reconstruction of explicit representations for solutions in these cases remains open.

Since the IP1 and IP2 are severely ill-posed the solution formulas cannot be applied to the real-life applications without prior regularization. Thus, the numerical analysis of the problems is another non-trivial open question to be considered.

Author Contributions: Formal analysis and investigation, N.K. and J.J.; writing–original draft preparation, N.K.; supervision and editing, J.J.

Funding: This research received no external funding.

Acknowledgments: We thank the referees for the valuable suggestions and remarks.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Sun, H.G.; Zhang, Y.; Baleanu, D.; Chen, W.; Chen, Y.Q. A new collection of real world applications of fractional calculus in science and engineering. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *64*, 213–231. [CrossRef]
2. Janno, J.; Kasemets, K. Identification of a kernel in an evolutionary integral equation occurring in subdiffusion. *J. Inverse Ill-Posed Probl.* **2017**, *25*, 777–798. [CrossRef]
3. Kinash, N.; Janno, J. Inverse problems for a generalized subdiffusion equation with final overdetermination. *Math. Model. Anal.* **2019**, *24*, 236–262.
4. Kochubei, A.N. General fractional calculus, evolution equations, and renewal processes. *Integr. Equ. Oper. Theory* **2011**, *71*, 583–600. [CrossRef]
5. Luchko, Y.; Yamamoto, M. General time-fractional diffusion equation: Some uniqueness and existence results for the initial-boundary-value problems. *Fract. Calc. Appl. Anal.* **2016**, *19*, 676–695. [CrossRef]
6. Janno, J.; Kinash, N. Reconstruction of an order of derivative and a source term in a fractional diffusion equation from final measurements. *Inverse Probl.* **2018**, *34*, 025007. [CrossRef]
7. Jin, B.; Rundell, W. A tutorial on inverse problems for anomalous diffusion processes. *Inverse Probl.* **2015**, *31*, 035003. [CrossRef]
8. Kinash, N.; Janno, J. Inverse problems for a perturbed time fractional diffusion equation with final overdetermination. *Math. Meth. Appl. Sci.* **2018**, *41*, 1925–1943. [CrossRef]
9. Kirane, M.; Malik, A.S.; Al-Gwaiz, M.A. An inverse source problem for a two dimensional time fractional diffusion equation with nonlocal boundary conditions. *Math. Meth. Appl. Sci.* **2013** , *36*, 1056–1069. [CrossRef]
10. Slodička, M.; Šiškova, K.; Van Bockstal, K. Uniqueness for an inverse source problem of determining a space dependent source in a time-fractional diffusion equation. *Appl. Math. Lett.* **2019**, *91*, 15–21. [CrossRef]
11. Wei, T.; Sun, L.; Li, Y. Uniqueness for an inverse space-dependent source term in a multi-dimensional time-fractional diffusion equation. *Appl. Math. Lett.* **2016**, *61*, 108–113. [CrossRef]
12. Lopushanska, H.; Lopushansky, A. Inverse problem with a time-integral condition for a fractional diffusion equation. *Math. Meth. Appl. Sci.* **2019**, *42*, 3327–3340. [CrossRef]
13. Šiškova, K.; Slodička, M. A source identification problem in a time-fractional wave equation with a dynamical boundary condition. *Comput. Math. Appl.* **2018**, *75*, 4337–4354. [CrossRef]

14. Kian, Y.; Yamamoto, M. Reconstruction and stable recovery of source terms and coefficients appearing in diffusion equations. *Inverse Probl.* **2019**, *35*, 115006. [CrossRef]
15. Konjik, S.; Oparnica, L.; Zorica, D.Z. Distributed-order fractional constitutive stress–strain relation in wave propagation modeling. *Angew. Math. Phys.* **2019**, *70*, 1–21. [CrossRef]
16. Mainardi, F.; Mura, A.; Pagnini, G.; Gorenflo, R. Time-fractional diffusion of distributed order. *J. Vib. Control* **2008**, *14*, 1267–1290. [CrossRef]
17. Sokolov, I.M.; Chechkin, A.V.; Klafter, J. Distributed order fractional kinetics. *Acta Phys. Pol. B* **2004**, *35*, 1323–1341.
18. Sibatov, R.T.; Uchaikin, V.V. Truncated Levy statistics for dispersive transport in disordered semiconductors. *Commun. Nonlinear Sci. Numer. Simulat.* **2011**, *16*, 4564–4571. [CrossRef]
19. Wu, X.; Deng, W.; Barkai, E. Tempered fractional Feynman-Kac equation: Theory and examples. *Phys. Rev. E* **2016**, *93*, 032151. [CrossRef]
20. Gajda, J.; Magdziarz, M. Fractional Fokker-Planck equation with tempered α-stable waiting times: Langevin picture and computer simulation. *Phys. Rev. E* **2010**, *82*, 011117. [CrossRef]
21. Atangana, A.; Alkahtani, B.S.T. Analysis of the Keller–Segel Model with a Fractional Derivative without Singular Kernel. *Entropy* **2015**, *17*, 4439–4453. [CrossRef]
22. Caputo, M.; Fabrizio, M. A new definition of fractional derivative without singular kernel. *Prog. Fract. Differ. Appl.* **2015**, *1*, 73–85.
23. Atangana, A. On the new fractional derivative and application to nonlinear Fisher's reaction-diffusion equation. *Appl. Math. Comput.* **2016**, *273*, 948–956. [CrossRef]
24. Gomez-Aguilar, J.F.; Escobar-Jimenez, R.F.; Lopez-Lopez, M.G.; Alvarado-Martinez, V.M. Atangana-Baleanu fractional derivative applied to electromagnetic waves in dielectric media. *J. Electromagn. Waves Appl.* **2016**, *30*, 1937–1952. [CrossRef]
25. Mainardi, F. *Fractional Calculus and Waves in Linear Viscoelasticity*; World Scientific: Singapore, 2010.
26. Povstenko, Y. Fractional heat conduction and associated thermal stress. *J. Therm. Stress.* **2004**, *28*, 83–102. [CrossRef]
27. Luchko, Y. Fractional wave equation and damped waves. *J. Math. Phys.* **2013**, *54*, 031505. [CrossRef]
28. Hanyga, A. Multidimensional solutions of time-fractional diffusion-wave equations. *Proc. R. Soc. Lond. A* **2002**, *458*, 933–957. [CrossRef]
29. Feng, Y.; Graef, J.R.; Kong, L.; Wang, M. The forward and inverse problems for a fractional boundary value problem. *Appl. Anal.* **2018**, *97*, 2474–2484. [CrossRef]
30. Straka, P.; Meerschaert, M.M.; McGough, R.J.; Zhou, Y. Fractional wave equations with attenuation. *Fract. Calc. Appl. Anal.* **2013**, *16*, 262–272. [CrossRef]
31. Baeumer, B.; Kurita, S.; Meerschaert, M.M. Inhomogeneous fractional diffusion equations. *Fract. Calc. Appl. Anal.* **2005**, *8*, 371–386.
32. Gorenflo, R.; Kilbas, A.A.; Mainardi, F.; Rogosin, S.V. *Mittag-Leffler Functions, Related Topics and Applications*; Springer: New York, NY, USA, 2014.
33. Jones, F. *Lebesgue Integration on Euclidean Space*; Jones and Bartlett Publishers: London, UK, 2001.
34. Pollard, H. The Convergence Almost Everywhere of Legendre Series. *Proc. Am. Math. Soc* **1972**, *35*, 442–444. [CrossRef]
35. Bojanic, R.; Vuilleumier, M. On the rate of convergence of Fourier-Legendre series of functions of bounded variation. *J. Approx. Theory* **1981**, *31*, 67–79. [CrossRef]