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Second Order Semilinear Volterra-Type Integro-Differential Equations with Non-Instantaneous Impulses

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Abstract: We consider a non-instantaneous system represented by a second order nonlinear differential equation in a Banach space E. We use the family of linear bounded operators introduced by Kozak, Darbo fixed point method and Kuratowski measure of noncompactness. A new set of sufficient conditions is formulated which guarantees the existence of the solution of the non-instantaneous system. An example is also discussed to illustrate the efficiency of the obtained results.

Keywords: second order differential equations; mild solution; non-instantaneous impulses; Kuratowski measure of noncompactness; Darbo fixed point

1. Introduction

The aim of this paper is to establish a result of the existence of mild solution for a class of the non-autonomous second order nonlinear differential equation with non-instantaneous impulses described in the form

$$\begin{cases} y''(t) = A(t)y(t) + f\left(t, y(t), \int_0^t g(t,s,y(s))ds\right), \ t \in (s_i, t_{i+1}], i = 0, \cdots, N, \\ y(t) = \gamma_i(t, y(t_i^-)), \quad t \in (t_i, s_i], \quad i = 1, \cdots, N, \\ y'(t) = \zeta_i(t, y(t_i^-)), \quad t \in (t_i, s_i], \quad i = 1, \cdots, N, \\ y(0) = y_0, \ y'(0) = y_1, \end{cases} \tag{1}$$

In this text, E is a reflexive Banach space endowed with a norm $|\cdot|$, $J = [0, a]$, $0 = s_0 < t_1 < s_1 < t_2, \cdots, t_N < s_N < t_{N+1} = a < \infty$. We consider in problem (1) that $y \in C((s_i, t_{i+1}), E)$, $i = 0, 1, \cdots, N$. The functions $\gamma_i(t, y(t_i^-))$ and $\zeta_i(t, y(t_i^-))$ represent noninstantaneous impulses during the intervals $(t_i, s_i]$, $i = 1, \cdots, N$, so impulses at t_i^- have some duration, namely on intervals $(t_i, s_i]$. Further, $A(t) : D(A(t)) \subset E \to E$ is a closed linear operator which generates a evolution system $\{S(t,s)\}_{(t,s)\in D}$ of linear bounded operators , $f : J \times E \times E \to E$, $g \in C(D \times E, E)$, $D = \{(t,s) \in J \times J : s \leq t\}$ and y_0, y_1 are given elements of E.

The theory and application of integrodifferential equations are important subjects in applied mathematics, see, for example [1–8] and recent development of the topic, see the monographs of [9]. In recent times there have been an increasing interest in studying the abstract autonomous second order, see for example [10–14]. Useful for the study of abstract second order equations is the existence of an evolution system $S(t,s)$ for the homogenous equation

$$y''(t) = A(t)y(t), \text{ for } t \geq 0. \tag{2}$$

For this purpose there are many techniques to show the existence of $S(t,s)$ which has been developed by Kozak [15]. In many problems, such as the transverse motion of an extensible beam, the vibration of hinged bars and many other physical phenomena, we deal with the second-order abstract differential equations in the infinite dimensional spaces. On the other hand, recently there exists an extensive literature for the non-autonomous second order see, for example, [16–22].

The dynamics of many evolving processes are subject to abrupt changes such as shocks, harvesting, and natural disaster. These phenomena involve short term perturbations from continuous and smooth dynamics, whose duration is negligible in comparison with the duration of an entire evolution. Particularly, the theory of instantaneous impulsive equations have wide applications in control, mechanics, electrical engineering, biological and medical fields. Recently, Hernandez et al. [23] use first time not instantaneous impulsive condition for semi-linear abstract differential equation of the form

$$\begin{cases} y'(t) = Ay(t) + f(t, y(t)), & t \in (s_i, t_{i+1}], i = 0, \cdots, N, \\ y(t) = g_i(t, y(t)), & t \in (t_i, s_i], i = 1, \cdots, N, \\ y(0) = y_0, \end{cases} \tag{3}$$

and introduced the concepts of mild and classical solution. Wang and Fečkan have changed the conditions $y(t) = g_i(t, y(t))$ in (3) as follows

$$y(t) = g_i(t, y(t_i^+)), \qquad t \in (t_i, s_i], i = 1, \cdots, N.$$

Of course then $y(t_i^+) = g_i(t, y(t_i^-))$, where $y(t_i^+)$ and $y(t_i^-)$ represent respectively the right and left limits of $y(t)$ at $t = t_i$. Motivated by above remark, Wang and Fečkan [24] have shown existence, uniqueness and stability of solutions of such general class of impulsive differential equations. To learn more about this kind of problems, we refer [25–34].

To deal with the above mentioned issues, we investigate necessary and sufficient conditions for the existence of a mild solution of system (1). By virtue of the theory of measure of noncompactness associated with Darbo's and Darbo-Sadovskii's fixed point theorem. This technique was considered by Banas and Goebel [35] and subsequently used in many papers; see, for example, [33,36–39].

A brief outline of this paper is given:. Some preliminaries are presented in Section 2. Section 3, we obtain necessary and sufficient conditions for System (1). An Appropriate example is given to illustrate our results.

2. Basic Definitions and Preliminaries

In this section, we review some basic concepts, notations, and properties needed to establish our main results.

Denote by $C(J, E)$ the space of all continuous E-valued functions on interval J which is a Banach space with the norm

$$\|y\| = \sup_{t \in J} |y(t)|.$$

To treat the impulsive conditions, we define the space of piecewise continuous functions

$PC(J, E) = \{y : J \to E : y \in C([0, t_1] \cup (t_k, s_k] \cup (s_k, t_{k+1}], E),\ k = 1, \ldots, N$
and there exist $y(t_k^-),\ y(t_k^+), y(s_k^-)$and $y(s_k^+)$ $k = 1, \ldots, N$ with $y(t_k^-) = y(t_k)$
and $y(s_k^-) = y(s_k)\}$.

It can be easily proved that $PC(J, E)$ is a Banach space endowed with

$$\|y\|_{PC} = \sup_{t \in J} |y(t)|.$$

For a positive number R, let

$$B_R = \{y \in PC(J, E) : \|y\|_{PC} \leq R\}.$$

be a bounded set in $PC(J, E)$.

$L^r(J, E)$ denotes the space of E-valued Bochner functions on $[0, a]$ with the norm

$$\|y\|_{L^r} = \left(\int_0^a |y(t)|^r dt\right)^{\frac{1}{r}}, \qquad r \geq 1.$$

$B(E)$ the Banach space of bounded linear operators from E into E.

First we recall the concept of the evolution operator $S(t, s)$ for problem (2), introduced by Kozak in [15] and recently used by Henríquez, Poblete and Pozo in [20].

Definition 1. *Let $S : D \to B(E)$. The family is said to be an evolution operator generated by the family $\{A(t) : t \in J\}$ if the following conditions are satisfied [15]:*

(e_1) *For each $y \in E$ the function $S(\cdot, \cdot)y : J \times J \to E$ is of class C^1 and*

- **(i)** *for each $t \in J$, $S(t, t) = 0$,*
- **(ii)** *for all $(t, s) \in D$ and for each $y \in E$,*

$$\frac{\partial}{\partial t} S(t, s)y\Big|_{t=s} = y, \;\; \frac{\partial}{\partial s} S(t, s)y\Big|_{t=s} = -y.$$

(e_2) *For each $(t, s) \in D$, if $y \in D(A(t))$, then $\frac{\partial}{\partial s} S(t, s)y \in D(A(t))$, the map $(t, s) \longmapsto S(t, s)y$ is of class C^2 and*

- **(i)** $\frac{\partial^2}{\partial t^2} S(t, s)y = A(t)S(t, s)y,$
- **(ii)** $\frac{\partial^2}{\partial s^2} S(t, s)y = S(t, s)A(s)y,$
- **(iii)** $\frac{\partial^2}{\partial s \partial t} S(t, s)y\Big|_{t=s} = 0.$

(e_3) *For all $(t, s) \in D$, if $y \in D(A(t))$, then $\frac{\partial}{\partial s} S(t, s)y \in D(A(t))$. Moreover, there exist $\frac{\partial^3}{\partial t^2 \partial s} S(t, s)y$, $\frac{\partial^3}{\partial s^2 \partial t} S(t, s)y$ and*

- **(i)** $\frac{\partial^3}{\partial t^2 \partial s} S(t, s)y = A(t) \frac{\partial}{\partial s} S(t, s)y,$
- **(ii)** $\frac{\partial^3}{\partial s^2 \partial t} S(t, s)y = \frac{\partial}{\partial t} S(t, s)A(s)y,$

and for all $y \in D(A)$ the function $(t, s) \longmapsto A(t) \frac{\partial}{\partial s} S(t, s)y$ is continuous in D.

Definition 2. *A function $f : J \times E \times E \to E$ is said to be a Carathéodory function if it satisfies:*

(i) *$t \to f(t, u, v)$ is measurable for each $u, v \in E \times E$,*
(ii) *$(u, v) \to f(t, u, v)$ is continuous for almost each $t \in J$.*

For W, a nonempty subset of E, we denote by $\overline{W}$ and $ConvW$ the closure and the closed convex hull of W, respectively. Finally, the standard algebraic operations on sets are denoted by aW and $Y + W$, respectively. Now, we recall some basic definitions and properties about Kuratowski measure of noncompactness that will be used in the proof of our main results.

Definition 3. *[35] The Kuratowski measure of noncompactness $\alpha_E(\cdot)$ defined on bounded set W of Banach space E is*

$$\alpha_E(W) = \inf\{\varepsilon > 0 : W = \cup_{i=1}^{n} W_i \text{ and } diam(W_i) \leq \varepsilon \text{ for } i = 1, 2, \cdots n\}.$$

Some basic properties of $\alpha_E(\cdot)$ are given in the following lemma.

Lemma 1. *Let Y and W be bounded sets of E and a be a real number [35]. The Kuratowski measure of noncompactness satisfies some properties:*

(p_1) *W is pre-compact if and only if $\alpha_E(W) = 0$,*
(p_2) $\alpha_E(\overline{W}) = \alpha_E(W)$,
(p_3) $\alpha_E(Y) \leq \alpha_E(W)$ *when* $Y \subset W$,
(p_4) $\alpha_E(Y + W) \leq \alpha_E(Y) + \alpha_E(W)$,
(p_5) $\alpha_E(aW) = |a|\alpha_E(W)$ *for any* $a \in \mathbf{R}$,
(p_6) $\alpha_E(ConvW) = \alpha_E(W)$.

The map $Q : X \subset E \to E$ is said to be a α-contraction if there exists a positive constant $\lambda < 1$ such that $\alpha_E(Q(W)) \leq \lambda\alpha_E(W)$ for any bounded closed subset $W \subset E$.

Lemma 2. *[40] Let E be a Banach space, $W \subset E$ be bounded. Then there exists a countable set $W_0 \subset W$, such that*

$$\alpha_E(W) \leq 2\alpha_E(W_0).$$

Lemma 3. *[41] Let E be a Banach space, $-\infty < a_1 < a_2 < +\infty$ for constants, and let $W = \{y_n\} \subset PC([a_1, a_2], E)$, be a bounded and countable set. Then $\alpha_E(W(t))$ is Lebesgue integral on $[a_1, a_2]$, and*

$$\alpha_E\Big(\Big\{\int_{a_1}^{a_2} y_n(t)dt : n \in \mathbb{N}\Big\}\Big) \leq 2\int_{a_1}^{a_2} \alpha_E(W(t))dt.$$

Denote by α_{PC} the Kuratowski measure of noncompactness of $PC(J, E)$. Before proving the existence results, we need the following Lemmas.

Lemma 4. *[35] If $W \subset PC(J; E)$ is bounded, then $\alpha_E(W(t)) \leq \alpha_{PC}(W)$, for all $t \in J$; here $W(t) = \{y(t); y \in W \subset E\}$. Furthermore if W is equicontinuous on J, then $\alpha_E(W(t))$ is continuous on J and*

$$\alpha_{PC}(W) = \sup_{t\in J} \alpha_E(W(t)).$$

Lemma 5. *[42] Let E, F be Banach spaces. If the map $\Psi : \mathcal{D}(\Psi) \subset E \to F$ is Lipschitz continuous with constant k, then $\alpha_E(\Psi(W)) \leq k\alpha_E(W)$ for any bounded subset $W \subset \mathcal{D}(\Psi)$.*

Theorem 1. *(Darbo) [43] Assume that W is a non-empty, closed and convex subset of a Banach space E and $0 \in W$. Let $Q : W \to W$ be a continuous mapping and α_E-contraction. If the set $\{y \in W : y = \lambda Qy\}$ is bounded for $0 < \lambda < 1$, then the map Q has at least one fixed point in W.*

Theorem 2. *(Darbo-Sadovskii) [35] Assume that W is a non-empty, closed, bounded, and convex subset of a Banach space E. Let $Q : W \to W$ be a continuous mapping and α_E-contraction. Then the map Q has at least one fixed point in W.*

3. Existence Results

In this section, we discuss the existence of mild solutions for system (1). Firstly, let us propose the definition of the mild solution of system (1).

Definition 4. *A function $y \in PC(J, E)$ is said to be a mild solution to the system (1), if it satisfies the following relations:*

$$y(0) = y_0,\ y'(0) = y_1,$$

the non-instantaneous conditions

$$y(t) = \gamma_i(t, y(t_i^-)), \quad y'(t) = \zeta_i(t, y(t_i^-)), \quad t \in (t_i, s_i],$$

and y is the solution of the following integral equations

$$y(t) = \begin{cases} -\dfrac{\partial}{\partial s} S(t,0)y_0 + S(t,0)y_1 \\ \qquad + \displaystyle\int_0^t S(t,s) f\left(s, y(s), \int_0^s g(s,\tau,y(\tau))d\tau\right) ds, \quad t \in [0, t_1], \\ -\dfrac{\partial}{\partial s} S(t,s_i)\gamma_i(s_i, y(t_i^-)) + S(t,s_i)\zeta_i(s_i, y(t_i^-)) \\ \qquad + \displaystyle\int_{s_i}^t S(t,s) f\left(s, y(s), \int_0^s g(s,\tau,y(\tau))d\tau\right) ds, \quad t \in (s_i, t_{i+1}]. \end{cases}$$

In this manuscript, we list the following hypotheses:

(H_1) There exist a pair of constants $M \geq 1$ and $\delta > 0$, such that

$$\|S(t,s)\|_{B(E)} \leq M e^{-\delta(t-s)} \quad \text{for any } (t,s) \in D.$$

(H_2) There exists a constant $\tilde{M} > 0$ such that:

$$\|\frac{\partial}{\partial s} S(t,s)\|_{B(E)} \leq \tilde{M} e^{-\delta(t-s)}, (t,s) \in D.$$

(H_3) $f : J \times E \times E \to E$ is of Carathéodory type and satisfies:

- (a) There exist $\Theta_f \in L^r(J, \mathbb{R}^+)$, $r \in [1, \infty)$ and a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ such that:

$$|f(t,y,z)| \leq \Theta_f(t)\psi(|y| + |z|) \text{ for a.a } t \in J \text{ and each } y, z \in E.$$

- (b) There exist integrable functions $\sigma, \varrho : J \to \mathbb{R}^+$, such that:

$$\alpha_E(f(t, W_1, W_2)) \leq \sigma(t)\alpha_E(W_1) + \varrho(t)\alpha_E(W_2)$$

$$\text{for a.a } t \in J \text{ and } W_1, W_2 \subset E.$$

(H_4) $g : D \times E \to E$ is a continuous function that satisfies:

- (a) There exist $\Theta_g \in L^1(J, \mathbb{R}^+)$, and a continuous nondecreasing function $\varphi : [0, \infty) \to (0, \infty)$ such that:

$$|g(t,s,y)| \leq \Theta_g(t)\varphi(|y|) \text{ for a.a } (t,s) \in D \text{ and each } y \in E.$$

- (b) There exists constant $K^* > 0$, such that

$$\alpha_E(g(t,s,W)) \leq K^* \alpha_E(W) \text{ for a.a } (t,s) \in D \text{ and } W \subset E.$$

(H_5) The functions $\gamma_i : (t_i, s_i] \times E \to E, i = 1, \cdots, N$, are continuous, and they satisfy the following conditions:

- (a) there exist positive constants $c_i, i = 1, \cdots, N$ such that

$$|\gamma_i(t, y_2) - \gamma_i(t, y_1)| \leq c_i\, |y_2 - y_1| \text{ for a.a } t \in (t_i, s_i] \text{ and each } y_1, y_2 \in E.$$

(b) there exist positive constants d_i, such that

$$d_i = \sup_{t\in[t_i,s_i]} \gamma_i(t,0).$$

(H_6) The functions $\zeta_i : (t_i, s_i] \times E \to E, i = 1, \cdots, N$, are continuous, and satisfy the following conditions:

(a) There exist constants $e_i, l_i > 0, i = 1, \cdots, N$ such that

$$|\zeta_i(t,y)| \leq e_i\,|y| + l_i \text{ for a.a } t \in (t_i;s_i] \text{ and each } y \in E.$$

(b) There exists constants $\overline{k}_i > 0, i = 1, \cdots, N$ such that

$$\alpha_E(\zeta_i(t,W)) \leq \overline{k}_i \alpha_E(W) \text{ for a.a } t \in (t_i,s_i] \text{ and any} W \subset E.$$

(H_7)

$$\max_{1\leq i\leq N}(k_i,1)\left(\max_{1\leq i\leq N}(\tilde{M}k_i + M\bar{k}_i) + 2M(\|\sigma\|_{L^1} + 2K^* a\|\varrho\|_{L^1})\right) < 1.$$

Remark 1. *From Lemma 5 and (H_5), there exist constants $k_i > 0$, such that*

$$\alpha_E(\gamma_i(t,W)) \leq k_i\alpha_E(W) \textit{ for a.a } t \in (t_i,s_i] \textit{ and each } y \in E.$$

Theorem 3. *Under the assumptions (H_1)–(H_7), the system (1) has at least one mild solution on J, provided that*

$$\int_0^a \max(\bar{M}\Theta_f(s), \Theta_g(s))ds \leq \int_{m_i}^\infty \frac{ds}{\psi(s)+\varphi(s)}, i = 2,3\cdots N \tag{4}$$

with

$$\bar{M} = \max_{2\leq i\leq N}\left\{\frac{M}{1-L_1}, \frac{M}{1-L_i}, \frac{Mc_i}{1-L_{i-1}}\right\},$$

and

$$m_i = \frac{d_i}{1-L_{i-1}} + \max_{2\leq i\leq N}\left\{\tilde{M}\,|y_0| + M\,|y_1|\,, \frac{\tilde{M}d_1}{1-L_1} + \frac{Ml_1}{1-L_1}, \frac{\tilde{M}d_i}{1-L_i} + \frac{Ml_i}{1-L_i}, \frac{\tilde{M}c_i d_{i-1}}{1-L_{i-1}} + \frac{Mc_i l_{i-1}}{1-L_{i-1}}\right\},$$

where

$$L_i = \tilde{M}c_i + Me_i < 1.$$

Proof. Define the mapping $\Lambda : PC(J,E) \to PC(J,E)$ by

$$(\Lambda y)(t) = \begin{cases} \gamma_i\left(t, -\dfrac{\partial}{\partial s}S(t,s_i)\gamma_{i-1}(s_{i-1}, y(t_{i-1}^-)) + S(t,s_{i-1})\zeta_{i-1}(s_{i-1}, y(t_{i-1}^-)) \right. \\ \qquad \left. + \displaystyle\int_{s_{i-1}}^{t_i} S(t,s) f\left(s, y(s), \int_0^s g(s,\tau,y(\tau))d\tau\right) ds\right), \quad t \in (t_i, s_i], \\ -\dfrac{\partial}{\partial s}S(t,0)y_0 + S(t,0)y_1 \\ \qquad + \displaystyle\int_0^t S(t,s) f\left(s, y(s), \int_0^s g(s,\tau,y(\tau))d\tau\right) ds, \quad t \in [0,t_1], \\ -\dfrac{\partial}{\partial s}S(t,s_i)\gamma_i(s_i, y(t_i^-)) + S(t,s_i)\zeta_i(s_i, y(t_i^-)) \\ \qquad + \displaystyle\int_{s_i}^t S(t,s) f\left(s, y(s), \int_0^s g(s,\tau,y(\tau))d\tau\right) ds, \quad t \in (s_i, t_{i+1}]. \end{cases} \tag{5}$$

It is obvious that the fixed point of Λ is the mild solution of (1). We shall show that Λ satisfies the assumptions of Theorem 1. The proof will be given in four steps.

Step 1. A priori bounds.

Let $\lambda \in (0,1)$ and let $y \in Y$ be a possible solution of $y = \lambda\Lambda(y)$ for some $0 < \lambda < 1$. Thus,

Case 1. For each $t \in [0, t_1]$, we get

$$y(t) = -\lambda \frac{\partial}{\partial s} S(t,0)y_0 + \lambda S(t,0)y_1 + \lambda \int_0^t S(t,s)f(s,y(s),\int_0^s g(s,\tau,y(\tau))d\tau)ds.$$

Then

$$\begin{aligned}
|y(t)| &\leq \left\| \frac{\partial}{\partial s} S(t,0) \right\|_{B(E)} |y_0| + \|S(t,0)\|_{B(E)} |y_1| \\
&+ \int_0^t \|S(t,s)\|_{B(E)} \Theta_f(s)\psi\left(|y(s)| + \int_0^s \Theta_g(\tau)\varphi(|y(\tau)|))d\tau\right) ds \\
&\leq \tilde{M}|y_0| e^{-\delta t} + M|y_1| e^{-\delta t} \\
&+ \int_0^t Me^{-\delta(t-s)}\Theta_f(s)\psi\left(|y(s)| + \int_0^s \Theta_g(\tau)\varphi(|y(\tau)|)d\tau\right) ds. \\
&\leq (\tilde{M}|y_0| + M|y_1|)e^{-\delta t} \\
&+ \int_0^t Me^{-\delta(t-s)}\Theta_f(s)\psi\left(|y(s)| + \int_0^s \Theta_g(\tau)\varphi(|y(\tau)|)d\tau\right) ds.
\end{aligned}$$

Case 2. For each $t \in (s_i, t_{i+1}]$, we have

$$\begin{aligned}
y(t) &= -\lambda \frac{\partial}{\partial s} S(t,s_i)\gamma_i(s_i,y(s_i)) + \lambda S(t,s_i)\zeta_i(s_i,y(s_i)) \\
&+ \lambda \int_{s_i}^t S(t,s)f(s,y(s),\int_0^s g(s,\tau,y(\tau))d\tau)ds,
\end{aligned}$$

then

$$\begin{aligned}
|y(t)| &\leq \left\| \frac{\partial}{\partial s} S(t,s_i) \right\|_{B(E)} |\gamma_i(s_i,y(s_i))| + \|S(t,s_i)\|_{B(E)} |\zeta_i(s_i,y(s_i))| \\
&+ \int_{s_i}^t \|S(t,s)\|_{B(E)} \Theta_f\psi\left(|y(s)| + \int_0^s \Theta_g(\tau)\varphi(|y(\tau)|))d\tau\right) ds \\
&\leq \tilde{M}c_i |y(s_i)| e^{-\delta(t-s_i)} + \tilde{M}d_i e^{-\delta(t-s_i)} \\
&+ Me_i |y(s_i)| e^{-\delta(t-s_i)} + Ml_i e^{-\delta(t-s_i)} \\
&+ \int_{s_i}^t Me^{-\delta(t-s)}\Theta_f(s)\psi\left(|y(s)| + \int_0^s \Theta_g(\tau)\varphi(|y(\tau)|)d\tau\right) ds. \\
&\leq \tilde{M}c_i |y(s_i)| + \tilde{M}d_i e^{-\delta(t-s_i)} \\
&+ Me_i |y(s_i)| + Ml_i e^{-\delta(t-s_i)} \\
&+ \int_{s_i}^t Me^{-\delta(t-s)}\Theta_f(s)\psi\left(|y(s)| + \int_0^s \Theta_g(\tau)\varphi(|y(\tau)|)d\tau\right) ds.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
\sup_{s\in[0,t]} |y(s)| &\leq \left(\frac{\tilde{M}d_i e^{s_i}}{1-L_i} + \frac{Ml_i e^{s_i}}{1-L_i} \right) e^{-\delta t} \\
&+ \int_{s_i}^t \frac{M}{1-L_i} e^{-\delta(t-s)}\Theta_f(s)\psi\left(\sup_{s\in[0,t]} |y(s)| + \int_0^s \Theta_g(\tau)\varphi(\sup_{s\in[0,t]} |y(s))| d\tau\right) ds.
\end{aligned}$$

Case 3. For each $t \in (s_i, t_i]$, we have,

$$\begin{aligned}
|y(t)| &= \lambda\Big|\gamma_i\Big(t, -\frac{\partial}{\partial s}S(t,s_{i-1})\gamma_{i-1}(s_{i-1},y(t_{i-1}^-)) + S(t,s_{i-1})\zeta_{i-1}(s_{i-1},y(t_{i-1}^-)) \\
&+ \int_{s_{i-1}}^{t_i} S(t,s)f\left(s,y(s),\int_0^s g(s,\tau,y(\tau))d\tau\right)ds\Big)\Big| \\
&\leq \lambda\Big|\gamma_i\Big(t, -\frac{\partial}{\partial s}S(t,s_i)\gamma_{i-1}(s_{i-1},y(t_{i-1}^-)) + S(t,s_{i-1})\zeta_{i-1}(s_{i-1},y(t_{i-1}^-)) \\
&+ \int_{s_{i-1}}^{t_i} S(t,s)f\left(s,y(s),\int_0^s g(s,\tau,y(\tau))d\tau\right)ds - \gamma_i(t,0)\Big)\Big| \\
&+ \lambda\,|\gamma_i(t,0)| \\
&\leq \lambda c_i\Big| -\frac{\partial}{\partial s}S(t,s_i)\gamma_{i-1}(s_{i-1},y(t_{i-1}^-)) + S(t,s_{i-1})\zeta_{i-1}(s_{i-1},y(t_{i-1}^-)) \\
&+ \int_{s_{i-1}}^{t_i} S(t,s)f\left(s,y(s),\int_0^s g(s,\tau,y(\tau))d\tau\right)ds\Big| \\
&+ \lambda d_i.
\end{aligned}$$

This implies

$$\begin{aligned}
\sup_{s\in[0,t]}|y(s)| &\leq \left(\frac{\tilde{M}c_i d_{i-1}e^{s_{i-1}}}{1-L_{i-1}} + \frac{Mc_i l_{i-1}e^{s_{i-1}}}{1-L_{i-1}}\right)e^{-\delta t} + \frac{d_i}{1-L_{i-1}} \\
&+ \int_{s_{i-1}}^{t_i}\frac{Mc_i}{1-L_{i-1}}e^{-\delta(t-s)}\Theta_f(s)\psi\left(\sup_{\tau\in[0,s]}|y(\tau)| + \int_0^s\Theta_g(\tau)\varphi(\sup_{z\in[0,\tau]}|y(z)|)d\tau\right)ds.
\end{aligned}$$

Then, for all $t \in J$, we have

$$\begin{aligned}
|y(t)| &\leq M_i^* e^{-\delta t} + \frac{d_i}{1-L_{i-1}} \\
&+ e^{-\delta t}\int_0^t \bar{M}e^{\delta s}\Theta_f(s)\psi\left(\sup_{\tau\in[0,s]}|y(\tau)| + \int_0^s\Theta_g(\tau)\varphi(\sup_{z\in[0,\tau]}|y(z)|)d\tau\right)ds.
\end{aligned}$$

where

$$M^* = \max_{2\leq i\leq N}\left\{\tilde{M}\,|y_0| + M\,|y_1|\,, \frac{\tilde{M}d_1e^{s_1}}{1-L_1} + \frac{Ml_1e^{s_1}}{1-L_1}, \frac{\tilde{M}d_ie^{s_i}}{1-L_i} + \frac{Ml_ie^{s_i}}{1-L_i},, \frac{\tilde{M}c_id_{i-1}e^{s_{i-1}}}{1-L_{i-1}} + \frac{Mc_il_{i-1}e^{s_{i-1}}}{1-L_{i-1}}\right\}.$$

Let us take the right-hand side of the above inequality as $\mu(t)$. Then

$$\mu(0) = M^* + \frac{d_i}{1-L_{i-1}},$$

$$\sup_{s\in[0,t]}|y(s)| \leq \mu(t),$$

and

$$\begin{aligned}
\mu'(t) &\leq -\delta\mu(t) + \bar{M}\Theta_f(t)\psi\left(\mu(t) + \int_0^t\Theta_g(s)\varphi(\mu(s))ds\right) \\
&\leq \bar{M}\Theta_f(t)\psi\left(\mu(t) + \int_0^t\Theta_g(s)\varphi(\mu(s))ds\right).
\end{aligned}$$

Let

$$\beta(t) = \mu(t) + \int_0^t\Theta_g(s)\varphi(\mu(s))ds.$$

Then

$$\begin{aligned}
\beta'(t) &= \mu'(t) + \Theta_g(t)\varphi(\mu(t)) \\
&\leq \bar{M}\Theta_f(t)\psi(\beta(t)) + \Theta_g(t)\varphi(\beta(t)).
\end{aligned}$$

This implies that

$$\int_{\beta(0)}^{\beta(t)} \frac{ds}{\psi(s)+\varphi(s)} \leq \int_{m_i}^{a} \max(\bar{M}\Theta_f(s), \Theta_g(s))ds < \int_{m_i}^{+\infty} \frac{ds}{\psi(s)+\varphi(s)}.$$

This above inequality implies that there exists a constant L such that $\beta(t) \leq L, t \in J$, and hence $\mu(t) \leq L, t \in J$. Since for every $t \in J, |y(t)| \leq \mu(t)$, we have $\|y\|_{PC} \leq L$.

Step 2. Λ is continuous.

Suppose that $(y_n)_{n\in\mathbb{N}}$ is a sequence in B_R which converges to y in B_R as $n \to \infty$. By the continuity of nonlinear term γ and ζ with respect to the second argument, for each $s \in J$, we have

$$\sup_{s\in J} |\gamma_i(s, y_n(s)) - \gamma_i(s, y(s))| \to 0 \quad \text{as} \quad n \to \infty, \tag{6}$$

$$\sup_{s\in J} |\zeta(s, y_n(s)) - \zeta(s, y(s))| \to 0 \quad \text{as} \quad n \to \infty. \tag{7}$$

By the Carathéodory character of nonlinear term f, for each $s \in J$, we have

$$\left| f\left(s, y_n(s), \int_0^s g(s,\tau,y_n(\tau))d\tau\right) - f\left(s, y(s), \int_0^s g(s,\tau,y(\tau))d\tau\right)\right| \to 0 \text{ as } n \to \infty. \tag{8}$$

Case 1. For the interval $(s_i, t_i]$, we obtain

$$\begin{aligned}
&|(\Lambda y_n)(t) - (\Lambda y)(t)| \\
&\leq \gamma_i \left(t, -\frac{\partial}{\partial s}S(t,s_i)\gamma_{i-1}(s_{i-1}, y_n(t_{i-1}^-)) + S(t,s_{i-1})\zeta_{i-1}(s_{i-1}, y_n(t_{i-1}^-))\right. \\
&\qquad \left. + \int_{s_{i-1}}^{t_i} S(t,s) f\left(s, y_n(s), \int_0^s g(s,\tau,y_n(\tau))d\tau\right) ds\right) \\
&-\gamma_i \left(t, -\frac{\partial}{\partial s}S(t,s_i)\gamma_{i-1}(s_{i-1}, y(t_{i-1}^-)) + S(t,s_{i-1})\zeta_{i-1}(s_{i-1}, y(t_{i-1}^-))\right. \\
&\qquad \left. + \int_{s_{i-1}}^{t_i} S(t,s) f\left(s, y(s), \int_0^s g(s,\tau,y(\tau))d\tau\right) ds\right).
\end{aligned}$$

Since the function γ_i is continuous and

$$\begin{aligned}
&\Big| -\frac{\partial}{\partial s}S(t,s_i)\gamma_{i-1}(s_{i-1}, y_n(t_{i-1}^-)) + S(t,s_{i-1})\zeta_{i-1}(s_{i-1}, y_n(t_{i-1}^-)) \\
&+ \int_{s_{i-1}}^{t_i} S(t,s) f\left(s, y_n(s), \int_0^s g(s,\tau,y_n(\tau))d\tau\right) ds + \frac{\partial}{\partial s}S(t,s_i)\gamma_{i-1}(s_{i-1}, y(t_{i-1}^-)) \\
&+S(t,s_{i-1})\zeta_{i-1}(s_{i-1}, y(t_{i-1}^-)) + \int_{s_{i-1}}^{t_i} S(t,s) f\left(s, y(s), \int_0^s g(s,\tau,y(\tau))d\tau\right) ds\Big| \\
&\leq \tilde{M}|\gamma_{i-1}(s_{i-1}, y_n(s_{i-1})) - \gamma_{i-1}(s_i, y(s_i))| + M|\zeta_{i-1}(s_{i-1}, y_n(s_{i-1})) - \zeta_{i-1}(s_{i-1}, y(s_{i-1}))| \\
&+M \int_{s_{i-1}}^{t} \left| f\left(s, y_n(s), \int_0^s g(s,\tau,y_n(\tau))d\tau\right) - f\left(s, y(s), \int_0^s g(s,\tau,y(\tau))d\tau\right)\right| ds. \\
&\to 0, \quad \text{as} \quad n \to \infty.
\end{aligned}$$

We can conclude that $\Lambda y_n \to \Lambda y$, as $n \to +\infty$.

Case 2. For the interval $[0, t_1]$, we obtain

$$\begin{aligned}
&|(\Lambda y_n)(t) - (\Lambda y)(t)| \\
&\leq M \int_0^t \left| f\left(s, y_n(s), \int_0^s g(s,\tau,y_n(\tau))d\tau\right) - f\left(s, y(s), \int_0^s g(s,\tau,y(\tau))d\tau\right)\right| ds \\
&\to 0, \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Case 3. For the interval $(s_i, t_{i+1}]$, we have

$$
\begin{aligned}
&|(\Lambda y_n)(t) - (\Lambda y)(t)| \\
&\leq \tilde{M}|\gamma_i(s_i, y_n(s_i)) - \gamma_i(s_i, y(s_i))| + M|\zeta_i(s_i, y_n(s_i)) - \zeta_i(s_i, y(s_i))| \\
&+M\int_{s_i}^{t}\left|f\left(s, y_n(s), \int_0^s g(s,\tau,y_n(\tau))d\tau\right) - f\left(s, y(s), \int_0^s g(s,\tau,y(\tau))d\tau\right)\right| ds \\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

As a consequence of Case 1–3, $\Lambda y_n \to \Lambda y$, as $n \to +\infty$. Hence the Λ is continuous.

Step 3. Λ is equicontinuous.

Case 1. For the interval $[0, t_1]$, $0 \leq \tilde{t}_1 \leq \tilde{t}_2 \leq t_1$, any $y \in B_R$, we have

$$
\begin{aligned}
&|(\Lambda y)(\tilde{t}_2) - (\Lambda y)(\tilde{t}_1)| \\
&\leq \|\frac{\partial}{\partial s}S(\tilde{t}_2, 0) - \frac{\partial}{\partial s}S(\tilde{t}_1, 0)\|_{B(E)}|y_0| \\
&+\|S(\tilde{t}_2, 0) - S(\tilde{t}_1, 0)\|_{B(E)}|y_1| \\
&+\left|\int_0^{\tilde{t}_1}(S(\tilde{t}_2, s) - S(\tilde{t}_1, s))f\left(s, y(s), \int_0^s g(s,\tau,y(\tau))d\tau\right)ds\right. \\
&\left.+\int_{\tilde{t}_1}^{\tilde{t}_2} S(\tilde{t}_2,\tau)f\left(s, y(s), \int_0^s g(s,\tau,y(\tau))d\tau\right)ds\right| \\
&\leq \int_0^{\tilde{t}_1}\|S(\tilde{t}_2,\tau) - S(\tilde{t}_1,\tau)\|_{B(E)}\,\Theta_f(\tau)\psi\left(|y(s)| + \int_0^s \Theta_g(\tau)\varphi(|y(\tau)|)d\tau\right)ds \\
&+M\int_{\tilde{t}_1}^{\tilde{t}_2}\Theta_f(s)\psi\left(|y(s)| + \int_0^s \Theta_g(\tau)\varphi(|y(\tau)|)d\tau\right)ds.
\end{aligned}
$$

It follows from the Hölder's inequality that

$$
\begin{aligned}
&|(\Lambda y)(\tilde{t}_2) - (\Lambda y)(\tilde{t}_1)| \\
&\leq \|\frac{\partial}{\partial s}S(\tilde{t}_2, 0) - \frac{\partial}{\partial s}S(\tilde{t}_1, 0)\|_{B(E)}|y_0| \\
&+\|S(\tilde{t}_2, 0) - S(\tilde{t}_1, 0)\|_{B(E)}|y_1| \\
&+\psi\left(R + \varphi(R)\|\Theta_g\|_{L^1}\right)\int_0^{\tilde{t}_1}\|S(\tilde{t}_2,\tau) - S(\tilde{t}_1,\tau)\|_{B(E)}\,\Theta_f(\tau)d\tau \\
&+\frac{M\|\Theta_f\|_{L^r}\psi\left(R + \varphi(R)\|\Theta_g\|_{L^1}\right)}{\delta^{1-\frac{1}{r}}}\left(e^{-\frac{r\delta}{r-1}(t-\tilde{t}_2)} - e^{-\frac{r\delta}{r-1}(t-\tilde{t}_1)}\right)^{1-\frac{1}{r}}.
\end{aligned}
$$

Case 2. For the interval $(s_i, t_{i+1}]$, $s_i \leq \tilde{t}_1 \leq \tilde{t}_2 \leq t_{i+1}$, any $y \in B_R$, then we get

$$
\begin{aligned}
|(\Lambda y)(\tilde{t}_2) - (\Lambda y)(\tilde{t}_1)| &\leq \|\frac{\partial}{\partial s}S(\tilde{t}_2, s_i) - \frac{\partial}{\partial s}S(\tilde{t}_1, s_i)\|_{B(E)}|\gamma_i(s_i, y(s_i))| \\
&+ \|S(\tilde{t}_2, s_i) - S(\tilde{t}_1, s_i)\|_{B(E)}|\zeta_i(s_i, y(s_i))| \\
&+ \left|\int_{s_i}^{\tilde{t}_1}(S(\tilde{t}_2, s) - S(\tilde{t}_1, s))f\left(s, y(s), \int_0^s g(s,\tau,y(\tau))d\tau\right)ds\right. \\
&+ \left.\int_{\tilde{t}_1}^{\tilde{t}_2} S(\tilde{t}_2,\tau)f\left(s, y(s), \int_0^s g(s,\tau,y(\tau))d\tau\right)ds\right| \\
&\leq \int_{s_i}^{\tilde{t}_1}\|S(\tilde{t}_2,\tau) - S(\tilde{t}_1,\tau)\|_{B(E)}\,\Theta_f(\tau)\psi\left(|y(s)| + \int_0^s \Theta_g(\tau)\varphi(|y(\tau)|)d\tau\right)ds \\
&+ M\int_{\tilde{t}_1}^{\tilde{t}_2}\Theta_f(s)\psi\left(|y(s)| + \int_0^s \Theta_g(\tau)\varphi(|y(\tau)|)d\tau\right)ds.
\end{aligned}
$$

It follows from the Hölder's inequality that

$$
\begin{aligned}
|(\Lambda y)(\tilde{t}_2)-(\Lambda y)(\tilde{t}_1)| \leq\ & \|\frac{\partial}{\partial s}S(\tilde{t}_2,s_i)-\frac{\partial}{\partial s}S(\tilde{t}_1,s_i)\|_{B(E)}|\gamma_i(s_i,y(s_i))| \\
+\ & \|S(\tilde{t}_2,s_i)-S(\tilde{t}_1,s_i)\|_{B(E)}|\zeta_i(s_i,y(s_i))| \\
+\ & \psi\left(R+\varphi(R)\|\Theta_g\|_{L^1}\right)\int_{s_i}^{\tilde{t}_1}\|S(\tilde{t}_2,\tau)-S(\tilde{t}_1,\tau)\|_{B(E)}\,p(\tau)d\tau \\
+\ & \frac{M\|\Theta_f\|_{L^r}\psi\left(R+\varphi(R)\|\Theta_g\|_{L^1}\right)}{\delta^{1-\frac{1}{r}}}\left(e^{-\frac{r\delta}{r-1}(t-\tilde{t}_2)}-e^{-\frac{r\delta}{r-1}(t-\tilde{t}_1)}\right)^{1-\frac{1}{r}}.
\end{aligned}
$$

Case 3. For the interval $(s_i,t_i]$, $s_i\leq\tilde{t}_1\leq\tilde{t}_2\leq t_i$, any $y\in B_R$, we have

$$
\begin{aligned}
&|(\Lambda y)(\tilde{t}_2)-(\Lambda y)(\tilde{t}_1)| \\
&=\Big|\gamma_i\Big(\tilde{t}_2,-\frac{\partial}{\partial s}S(\tilde{t}_2,s_i)\gamma_{i-1}(s_{i-1},y(t_{i-1}^-))+S(\tilde{t}_2,s_{i-1})\zeta_{i-1}(s_{i-1},y(t_{i-1}^-)) \\
&+\int_{s_{i-1}}^{\tilde{t}_2}S(t,s)f\left(s,y(s),\int_0^s g(s,\tau,y(\tau))d\tau\right)ds\Big) \\
&-\gamma_i\Big(\tilde{t}_1,-\frac{\partial}{\partial s}S(\tilde{t}_1,s_i)\gamma_{i-1}(s_{i-1},y(t_{i-1}^-))+S(\tilde{t}_1,s_{i-1})\zeta_{i-1}(s_{i-1},y(t_{i-1}^-)) \\
&+\int_{s_{i-1}}^{\tilde{t}_1}S(\tilde{t}_1,s)f\left(s,y(s),\int_0^s g(s,\tau,y(\tau))d\tau\right)ds\Big)\Big|.
\end{aligned}
$$

then

$$
\begin{aligned}
&|(\Lambda y)(\tilde{t}_2)-(\Lambda y)(\tilde{t}_1)| \\
&\leq c_i\Big|-\frac{\partial}{\partial s}S(\tilde{t}_2,s_i)\gamma_{i-1}(s_{i-1},y(t_{i-1}^-))+S(\tilde{t}_2,s_{i-1})\zeta_{i-1}(s_{i-1},y(t_{i-1}^-)) \\
&+\int_{s_{i-1}}^{\tilde{t}_2}S(t,s)f\left(s,y(s),\int_0^s g(s,\tau,y(\tau))d\tau\right)ds \\
&+\frac{\partial}{\partial s}S(\tilde{t}_1,s_i)\gamma_{i-1}(s_{i-1},y(t_{i-1}^-))-S(\tilde{t}_1,s_{i-1})\zeta_{i-1}(s_{i-1},y(t_{i-1}^-)) \\
&-\int_{s_{i-1}}^{\tilde{t}_1}S(\tilde{t}_1,s)f\left(s,y(s),\int_0^s g(s,\tau,y(\tau))d\tau\right)ds\Big|.
\end{aligned}
$$

Similarly, one can easily see that

$$
\begin{aligned}
|(\Lambda y)(\tilde{t}_2)-(\Lambda y)(\tilde{t}_1)| \leq\ & c_i\|\frac{\partial}{\partial s}S(\tilde{t}_2,s_{i-1})-\frac{\partial}{\partial s}S(\tilde{t}_1,s_{i-1})\|_{B(E)}|\gamma_{i-1}(s_{i-1},y(t_{i-1}^-))| \\
+\ & c_i\|S(\tilde{t}_2,s_{i-1})-S(\tilde{t}_1,s_{i-1})\|_{B(E)}|\zeta_{i-1}(s_{i-1},y(t_{i-1}^-))| \\
+\ & c_i\psi\left(R+\varphi(R)\|\Theta_g\|_{L^1}\right)\int_{s_{i-1}}^{\tilde{t}_1}\|S(\tilde{t}_2,\tau)-S(\tilde{t}_1,\tau)\|_{B(E)}\,\Theta_f(\tau)d\tau \\
+\ & \frac{Mc_i\|\Theta_f\|_{L^r}\psi\left(R+\varphi(R)\|\Theta_g\|_{L^1}\right)}{\delta^{1-\frac{1}{r}}}\left(e^{-\frac{r\delta}{r-1}(t-\tilde{t}_2)}-e^{-\frac{r\delta}{r-1}(t-\tilde{t}_1)}\right)^{1-\frac{1}{r}}.
\end{aligned}
$$

In view of Case 1–3, as a result, $\|(\Lambda y)(\tilde{t}_2)-(\Lambda y)(\tilde{t}_1)\|\to 0$ as $\tilde{t}_2\to\tilde{t}_1$, which meansthat Λ is equicontinuous.

Step 4. Λ is a α_{PC}-contraction operator.

For every bounded subset $B\subset PC(J,E)$, then we know that there exists a countable set $B_1=\{y\}_{n=1}^{\infty}\subset B$ (see Lemma 2), such that for any $t\in J$, we have

$$\alpha_E(\Lambda(B)(t))\leq 2\alpha_E(\Lambda(B_1)(t)). \tag{9}$$

Note that B and ΛB are equicontinuous, we can get from Lemma 2, Lemma 3, Lemma 4 and using the assumptions (H_1)–(H_6), we obtain
Case 1. For the interval $(t_i, s_i]$, we have

$$\begin{aligned}
\alpha_E(\Lambda B_1(t)) \quad &\leq \quad \tilde{M}k_i \left\{\alpha_E\left(\gamma_{i-1}(s_{i-1}, y_n(t_{i-1}^-))\right)\right\}_{n=0}^{\infty} \\
&+ \quad Mk_i \left\{\alpha_E\left(\zeta_{i-1}(s_{i-1}, y_n(t_{i-1}^-))\right)\right\}_{n=0}^{\infty} \\
&+ \quad k_i\alpha_E\left(\left\{\int_{s_{i-1}}^{t} S(t,s)f(s, y_n(s), \int_0^s g(s,\tau, y_n(s))d\tau)ds\right\}_{n=0}^{\infty}\right) \\
&\leq \quad \tilde{M}k_ik_{i-1}\left\{\alpha_E(y_n(t_i^-))\right\}_{n=0}^{\infty}) + Mk_i\bar{k}_{i-1}\left\{\alpha_E(y_n(t_{i-1}^-))\right\}_{n=0}^{\infty}) \\
&+ \quad 2Mk_i\int_{s_{i-1}}^{t} \left\{\alpha_E\left(f(s, y_n(s), \int_0^s g(s,\tau,y_n(\tau))d\tau)ds)\right)\right\}_{n=0}^{\infty} ds \\
&\leq \quad \tilde{M}k_ik_{i-1}\left\{\alpha_E(y_n(t_i^-))\right\}_{n=0}^{\infty}) + Mk_i\bar{k}_{i-1}\left\{\alpha_E(y_n(t_{i-1}^-))\right\}_{n=0}^{\infty}) \\
&+ \quad 2Mk_i\int_{s_{i-1}}^{t} \sigma_1(s)\left\{\alpha_E(y_n(s))\right\}_{n=0}^{\infty} \\
&+ \quad \varrho_i(s)\left\{\alpha_E\left(\int_0^s g(s,\tau,y_n(\tau))d\tau\right)\right\}_{n=0}^{\infty}\Bigg) ds \\
&\leq \quad \tilde{M}k_ik_{i-1}\left\{\alpha_E(y_n(t_i^-))\right\}_{n=0}^{\infty}) + Mk_i\bar{k}_{i-1}\left\{\alpha(y_n(t_{i-1}^-))\right\}_{n=0}^{\infty}) \\
&+ \quad 2Mk_i\int_{s_{i-1}}^{t} \sigma_i(s)\left\{\alpha_E(y_n(s))\right\}_{n=0}^{\infty}) \\
&+ \quad 2K^*\varrho_i(s)\left\{\int_0^s \alpha_E(y_n(\tau))d\tau\right\}_{n=0}^{\infty}\Bigg) ds \\
&\leq \quad (\tilde{M}k_ik_{i-1} + Mk_i\bar{k}_{i-1})\alpha_E(B_1(t_i^-)) \\
&+ \quad 2Mk_i\int_{s_{i-1}}^{t}\left(\sigma_i(s)\alpha_E(B_1(s)) + 2K^*\varrho_i(s)\int_0^s \alpha_E(B_1(\tau)d\tau\right) ds. \\
&\leq \quad (\tilde{M}k_ik_{i-1} + k_i\bar{k}_{i-1})\sup\nolimits_{s\in(t_i,si]}\alpha_E(B(t)) \\
&+ \quad 2Mk_i\int_{s_{i-1}}^{t}\left(\sigma(s)\alpha_E(B_1(s)) + 2K^*\varrho(s)s\sup_{\tau\in[0,s]}\alpha_E(B_1(\tau)\right) ds. \\
&\leq \quad (\tilde{M}k_ik_{i-1} + Mk_i\bar{k}_{i-1})\sup\nolimits_{s\in(s_i,t_{i+1}]}\alpha_E(B(t)) \\
&+ \quad 2M\int_{s_{i-1}}^{t}\left(\sigma(s)\sup_{s\in(s_i,t_{i+1}]}\alpha_E(B_1(s)) + 2K^*\varrho(s)s\sup_{\tau\in(t_i,s_i]}\alpha_E(B_1(\tau)\right) ds. \\
&\leq \quad (\tilde{M}k_ik_{i-1} + Mk_i\bar{k}_{i-1})\sup\nolimits_{s\in(t_i,s_i]}\alpha_E(B(t)) \\
&+ \quad 2Mk_i\int_{s_{i-1}}^{t}(\sigma(s) + 2K^*s\varrho(s))\sup_{s\in(t_i,s_i]}\alpha_E(B_1(s))ds \\
&\leq \quad k_i(\tilde{M}k_{i-1} + M\bar{k}_{i-1} + 2M(\|\sigma\|_{L^1} + 2K^*s_i\|\varrho\|_{L^1}))\sup\nolimits_{t\in(t_i,s_i]}\alpha_E(B(t)) \\
&\leq \quad k_i(\tilde{M}k_{i-1} + M\bar{k}_{i-1} + 2M(\|\sigma\|_{L^1} + 2K^*a\|\varrho\|_{L^1})\sup\nolimits_{t\in(t_i,s_i]}\alpha_E(B(t)).
\end{aligned}$$

Then

$$\alpha_E(N(B(t))) \leq k_i\left(\tilde{M}k_{i-1} + M\bar{k}_{i-1} + 2M(\|\sigma\|_{L^1} + 2K^*a\|\varrho\|_{L^1}\right)\alpha_{PC}(B(t)). \tag{10}$$

Case 2. For the interval $[0, t_1]$, we have

$$\alpha_E(\Lambda B_1(t)) \quad \leq \quad \alpha_E\left(\left\{\int_0^t S(t,s)f(s, y_n(s), \int_0^s g(s,\tau,y_n(s))d\tau)ds\right\}_{n=0}^{\infty}\right)$$

$$
\begin{aligned}
&\leq 2M\int_0^t \left\{\alpha_E\left(f(s,y_n(s),\int_0^s g(s,\tau,y_n(\tau))d\tau)ds)\right)\right\}_{n=0}^{\infty} ds \\
&\leq 2M\int_0^t \sigma_1(s)\left\{\alpha_E(y_n(s))\right\}_{n=0}^{\infty} ds \\
&\quad + \varrho_i(s)\left\{\alpha_E\left(\int_0^s g(s,\tau,y_n(\tau))d\tau\right)\right\}_{n=0}^{\infty}\Bigg) ds \\
&\leq 2M\int_0^t \sigma_i(s)\left\{\alpha_E(y_n(s))\right\}_{n=0}^{\infty}) \\
&\quad + 2K^*\varrho_i(s)\left\{\int_0^s \alpha_E(y_n(\tau))d\tau\right\}_{n=0}^{\infty}\Bigg) ds \\
&\leq 2M\int_0^t \left(\sigma_i(s)\alpha_E(B_1(s)) + 2K^*\varrho_i(s)\int_0^s \alpha_E(B_1(\tau)d\tau\right) ds \\
&\leq 2M\int_0^t \left(\sigma(s)\alpha_E(B_1(s)) + 2K^*\varrho(s)s\sup_{\tau\in[0,s]}\alpha_E(B_1(\tau)\right) ds \\
&\leq 2M\int_0^t \left(\sigma(s)\sup_{s\in[0;t_1]}\alpha_E(B_1(s)) + 2K^*\varrho(s)s\sup_{\tau\in[0;t_1]}\alpha_E(B_1(\tau)\right) ds \\
&\leq 2M\int_0^t (\sigma(s)+2K^*s\varrho(s))\sup_{s\in[0;t_1]}\alpha_E(B_1(s))ds. \\
&\leq 2M(\|\sigma\|_{L^1} + 2K^*t_1\|\varrho\|_{L^1})\sup_{t\in[0;t_1]}\alpha_E(B(t)) \\
&\leq 2M(\|\sigma\|_{L^1} + 2K^*a\|\varrho\|_{L^1})\sup_{t\in[0;t_1]}\alpha_E(B(t)).
\end{aligned}
$$

Then

$$\alpha_E(\Lambda(B(t))) \leq 2M(\|\sigma\|_{L^1} + 2K^*a\|\varrho\|_{L^1}))\alpha_{PC}(B(t)). \tag{11}$$

Case 3. For the interval $(s_i, t_{i+1}]$, we have

$$
\begin{aligned}
\alpha_E(\Lambda B_1(t)) &\leq \tilde{M}\left\{\alpha_E\left(\gamma_i(s,y_n(t_i^-))\right)\right\}_{n=0}^{\infty} + M\left\{\alpha_E\left(\zeta_i(s,y_n(t_i^-))\right)\right\}_{n=0}^{\infty} \\
&+ \alpha_E\left(\left\{\int_{s_i}^t S(t,s)f(s,y_n(s),\int_0^s g(s,\tau,y_n(s))d\tau)ds\right\}_{n=0}^{\infty}\right) \\
&\leq \tilde{M}k_i\left\{\alpha_E(y_n(t_i^-))\right\}_{n=0}^{\infty}) + M\bar{k}_i\left\{\alpha_E(y_n(t_i^-))\right\}_{n=0}^{\infty}) \\
&+ 2M\int_{s_i}^t \left\{\alpha_E\left(f(s,y_n(s),\int_0^s g(s,\tau,y_n(\tau))d\tau)ds)\right)\right\}_{n=0}^{\infty} ds \\
&\leq \tilde{M}k_i\left\{\alpha_E(y_n(t_i^-))\right\}_{n=0}^{\infty}) + M\bar{k}_i\left\{\alpha_E(y_n(t_i^-))\right\}_{n=0}^{\infty}) \\
&+ 2M\int_{s_i}^t \sigma_1(s)\left\{\alpha_E(y_n(s))\right\}_{n=0}^{\infty} ds \\
&+ \varrho_i(s)\left\{\alpha_E\left(\int_0^s g(s,\tau,y_n(\tau))d\tau\right)\right\}_{n=0}^{\infty}\Bigg) ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad \tilde{M}k_i \left\{\alpha_E(y_n(t_i^-))\right\}_{n=0}^{\infty}) + M\bar{k}_i \left\{\alpha_E(y_n(t_i^-))\right\}_{n=0}^{\infty}) \\
&+ \quad 2M \int_{s_i}^{t} \sigma_i(s) \left\{\alpha_E(y_n(s))\right\}_{n=0}^{\infty}) \\
&+ \quad 2K^* \varrho_i(s) \left\{ \int_0^s \alpha_E(y_n(\tau)) d\tau \right\}_{n=0}^{\infty} \Bigg) ds \\
&\leq \quad (\tilde{M}k_i + M\bar{k}_i)\alpha_E(B(t_i^-)) \\
&\quad\quad +2M \int_{s_i}^{t} \left(\sigma(s)\alpha_E(B_1(s)) + 2K^* \varrho(s) \int_0^s \alpha_E(B_1(\tau) d\tau \right) ds \\
&\leq \quad (\tilde{M}k_i + M\bar{k}_i) \sup_{s\in(s_i,t_{i+1}]} \alpha_E(B_1(s)) \\
&+ \quad 2M \int_{s_i}^{t} \left(\sigma(s)\alpha_E(B_1(s)) + 2K^* \varrho(s) s \sup_{\tau\in[0,s]} \alpha_E(B_1(\tau) \right) ds \\
&\leq \quad (\tilde{M}k_i + M\bar{k}_i) \sup_{s\in(s_i,t_{i+1}]} \alpha_E(B(s)) \\
&+ \quad 2M \int_{s_i}^{t} \left(\sigma(s) \sup_{s\in(s_i,t_{i+1}]} \alpha_E(B_1(s)) + 2K^* \varrho(s) s \sup_{\tau\in(s_i,t_{i+1}]} \alpha_E(B_1(\tau) \right) ds \\
&\leq \quad (\tilde{M}k_i + M\bar{k}_i)\alpha_E(B_1(t_i^-)) \\
&+ \quad 2M \int_{s_i}^{t} (\sigma(s) + 2K^* s \varrho(s)) \sup_{s\in(s_i,t_{i+1}]} \alpha_E(B_1(s)) ds. \\
&\leq \quad \left(\tilde{M}k_i + M\bar{k}_i + 2M(\|\sigma\|_{L^1} + 2K^* t_{i+1} \|\varrho\|_{L^1})\right) \sup_{t\in(s_i,t_{i+1}]} \alpha_E(B(t)) \\
&\leq \quad \left(\tilde{M}k_i + M\bar{k}_i + 2M(\|\sigma\|_{L^1} + 2K^* a \|\varrho\|_{L^1})\right) \sup_{t\in(s_i,t_{i+1}]} \alpha_E(B(t)).
\end{aligned}
$$

Then

$$\alpha_E(\Lambda(B(t))) \leq \left(\tilde{M}k_i + M\bar{k}_i + 2M(\|\sigma\|_{L^1} + 2K^* a \|\varrho\|_{L^1})\right) \alpha_{PC}(B(t)). \tag{12}$$

From the above cases (10)–(12), for all $t \in J$, we obtain

$$\alpha_{PC}(\Lambda(B)) \leq \max_{1\leq i\leq N}(k_i, 1) \left(\max_{1\leq i\leq N}(\tilde{M}k_i + M\bar{k}_i) + 2M(\|\sigma\|_{L^1} + 2K^* a \|\varrho\|_{L^1}) \right) \alpha_{PC}(B).$$

Thus, we find that Λ is α_{PC}-contraction operator. Applying now theorem 1, we conclude that Λ has a fixed point which is an solution of the system (1). □

Next, we present another existence result for the mild solution of the system (1).

Theorem 4. *Assume that hypotheses* (H_1)–(H_6) *are fulfilled and*

$$\lim_{R\to+\infty} \inf \frac{\psi(R + \|\Theta_g\|_{L^1} \varphi(R)) \|\Theta_f\|_{L^r}}{R} = \rho < \infty,$$

and

$$\tilde{M}c_i + Me_i + \frac{M\rho \|\Theta_f\|_{L^r}}{\delta^{1-\frac{1}{r}}} \leq 1, i = 1, \cdots, N. \tag{13}$$

Then, there exists a mild solution of system (1).

Proof. Following the proof of Theorem 3 we conclude that the map $\Lambda : B_R \to B_R$ given by Equation (5) is continuous. Next, we show that there exists $R > 0$ such that $\Lambda(B_R) \subset B_R$. In fact, if it is not true, then for each positive number R , there exists a function $\check{y} \in B_R$ and $\check{t} \in J$ such that $R \leq \left|(\Lambda y)(\check{t})\right|$. Therefore for

Case 1. For $\check{t} \in (s_i, t_i]$, and $\check{y} \in B_R$, we have,

$$\begin{aligned}
\left|(\Lambda\check{y})(\check{t})\right| &\leq \left\|\frac{\partial}{\partial s}S(\check{t}, s_{i-1})\right\|_{B(E)} |\gamma_i(s_{i-1}, \check{y}(s_{i-1}))| \\
&+ \|S(\check{t}, s_{i-1})\|_{B(E)} |\zeta_i(s_{i-1}, \check{y}(s_{i-1}))| \\
&+ \int_{s_{i-1}}^{\check{t}} \|S(t,s)\|_{B(E)} \Theta_f(s)\psi\left(|\check{y}(s)| + \int_0^s \Theta_g(\tau)\varphi(|\check{y}(\tau)|))d\tau\right) ds \\
&\leq \tilde{M}c_{i-1}|\check{y}(s_{i-1})| + \tilde{M}d_{i-1} \\
&+ Me_{i-1}|\check{y}(s_{i-1})| + Ml_{i-1} \\
&+ \int_{s_{i-1}}^{\check{t}} Me^{-\delta(t-s)}\Theta_f(s)\psi\left(|\check{y}(s)| + \int_0^s \Theta_g(\tau)\varphi(|\check{y}(\tau)|)d\tau\right) ds.
\end{aligned}$$

Then

$$\begin{aligned}
\left|(\Lambda y)(\check{t})\right| &\leq \tilde{M}c_{i-1}R + \tilde{M}d_{i-1} \\
&+ Me_{i-1}R + Ml_{i-1} \\
&+ \int_{s_{i-1}}^{\check{t}} Me^{-\delta(t-s)}\Theta_f(s)\psi\left(|\check{y}(s)| + \int_0^s \Theta_g(\tau)\varphi(|\check{y}(\tau)|)d\tau\right) ds. \\
&\leq (\tilde{M}c_{i-1} + Me_{i-1})R + \tilde{M}l_{i-1} + Ml_{i-1} \\
&+ M\psi(R + \|\Theta_g\|_{L^1}\varphi(R)) \int_{s_{i-1}}^{\check{t}} e^{-\delta(t-s)}\Theta_f(s)ds.
\end{aligned}$$

It follows from the Hölder's inequality that

$$\begin{aligned}
\left|(\Lambda y)(\check{t})\right| &\leq (\tilde{M}c_{i-1} + Me_{i-1})R + \tilde{M}d_{i-1} + Ml_{i-1} \\
&+ \frac{M\psi(R + \|\Theta_g\|_{L^1}\varphi(R))\|\Theta_f\|_{L^r}}{\delta^{1-\frac{1}{r}}}.
\end{aligned}$$

Case 2. For $\check{t} \in [0; t_1]$, and $\check{y} \in B_R$, we get,

$$\begin{aligned}
\left|(\Lambda y)(\check{t})\right| &\leq \left\|\frac{\partial}{\partial s}S(t, 0)\right\|_{B(E)} |y_0| \\
&+ \|S(t,s)\|_{B(E)} |y_1| \\
&+ \int_0^{\check{t}} \|S(t,s)\|_{B(E)} \Theta_f(s)\psi\left(|\check{y}(s)| + \int_0^s \Theta_g(\tau)\varphi(|\check{y}(\tau)|)d\tau\right) ds \\
&\leq \tilde{M}|y_1| + M|y_0| + M\psi(R + \|\Theta_g\|_{L^1}\varphi(R)) \int_0^{\check{t}} e^{-\delta(t-s)}\Theta_f(s)ds.
\end{aligned}$$

It follows from the Hölder's inequality that

$$\begin{aligned}
\left|(\Lambda y)(\check{t})\right| &\leq \tilde{M}|y_0| + M|y_1| + \frac{M\psi(R + \|\Theta_g\|_{L^1}\varphi(R))\|\Theta_f\|_{L^r}}{\delta^{1-\frac{1}{r}}}(1 - e^{-\frac{r\delta}{r-1}\check{t}})^{1-\frac{1}{r}} \\
&\leq \tilde{M}|y_0| + M|y_1| + \frac{M\psi(R + \|\Theta_g\|_{L^1}\varphi(R))\|\Theta_f\|_{L^r}}{\delta^{1-\frac{1}{r}}}.
\end{aligned}$$

Case 3. For $\check{t} \in (s_i, t_{i+1}]$, and $\check{y} \in B_R$, we have,

$$\begin{aligned} \left|(\Lambda y)(\check{t})\right| &\leq \tilde{M}c_i \left|y(\check{t})\right| + \tilde{M}d_i + Me_i \left|y(\check{t})\right| + Ml_i \\ &+ \int_{s_i}^{t} \left\|S(\check{t},s)\right\|_{B(E)} \Theta_f(s)\psi\left(|\check{y}(s)| + \int_0^s \Theta_g(\tau)\varphi(|\check{y}(\tau)|)d\tau\right) ds. \\ &\leq \tilde{M}c_i R + \tilde{M}d_i + Me_i R + Ml_i \\ &+ M\psi(R + \|\Theta_g\|_{L^1}\varphi(R)) \int_{s_i}^{\check{t}} e^{-\delta(\check{t}-s)}\Theta_f(s)ds. \end{aligned}$$

It follows from the Hölder's inequality that

$$\begin{aligned} |(\Lambda y)(t)| &\leq \tilde{M}d_i + Ml_i \\ &+ (\tilde{M}c_i + Me_i)R \\ &+ \frac{M\psi(R + \|\Theta_g\|_{L^1}\varphi(R))\|\Theta_f\|_{L^r}}{\delta^{1-\frac{1}{r}}}. \end{aligned}$$

Therefore for all $\check{t} \in J$, we have

$$\begin{aligned} R < \left|(\Lambda y)(\check{t})\right| &\leq (\tilde{M}c_i + Me_i)R \\ &+ \max(\tilde{M}d_i + Ml_i, \tilde{M}|y_0| + M|y_1|) \\ &+ \frac{M\psi(R + \|\Theta_g\|_{L^1}\varphi(R))\|\Theta_f\|_{L^r}}{\delta^{1-\frac{1}{r}}}. \end{aligned}$$

Dividing both sides by R and taking the $\liminf$ as $R \to +\infty$, we have

$$\tilde{M}c_i + Me_i + \frac{M\rho\|\Theta_f\|_{L^r}}{\delta^{1-\frac{1}{r}}} > 1, i = 0, \cdots, N.$$

which contradicts (13). Hence, the operator Λ transforms the set B_R into itself.

The proof of $\Lambda : B_R \to B_R$ is α_E-contraction is similar to those in Theorem 3. Therefore, we omit the details. By the Darbo-Sadovskii fixed point theorem 2 we deduce that Λ has a fixed point which is a mild solution of system (1). $\square$

4. An Example

In this section, we give an example to illustrate the above theoretical result.

Set $E = L^2([0,\pi],\mathbb{R})$ be the space of all square integrable functions from $[0,\pi]$ into $\mathbb{R}$. We denote by $\mathbb{H}^2([0,\pi],\mathbb{R})$ the Sobolev space of functions $u : [0,\pi] \to \mathbb{R}$, such that $u'' \in L^2([0,\pi],\mathbb{R})$. Define the operator $\mathbb{A} :\ D(\mathbb{A}) \to E$ by

$$\mathbb{A}u(\tau) = u''(\tau),$$

with domain

$$D(\mathbb{A}) = \left\{\omega \in E :\ \omega, \omega' \text{ are absolutely continuous, } \omega'' \in E,\ \omega(0) = \omega(\pi) = 0\right\}.$$

It is well known that $\mathbb{A}$ is the infinitesimal generator of a C_0-semigroup and of a strongly continuous cosine function on E, which will be denoted by $(C(t))$. From [14], for all $x \in \mathbb{H}^2([0,\pi],\mathbb{R}), t \in \mathbb{R}$, $\|C(t)\|_{B(E)} \leq 1$. Define also the operator $\mathbb{B} : \mathbb{H}^1([0,\pi],\mathbb{R}) \to E$ by

$$\mathbb{B}(t)u(s) = a(t)u'(s),$$

where $a : [0,1] \to \mathbb{R}$ is a Hölder continuous function.

Consider the closed linear operator $\mathcal{A}(t) = \mathbb{B}(t) + \mathbb{A}$. It has been proved by Henríquez in [44] that the family $\{\mathcal{A}(t) : t \in J\}$ generates an evolution operator $\{S(t,s)\}_{(t,s)\in D}$. Moreover, $S(\cdot,\cdot)$ is well defined and satisfies the conditions $(H1)$ and $(H2)$, with $M = \tilde{M} = 1$ and $\delta = 1$.

We consider the following system:

$$\begin{cases} \frac{\partial^2}{\partial t^2}u(t,\tau) & = \frac{\partial^2}{\partial \tau^2}u(t,\tau) + a(t)\frac{\partial}{\partial t}u(t,\tau) \\ & + \dfrac{u(t,\tau)}{12(\sqrt{t}+1)(1+|u(t,\tau)|)} \\ & + \dfrac{e^{-t}}{(\sqrt{t}+1)(t+1)}\displaystyle\int_0^t \frac{\sqrt{t}u(s,\tau)}{8(1+s^2+t)(1+u^2(s,\tau))}ds, & t \in \left(0, \frac{1}{\sqrt{3}}\right] \cup \left(\frac{2}{\sqrt{3}}, 1\right], \\ u(t,\tau) & = \frac{1}{12}\cos \pi t\, u\left(\frac{1}{\sqrt{3}}^-, \tau\right), & t \in \left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right], \tau \in [0,\pi], \\ \frac{\partial}{\partial t}u(t,\tau) & = \dfrac{1}{12}\sin \pi t\, u\left(\dfrac{1}{\sqrt{3}}^-, \tau\right), & t \in \left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right], \tau \in [0,\pi], \\ u(t,0) = & u(t,\pi) = 0, & t \in [0,1], \\ u(0,\tau) & = y_0, & \tau \in [0,\pi], \\ \frac{\partial}{\partial t}u(0,\tau) & = y_1, & \tau \in [0,\pi]. \end{cases} \tag{14}$$

Take $a = t_2 = 1$, $t_0 = s_0 = 0$, $t_1 = \frac{1}{\sqrt{3}}$, $s_1 = \frac{2}{\sqrt{3}}$. The system (14) can be written in the abstract form:

$$\begin{cases} y''(t) = A(t)y(t) + f\left(t, y(t), \displaystyle\int_0^t g(t,s,y(s))ds\right), \ t \in (s_i, t_{i+1}], i = 1,2 \\ y(t) = \gamma_i(t, y(t_i^-)), \quad t \in (t_i, s_i], \quad i = 1, \\ y'(t) = \zeta_i(t, y(t_i^-)), \quad t \in (t_i, s_i], \quad i = 1, \\ y(0) = y_0,\ y'(0) = y_1, \end{cases} \tag{15}$$

where $y(t) = u(t,\cdot)$, that is $y(t)(\tau) = u(t,\tau),\ \tau \in [0,\pi]$.

The function $f : J \times E \times E \to E$, is given by

$$f(t,y,z)(\tau) = \frac{|y(t)(\tau)|}{12(\sqrt{t}+1)(1+|y(t)(\tau)|)} + \frac{e^{-t}}{(\sqrt{t}+1)(t+1)}z(t)(\tau),$$

The function $g : D \times E \to E$, is given by

$$g(t,s,y)(\tau) = \frac{\sqrt{t}y(t)(\tau)}{8(1+s^2+t)(1+y^2(t)(\tau))},$$

Functions

$$\gamma_1(t, y(t_1^-))(\tau) = \frac{1}{12}\cos \pi t\, y\left(\frac{1}{\sqrt{3}}^-\right)(\tau), \tag{16}$$

and

$$\zeta_1(t, y(t_1^-))(\tau) = \frac{1}{12}\sin \pi t\, y\left(\frac{1}{\sqrt{3}}^-\right)(\tau), \tag{17}$$

represent noninstantaneous impulses during interval $\left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right]$. We have

$$|f(t,y,z)(\tau)| \le \frac{1}{1+\sqrt{t}}\psi(|y(t)(\tau)| + |z(t)(\tau)|), \tag{18}$$

and

$$|g(t,s,y)(\tau)| \le \frac{\sqrt{t}}{8+8t}|y(t)(\tau)|. \tag{19}$$

From the above discussion, we obtain

$$\psi(t) = t, \varphi(t) = t, \ \Theta_f(t) = \frac{1}{1+\sqrt{t}}, \quad \Theta_g(t) = \frac{\sqrt{t}}{8+8t}.$$

For each $t \in J$, and $W_1, W_2 \subset E$, we get

$$\alpha_E(f(t, W_1, W_2)) \leq \frac{1}{12(\sqrt{t}+1)}\alpha_E(W_1) + \frac{e^{-t}}{(\sqrt{t}+1)(t+1)}\alpha_E(W_2),$$

We shall show that condition (H_3) holds with

$$\sigma(t) = \frac{1}{12(\sqrt{t}+1)}, \quad \rho(t) = \frac{e^{-t}}{(\sqrt{t}+1)(t+1)}.$$

Moreover

$$\|\sigma\|_{L^1} \leq \frac{1}{12}, \quad \|\rho\|_{L^1} \leq 1.$$

By (19), for any $t \in J$ and $W \subset E$, we get

$$\alpha_E(g(t, s, W)) \leq \frac{1}{8} \sup_{t \in [0,1]} \frac{\sqrt{t}}{1+t}\alpha_E(W),$$

then

$$\alpha_E(g(t, s, W)) \leq \frac{\sqrt{2}}{24}\alpha_E(W).$$

Hence (H_5) is satisfied with $K^* = \dfrac{\sqrt{2}}{24}$.

Next, let us observe that, in view of (16) and (17), the mapping γ_1 and ζ_1 fulfil the hypothes (H_5) and (H_6) with $c_1 = e_1 = k_1 = \bar{k}_1 = \frac{1}{12}$ and $d_1 = l_1 = 0$. Furthermore, we have

$$\max(k_1, 1)\left(\tilde{M}k_1 + M\bar{k}_1 + 2M(\|\sigma\|_{L^1} + 2K^* a\|\varrho\|_{L^1})\right) = \frac{2+\sqrt{2}}{6} < 1.$$

Clearly all the conditions of theorem 3 are satisfied. Hence by the conclusion of Theorem 3, it follows that problem (14) has a solution.

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