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Common Fixed Point Results for Generalized Wardowski Type Contractive Multi-Valued Mappings

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Abstract: In this paper, we introduce generalized Wardowski type quasi-contractions called α-(φ, Ω)-contractions for a pair of multi-valued mappings and prove the existence of the common fixed point for such mappings. An illustrative example and an application are given to show the usability of our results.

Keywords: common fixed point; Ω-contraction; α-admissible; multi-valued mapping; integral inclusion

1. Introduction

For a metric space (Λ, d), let $\mathcal{CB}(\Lambda)$ be the class of all nonempty closed and bounded subsets of Λ and $\mathcal{K}(\Lambda)$ be the class of all nonempty compact subsets of Λ (it is well known that $\mathcal{K}(\Lambda) \subseteq \mathcal{CB}(\Lambda)$). The mapping $\mathfrak{H} : \mathcal{CB}(\Lambda) \times \mathcal{CB}(\Lambda) \to \mathbb{R}^+ \cup \{0\}$ defined by

$$\mathfrak{H}(P, Q) = \max\{\sup_{p\in P} d(p, Q), \sup_{q\in Q} d(q, P)\}, \ for\ any\ P, Q \in \mathcal{CB}(\Lambda)$$

is called the Pompeiu–Hausdorff metric induced by d, where $d(p, Q) = \inf\{d(p, q) : q \in Q\}$ is the distance from p to $Q \subseteq \Lambda$. For example, if we consider the set of real numbers with the usual metric $d(\eta, \varrho) = |\eta - \varrho|$, then, for any two closed intervals $[a, b]$ and $[c, d]$, we have $\mathfrak{H}([a, b], [c, d]) = \max\{|a - c|, |b - d|\}$.

In 1969, Nadler [1] extended the Banach contraction principle as follows:

Theorem 1 ([1])**.** *Let (Λ, d) be a complete metric space and $\Upsilon : \Lambda \to \mathcal{CB}(\Lambda)$ be a multi-valued mapping such that*

$$\mathfrak{H}(\Upsilon\eta, \Upsilon\varrho) \leq kd(\eta, \varrho) \tag{1}$$

for all $\eta, \varrho \in \Lambda$, where $k \in [0, 1)$. Then, Υ has at least one fixed point.

Recently, Wardowski [2] gave a new generalization of Banach contraction to show the existence of the fixed point for such contraction by a more simple method of proof than the Banach's one. After that, several authors studied different variations of Wardowski contraction for single-valued and multivalued mappings—for example, see [3–8]. On the other hand, Aydi et al. [9] studied a common fixed point for generalized multi-valued contractions. In this paper, we introduce the concept of α-(φ, Ω)-contraction for a pair of multi-valued mappings and prove the existence of common fixed point for such mappings. Our results generalize and improve many existing results in the literature (for instance, [7,9]). In addition, an illustrative example and an application to the system of Volterra-type integral inclusions are given.

2. Preliminaries

In the sequel, we recall some definitions and results which will be used in this article. Following [2], denote by Ξ the collection of all functions $\Omega : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions :

(Ω1) Ω is strictly increasing,
(Ω2) For each sequence $\{\sigma_n\}$ in $(0, +\infty)$, $\lim_{n\to\infty} \sigma_n = 0$ if and only if $\lim_{n\to\infty} \Omega(\sigma_n) = -\infty$,
(Ω3) There exists $k \in (0,1)$ such that $\lim_{\sigma\to 0^+} \sigma^k \Omega(\sigma) = 0$.

Definition 1 ([2])**.** *Let (Λ, d) be a metric space. A mapping $\Upsilon : \Lambda \to \Lambda$ is said to be an Ω-contraction if there exist $\tau \in \mathbb{R}^+$ and $\Omega \in \Xi$ such that for all $\eta, \varrho \in \Lambda$,*

$$d(\Upsilon\eta, \Upsilon\varrho) > 0 \Longrightarrow \tau + \Omega(d(\Upsilon\eta, \Upsilon\varrho)) \leq \Omega(d(\eta, \varrho)). \tag{2}$$

It should be noted that any contraction is an Ω-contraction. To see this, suppose that Υ is a contraction on a metric space (Λ, d) with constant $k \in [0, 1)$ that is, $d(\Upsilon\eta, \Upsilon\varrho) \leq kd(\eta, \varrho)$, for all $\eta, \varrho \in \Lambda$. If $k = 0$, $d(\Upsilon\eta, \Upsilon\varrho) = 0$ and we have nothing to prove. In the case where $k \in (0, 1)$, taking ln on both sides of the contraction, we get

$$-lnk + ln(d(\Upsilon\eta, \Upsilon\varrho)) \leq ln(d(\eta, \varrho)) \tag{3}$$

for all $\eta, \varrho \in \Lambda$ with $d(\Upsilon\eta, \Upsilon\varrho) > 0$. Putting $\tau = -lnk$ and $\Omega(t) = lnt$ in the above inequality, we have an Ω-contraction.

Example 1 ([2])**.** *The functions $\Omega : \mathbb{R}^+ \to \mathbb{R}$ defined by*

(1) $\Omega(\sigma) = \ln \sigma$,
(2) $\Omega(\sigma) = \ln \sigma + \sigma$,
(3) $\Omega(\sigma) = \frac{-1}{\sqrt{\sigma}}$,
(4) $\Omega(\sigma) = \ln(\sigma^2 + \sigma)$,

belong to Ξ.

Theorem 2 ([2])**.** *Let (Λ, d) be a complete metric space and $\Upsilon : \Lambda \to \Lambda$ be an Ω-contraction. Then, Υ has a unique fixed point μ in Λ and for any point $\eta \in \Lambda$ the sequence $\{\Upsilon^n\eta\}$ converges to μ.*

In 2012, Samet et al. [10] introduced the notion of α-admissible mapping as follows:

Let Λ be a nonempty set. The selfmap Υ on Λ is called α-admissible whenever there exists a map $\alpha : \Lambda \times \Lambda \to [0, \infty)$ such that $\alpha(\eta, \varrho) \geq 1$ implies $\alpha(\Upsilon\eta, \Upsilon\varrho) \geq 1$, for all $\eta, \varrho \in \Lambda$. In addition, it is well known that Λ is called α-regular, if for any sequence $\{\eta_n\}$ in Λ that $\eta_n \to \eta$ and $\alpha(\eta_n, \eta_{n+1}) \geq 1$ for all n, then $\alpha(\eta_n, \eta) \geq 1$ for all n. In 2013, Mohammadi et al. introduced the notion of α-admissiblity for multi-valued mappings as follows:

Definition 2 ([11])**.** *Let Λ be a nonempty set and 2^Λ is the set of all nonempty subsets of Λ. A multi-valued mapping $\Upsilon : \Lambda \to 2^\Lambda$ is called α-admissible, if there exists a function $\alpha : \Lambda \times \Lambda \to [0, \infty)$ such that, for each $\eta \in \Lambda$ and $\varrho \in \Upsilon\eta$ with $\alpha(\eta, \varrho) \geq 1$, then $\alpha(\varrho, \mu) \geq 1$ for all $\mu \in \Upsilon\varrho$.*

3. Main Results

Let Φ denote the set of all the functions $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying:

(φ_1) $\lim_{n\to\infty} \frac{\varphi^n(t)}{n} < 0$ for all $t > 0$;
(φ_2) $\varphi(t) < t$ for all $t \geq 0$.
(φ_3) φ is nondecreasing and upper semi-continuous.

Example 2. *The functions $\varphi : \mathbb{R} \to \mathbb{R}$ defined by*

(1) $\varphi_1(t) = t - \tau$ *where* $\tau > 0$,

(2) $\varphi_2(t) = \begin{cases} t^3 - 1, & t < 1, \\ \sqrt{t} - 1, & t > 1. \end{cases}$

belong to Φ.

It is easy to see that any function φ satisfying (φ_1) has the property that $\lim\limits_{n\to\infty} \varphi^n(t) = -\infty$ for all $t > 0$.

Definition 3. *Let Λ be a nonempty set. We say that a pair (Υ, Γ) of multi-valued mappings $\Upsilon, \Gamma : \Lambda \to 2^{\Lambda}$ is α-admissible, if there exists a function $\alpha : \Lambda \times \Lambda \to [0, \infty)$ such that*

(α_1) for each $\eta \in \Lambda$ and $\varrho \in \Upsilon\eta$ with $\alpha(\eta, \varrho) \geq 1$, then $\alpha(\varrho, \mu) \geq 1$ for all $\mu \in \Gamma\varrho$,
(α_2) for each $\eta \in \Lambda$ and $\varrho \in \Gamma\eta$ with $\alpha(\eta, \varrho) \geq 1$, then $\alpha(\varrho, \mu) \geq 1$ for all $\mu \in \Upsilon\varrho$.

It is well known that a function $\alpha : \Lambda \times \Lambda \to [0, \infty)$ is called symmetric if $\alpha(\eta, \varrho) \geq 1$ implies $\alpha(\varrho, \eta) \geq 1$ for all $\eta, \varrho \in \Lambda$. We say that a pair (Υ, Γ) of multi-valued mappings $\Upsilon, \Gamma : \Lambda \to 2^{\Lambda}$ is symmetric α-admissible if there exists a symmetric function $\alpha : \Lambda \times \Lambda \to [0, \infty)$ such that (Υ, Γ) is α-admissible.

Definition 4. *We say that a pair of mappings $\Upsilon, \Gamma : \Lambda \to \mathcal{CB}(\Lambda)$ is α-(φ, Ω)-contraction whenever there exist $\alpha : \Lambda \times \Lambda \to [0, \infty)$, $\varphi \in \Phi$ and $\Omega \in \Xi$ such that*

$$\Omega(\mathfrak{H}(\Upsilon\eta, \Gamma\varrho)) \leq \varphi(\Omega(M(\eta, \varrho))), \tag{4}$$

for all $\eta, \varrho \in \Lambda$ with $\alpha(\eta, \varrho) \geq 1$ and $\mathfrak{H}(\Upsilon\eta, \Gamma\varrho) > 0$ where

$$M(\eta, \varrho) = \max\{d(\eta, \varrho), d(\eta, \Upsilon\eta), d(\varrho, \Gamma\varrho), \frac{d(\eta, \Gamma\varrho) + d(\varrho, \Upsilon\eta)}{2}\}.$$

Theorem 3. *Let (Λ, d) be a complete metric space and $\Upsilon, \Gamma : \Lambda \to \mathcal{K}(\Lambda)$ be two mappings such that (Υ, Γ) is an α-(φ, Ω)-contraction. Assume that the following assertions hold:*

(i) There exists $\eta_0 \in \Lambda$ and $\eta_1 \in \Upsilon\eta_0$ such that $\alpha(\eta_0, \eta_1) \geq 1$,
(ii) (Υ, Γ) is a symmetric α-admissible pair.

Then, Υ and Γ have a common fixed point provided that one of the following holds:

(C1) Υ and Γ are continuous,
(C2) Ω is continuous and Λ is α-regular.

Proof. It is easy to check that, if $M(\eta, \varrho) = 0$, then $\eta = \varrho$ and it is a common fixed point of Υ and Γ. Let η_0, η_1 be as in the assumption (i) that is, $\eta_0 \in \Lambda$ and $\eta_1 \in \Upsilon\eta_0$ be such that $\alpha(\eta_0, \eta_1) \geq 1$. We consider the following steps:

Step 1: If $M(\eta_0, \eta_1) = 0$, then $\eta_0 = \eta_1$ is a common fixed point of Υ and Γ. Thus, we may assume that $M(\eta_0, \eta_1) > 0$. Now, we have

$$\begin{aligned} M(\eta_0, \eta_1) &= \max\{d(\eta_0, \eta_1), d(\eta_0, \Upsilon\eta_0), d(\eta_1, \Gamma\eta_1), \frac{d(\eta_0, \Gamma\eta_1) + d(\eta_1, \Upsilon\eta_0)}{2}\} \\ &= \max\{d(\eta_0, \eta_1), d(\eta_1, \Gamma\eta_1)\}. \end{aligned}$$

Consider the following two cases:

(Case a): $d(\eta_1, \Gamma\eta_1) = 0$, that is, $\eta_1 \in \Gamma\eta_1$. In this case, since (Υ, Γ) is symmetric α-admissible pair, $\eta_1 \in \Upsilon\eta_0$ and $\alpha(\eta_0, \eta_1) \geq 1$, by (α_1), we have $\alpha(\eta_1, \eta_1) \geq 1$. If $d(\eta_1, \Upsilon\eta_1) > 0$, then by α-(φ, Ω)-contractivity of the pair (Υ, Γ), we have

$$\begin{aligned} \Omega(d(\eta_1, \Upsilon\eta_1)) &\leq \Omega(\mathfrak{H}(\Gamma\eta_1, \Upsilon\eta_1)) \\ &\leq \varphi(\Omega(M(\eta_1, \eta_1))) \\ &= \Omega(d(\eta_1, \Upsilon\eta_1)), \end{aligned}$$

which is a contradiction. Hence, $\eta_1 \in \Upsilon\eta_1$ and so η_1 is a common fixed point of Υ and Γ.

(Case b): $d(\eta_1, \Gamma\eta_1) > 0$. In this case, we have $\mathfrak{H}(\Upsilon\eta_0, \Gamma\eta_1) \geq d(\eta_1, \Gamma\eta_1) > 0$. Since $\alpha(\eta_0, \eta_1) \geq 1$ and the pair (Υ, Γ) is α-(φ, Ω)-contraction, we have

$$\begin{aligned} \Omega(d(\eta_1, \Gamma\eta_1)) &\leq \Omega(\mathfrak{H}(\Upsilon\eta_0, \Gamma\eta_1)) \\ &\leq \varphi(\Omega(M(\eta_0, \eta_1))) \\ &= \varphi(\Omega(\max\{d(\eta_0, \eta_1), d(\eta_1, \Gamma\eta_1)\})). \end{aligned} \tag{5}$$

In the case $\max\{d(\eta_0, \eta_1), d(\eta_1, \Gamma\eta_1)\} = d(\eta_1, \Gamma\eta_1)$, we have $\Omega(d(\eta_1, \Gamma\eta_1)) \leq \varphi(\Omega(d(\eta_1, \Gamma\eta_1))$, which contradicts with (φ_2). Hence, $\max\{d(\eta_0, \eta_1), d(\eta_1, \Gamma\eta_1)\} = d(\eta_0, \eta_1)$ and then we have

$$\Omega(d(\eta_1, \Gamma\eta_1)) \leq \varphi(\Omega(d(\eta_0, \eta_1))). \tag{6}$$

On the other hand, since $\Gamma\eta_1$ is compact, there exists $\eta_2 \in \Gamma\eta_1$ such that $d(\eta_1, \eta_2) = d(\eta_1, \Gamma\eta_1)$. Substituting in (6), we get

$$\Omega(d(\eta_1, \eta_2)) \leq \varphi(\Omega(d(\eta_0, \eta_1))). \tag{7}$$

Note that, since (Υ, Γ) is symmetric α-admissible pair, we have $\alpha(\eta_1, \eta_2) \geq 1$.

Step 2: If $M(\eta_2, \eta_1) = 0$, then $\eta_1 = \eta_2$ is a common fixed point of Υ and Γ. Thus, we may assume that $M(\eta_2, \eta_1) > 0$. Now, we have

$$\begin{aligned} M(\eta_2, \eta_1) &= \max\{d(\eta_1, \eta_2), d(\eta_2, \Upsilon\eta_2), d(\eta_1, \Gamma\eta_1), \frac{d(\eta_1, \Upsilon\eta_2) + d(\eta_2, \Gamma\eta_1)}{2}\} \\ &= \max\{d(\eta_1, \eta_2), d(\eta_2, \Upsilon\eta_2)\}. \end{aligned}$$

Consider two cases:

(Case c): $d(\eta_2, \Upsilon\eta_2) = 0$ that is, $\eta_2 \in \Upsilon\eta_2$. In this case, since (Υ, Γ) is symmetric α-admissible pair, $\eta_2 \in \Gamma\eta_1$ and $\alpha(\eta_1, \eta_2) \geq 1$, by (α_2), we have $\alpha(\eta_2, \eta_2) \geq 1$. If $d(\eta_2, \Gamma\eta_2) > 0$, then, by α-(φ, Ω)-contractivity of the pair (Υ, Γ), we have

$$\begin{aligned} \Omega(d(\eta_2, \Gamma\eta_2)) &\leq \Omega(\mathfrak{H}(\Upsilon\eta_2, \Gamma\eta_2)) \\ &\leq \varphi(\Omega(M(\eta_2, \eta_2))) \\ &= \Omega(d(\eta_2, \Gamma\eta_2)), \end{aligned}$$

which is a contradiction. Hence, $\eta_2 \in \Gamma\eta_2$ and so η_2 is a common fixed point of Υ and Γ.

(Case d): $d(\eta_2, \Upsilon\eta_2) > 0$. In this case, we have $\mathfrak{H}(\Upsilon\eta_2, \Gamma\eta_1) \geq d(\eta_2, \Upsilon\eta_2) > 0$. Since $\alpha(\eta_1, \eta_2) \geq 1$ and the pair (Υ, Γ) is α-(φ, Ω)-contraction, we have

$$\begin{aligned} \Omega(d(\eta_2, \Upsilon\eta_2)) &\leq \Omega(\mathfrak{H}(\Upsilon\eta_2, \Gamma\eta_1)) \\ &\leq \varphi(\Omega(M(\eta_2, \eta_1))) \\ &= \varphi(\Omega(\max\{d(\eta_1, \eta_2), d(\eta_2, \Upsilon\eta_2)\})). \end{aligned} \tag{8}$$

In the case $\max\{d(\eta_1, \eta_2), d(\eta_2, \Upsilon\eta_2)\} = d(\eta_2, \Upsilon\eta_2)$, we have $\Omega(d(\eta_2, \Upsilon\eta_2)) \leq \varphi(\Omega(d(\eta_2, \Upsilon\eta_2))$, which contradicts with (φ_2). Hence, $\max\{d(\eta_1, \eta_2), d(\eta_2, \Upsilon\eta_2)\} = d(\eta_1, \eta_2)$, and so

$$\Omega(d(\eta_2, \Upsilon\eta_2)) \leq \varphi(\Omega(d(\eta_1, \eta_2))). \tag{9}$$

On the other hand, since $\Upsilon\eta_2$ is compact, there exists $\eta_3 \in \Upsilon\eta_2$ such that $d(\eta_2, \eta_3) = d(\eta_2, \Upsilon\eta_2)$. Substituting in (9), we get

$$\Omega(d(\eta_2, \eta_3)) \leq \varphi(\Omega(d(\eta_1, \eta_2))). \tag{10}$$

Substituting (7) in (10), we get

$$\Omega(d(\eta_2, \eta_3)) \leq \varphi^2(\Omega(d(\eta_0, \eta_1))). \tag{11}$$

Continuing this process, either we find a common fixed point of Υ and Γ or we can construct a sequence $\{\eta_n\}$ in Λ such that $\eta_{2n+1} \in \Upsilon\eta_{2n}$, $\eta_{2n+2} \in \Gamma\eta_{2n+1}$, $d(\eta_n, \eta_{n+1}) > 0$, $\alpha(\eta_n, \eta_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and

$$\Omega(d(\eta_n, \eta_{n+1})) \leq \varphi^n(\Omega(d(\eta_0, \eta_1))) \tag{12}$$

for all $n \in \mathbb{N}$.

Put $\gamma_n = d(\eta_n, \eta_{n+1})$. Then, from (12), we have

$$\Omega(\gamma_n) \leq \varphi^n(\Omega(\gamma_0)) \to -\infty,$$

as $n \to \infty$. Thus, $\lim_{n\to\infty} \Omega(\gamma_n) = -\infty$. From $(\Omega 2)$, $\lim_{n\to\infty} \gamma_n = 0$. Then, for any $n \in \mathbb{N}$, we have

$$\gamma_n{}^k(\Omega(\gamma_n)) \leq \gamma_n{}^k \varphi^n(\Omega(\gamma_0)).$$

Taking the limit on both sides of the above inequality, we obtain $\lim_{n\to\infty} \gamma_n{}^k \varphi^n(\Omega(\gamma_0)) = 0$. In addition, from (φ_1), there exists $\lambda > 0$ such that $|\frac{\varphi^n(\Omega(\gamma_0))}{n}| > \lambda$. Now, we have

$$n\gamma_n^k \lambda \leq n\gamma_n^k |\frac{\varphi^n(\Omega(\gamma_0))}{n}| = |\gamma_n^k \varphi^n(\Omega(\gamma_0))|.$$

Taking the limit on both sides of the above inequality, we obtain $\lim_{n\to\infty} n\gamma_n^k \lambda = 0$, and so $\lim_{n\to\infty} n\gamma_n^k = 0$. Therefore, there exists $N \in \mathbb{N}$ such that $\gamma_n \leq \frac{1}{n^{\frac{1}{k}}}$ for all $n \geq N$. Now, for any $m, n \in \mathbb{N}$ with $m > n$, we have

$$d(\eta_n, \eta_m) \leq \sum_{i=n}^{m-1} \gamma_i \leq \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{k}}} \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}.$$

From the above and from the convergence of the series $\sum_{i=1}^{\infty} 1/i^{\frac{1}{k}}$, we receive that $\{\eta_n\}$ is a Cauchy sequence. From the completeness of Λ, there exists $\mu \in \Lambda$ such that $\lim_{n\to\infty} \eta_n = \mu$.

Suppose that the condition $(C1)$ is satisfied. Then,

$$d(\mu, \Upsilon\mu) = \lim_{n\to\infty} d(\eta_{2n+1}, \Upsilon\mu) \leq \lim \mathfrak{H}(\Upsilon\eta_{2n}, \Upsilon\mu) = 0$$

and

$$d(\mu, \Gamma\mu) = \lim_{n\to\infty} d(\eta_{2n+2}, \Gamma\mu) \leq \lim_{n\to\infty} \mathfrak{H}(\Gamma\eta_{2n+1}, \Gamma\mu) = 0.$$

Thus, μ is a common fixed point of Υ and Γ.

Now, suppose that $(C2)$ holds. Since Λ is α-regular, we have $\alpha(\eta_n, \mu) \geq 1$. Then, we consider two cases:

(i) There exists $N \in \mathbb{N}$ such that for all $n \geq N$ one has $\Upsilon\eta_{2n} = \Gamma\mu$. Then, $\eta_{2n+1} \in \Upsilon\eta_{2n} = \Gamma\mu$. Since $\eta_{2n+1} \to \mu$ and $\Gamma\mu$ is closed we get $\mu \in \Gamma\mu$.

(ii) There exists a subsequence $\{\eta_{2n_i}\}$ of $\{\eta_{2n}\}$ such that $\Upsilon\eta_{2n_i} \neq \Gamma\mu$. In this case, suppose, on the contrary, that $d(\mu, \Gamma\mu) > 0$. Then,

$$\begin{aligned}
\Omega(d(\eta_{2n_i+1}, \Gamma\mu)) &\leq \Omega(\mathfrak{H}(\Upsilon\eta_{2n_i}, \Gamma\mu)) \\
&\leq \varphi(\Omega(M(\eta_{2n_i}, \mu))) \\
&= \varphi(\Omega(\max\{d(\eta_{2n_i}, \mu), d(\eta_{2n_i}, \Upsilon\eta_{2n_i}), d(\mu, \Gamma\mu), \frac{d(\eta_{2n_i}, \Gamma\mu) + d(\mu, \Upsilon\eta_{2n_i})}{2}\})).
\end{aligned}$$

Taking the limit on both sides of the above inequality, we obtain $\Omega(d(\mu, \Gamma\mu)) \leq \varphi(\Omega d(\mu, \Gamma\mu))$, a contradiction. Thus, $d(\mu, \Gamma\mu) = 0$ and so $\mu \in \Gamma\mu$.

A similar technique can be used to show that $\mu \in \Upsilon\mu$. $\square$

Taking $\varphi(t) = t - \tau$ in Theorem 3, we obtain the following result.

Corollary 1. *Let (Λ, d) be a complete metric space and $\Upsilon, \Gamma : \Lambda \to \mathcal{K}(\Lambda)$ be two mappings satisfying*

$$\tau + \Omega(\mathfrak{H}(\Upsilon\eta, \Gamma\varrho)) \leq \Omega(M(\eta, \varrho))$$

for all $\eta, \varrho \in \Lambda$ with $\alpha(\eta, \varrho) \geq 1$ and $\mathfrak{H}(\Upsilon\eta, \Gamma\varrho) > 0$, where $\tau > 0$, $\Omega \in \Xi$ and

$$M(\eta, \varrho) = \max\{d(\eta, \varrho), d(\eta, \Upsilon\eta), d(\varrho, \Gamma\varrho), \frac{d(\eta, \Gamma\varrho) + d(\varrho, \Upsilon\eta)}{2}\}.$$

Assume that the following assertions hold:

(i) There exists $\eta_0 \in \Lambda$ and $\eta_1 \in \Upsilon\eta_0$ such that $\alpha(\eta_0, \eta_1) \geq 1$,

(ii) (Υ, Γ) is a symmetric α-admissible pair.

Then, Υ and Γ have a common fixed point provided that one of (C1) and (C2) holds.

Taking $\Omega(t) = \ln t + t$ in Corollary 1, we obtain the following result.

Corollary 2. *Let (Λ, d) be a complete metric space and $\Upsilon, \Gamma : \Lambda \to \mathcal{K}(\Lambda)$ be two mappings satisfying*

$$\frac{\mathfrak{H}(\Upsilon\eta, \Gamma\varrho)}{M(\eta, \varrho)} e^{\mathfrak{H}(\Upsilon\eta, \Gamma\varrho) - M(\eta, \varrho)} \leq e^{-1},$$

for all $\eta, \varrho \in \Lambda$ with $\alpha(\eta, \varrho) \geq 1$ and $\mathfrak{H}(\Upsilon\eta, \Upsilon\varrho) > 0$, where

$$M(\eta, \varrho) = \max\{d(\eta, \varrho), d(\eta, \Upsilon\eta), d(v, \Upsilon\varrho), \frac{d(\eta, \Upsilon\varrho) + d(\varrho, \Upsilon\eta)}{2}\}.$$

Assume that the following assertions hold:

(i) *There exists* $\eta_0 \in \Lambda$ *and* $\eta_1 \in \Upsilon\eta_0$ *such that* $\alpha(\eta_0, \eta_1) \geq 1$,

(ii) (Υ, Γ) *is a symmetric* α*-admissible pair.*

Then, Υ *and* Γ *have a common fixed point provided that one of (C1) and (C2) holds.*

Example 3. *Let* $\Lambda = \{\kappa_n = \frac{n(n+1)}{2} : n = 1, 2, \ldots\} \cup \{0\}$ *and* $d(\eta, \varrho) = |\eta - \varrho|$. *Define* $\Upsilon, \Gamma : \Lambda \to \mathcal{K}(\Lambda)$ *by*

$$\Upsilon\eta = \begin{cases} \{0\}, & \eta = 0, \\ \{\kappa_1\}, & \eta = \kappa_1, \kappa_2, \\ \{\kappa_1, \kappa_2, \ldots, \kappa_{n-1}\}, & \eta = \kappa_n,\ n \geq 3, \end{cases}$$

and

$$\Gamma\eta = \begin{cases} \{0\}, & \eta = 0, \\ \{\kappa_1\}, & \eta = \kappa_1, \kappa_2, \\ \{\kappa_2, \kappa_3, \ldots, \kappa_{n-1}\}, & \eta = \kappa_n,\ n \geq 3. \end{cases}$$

Define a function $\alpha : \Lambda \times \Lambda \to [0, \infty)$ *by* $\alpha(\eta, \varrho) = 1$ *if* $\eta, \varrho \in \{\kappa_n : n = 1, 2, \ldots\}$ *and* $\alpha(\eta, \varrho) = 0$, *otherwise. Then, for any* $(\eta, \varrho) \in \Lambda$ *with* $\alpha(\eta, \varrho) \geq 1$ *and* $\mathfrak{H}(\Upsilon\eta, \Gamma\varrho) \neq 0$, *we have the following cases:*

Case 1: $\eta = \kappa_1$ *and* $\varrho = \kappa_n,\ n \geq 3$. *Then,*

$$\mathfrak{H}(\Upsilon\eta, \Gamma\varrho) = \mathfrak{H}(\{\kappa_1\}, \{\kappa_2, ..., \kappa_{n-1}\}) = |\kappa_{n-1} - 1|$$

and $M(\eta, \varrho) = |\kappa_n - 1|$. *Hence, we have*

$$\begin{aligned} \frac{\mathfrak{H}(\Upsilon\kappa_1, \Gamma\kappa_n)}{M(\kappa_1, \kappa_n)} e^{\mathfrak{H}(\Upsilon\kappa_1, \Gamma\kappa_n) - M(\kappa_1, \kappa_n)} &= \frac{\kappa_{n-1} - 1}{\kappa_n - 1} e^{\kappa_{n-1} - \kappa_n} \\ &= \frac{\frac{n(n-1)}{2} - 1}{\frac{n(n+1)}{2} - 1} e^{-n} \\ &< e^{-1}. \end{aligned}$$

Case 2: $\eta = \kappa_n,\ n \geq 3$ *and* $\varrho = \kappa_1$. *Then,*

$$\mathfrak{H}(\Upsilon\eta, \Gamma\varrho) = \mathfrak{H}(\{\kappa_1, \kappa_2, ..., \kappa_{n-1}\}, \{\kappa_1\}) = |\kappa_{n-1} - 1|$$

and $M(\eta, \varrho) = |\kappa_n - 1|$. *Hence, we have*

$$\begin{aligned} \frac{\mathfrak{H}(\Upsilon\kappa_n, \Gamma\kappa_1)}{M(\kappa_n, \kappa_1)} e^{\mathfrak{H}(\Upsilon\kappa_n, \Gamma\kappa_1) - M(\kappa_n, \kappa_1)} &= \frac{\kappa_{n-1} - 1}{\kappa_n - 1} e^{\kappa_{n-1} - \kappa_n} \\ &= \frac{\frac{n(n-1)}{2} - 1}{\frac{n(n+1)}{2} - 1} e^{-n} \\ &< e^{-1}. \end{aligned}$$

Case 3: $\eta = \kappa_2$ *and* $\varrho = \kappa_n,\ n \geq 3$. *Then,*

$$\mathfrak{H}(\Upsilon\eta, \Gamma\varrho) = \mathfrak{H}(\{\kappa_1\}, \{\kappa_2, ..., \kappa_{n-1}\}) = |\kappa_{n-1} - \kappa_1|$$

and $M(\eta,\varrho) = |\kappa_n - \kappa_2|$. *Hence, we have*

$$\begin{aligned}\frac{\mathfrak{H}(\Upsilon\kappa_2,\Gamma\kappa_n)}{M(\kappa_2,\kappa_n)}e^{\mathfrak{H}(\Upsilon\kappa_2,\Gamma\kappa_n)-M(\kappa_2,\kappa_n)} &= \frac{\kappa_{n-1}-1}{\kappa_n-3}e^{\kappa_{n-1}-\kappa_n+2}\\ &= \frac{\frac{n(n-1)}{2}-1}{\frac{n(n+1)}{2}-3}e^{-n+2}\\ &\leq e^{-1}.\end{aligned}$$

Case 4: $\eta = \kappa_n,\ n \geq 3$ *and* $\varrho = \kappa_2$. *Then,*

$$\mathfrak{H}(\Upsilon\eta,\Gamma\varrho) = \mathfrak{H}(\{\kappa_1,\kappa_2,...,\kappa_{n-1}\},\{\kappa_1\}) = |\kappa_{n-1}-1|$$

and $M(\eta,\varrho) = |\kappa_n - \kappa_2|$. *Hence, we have*

$$\begin{aligned}\frac{\mathfrak{H}(\Upsilon\kappa_n,\Gamma\kappa_2)}{M(\kappa_n,\kappa_2)}e^{\mathfrak{H}(\Upsilon\kappa_n,\Gamma\kappa_2)-M(\kappa_n,\kappa_2)} &= \frac{\kappa_{n-1}-1}{\kappa_n-3}e^{\kappa_{n-1}-\kappa_n+2}\\ &= \frac{\frac{n(n-1)}{2}-1}{\frac{n(n+1)}{2}-3}e^{-n+2}\\ &\leq e^{-1}.\end{aligned}$$

Case 5: $\eta = \kappa_n$ *and* $\varrho = \kappa_m,\ n > m$. *Then,*

$$\begin{aligned}\mathfrak{H}(\Upsilon\eta,\Gamma\varrho) &= \mathfrak{H}(\{\kappa_1,\kappa_2,...,\kappa_{n-1}\},\{\kappa_2,...,\kappa_{m-1}\}) = \kappa_{n-1}-\kappa_{m-1}\\ &= \frac{n(n-1)}{2}-\frac{m(m-1)}{2} = \frac{(n-m)(n+m-1)}{2}\end{aligned}$$

and

$$M(\eta,\varrho) \geq |\kappa_n-\kappa_m| = \frac{n(n+1)}{2}-\frac{m(m+1)}{2} = \frac{(n-m)(n+m+1)}{2}.$$

Hence, we have

$$\frac{\mathfrak{H}(\Upsilon\kappa_n,\Gamma\kappa_m)}{M(\kappa_n,\kappa_m)}e^{\mathfrak{H}(\Upsilon\kappa_n,\Gamma\kappa_m)-M(\kappa_n,\kappa_m)} \leq \frac{n+m-1}{n+m+1}e^{-(n-m)} < e^{-1}.$$

Case 6: $\eta = \kappa_m$ *and* $\varrho = \kappa_n,\ n > m$. *Then,*

$$\begin{aligned}\mathfrak{H}(\Upsilon\eta,\Gamma\varrho) &= \mathfrak{H}(\{\kappa_1,\kappa_2,...,\kappa_{m-1}\},\{\kappa_2,...,\kappa_{n-1}\}) = \kappa_{n-1}-\kappa_{m-1}\\ &= \frac{n(n-1)}{2}-\frac{m(m-1)}{2} = \frac{(n-m)(n+m-1)}{2}\end{aligned}$$

and

$$M(\eta,\varrho) \geq |\kappa_n-\kappa_m| = \frac{n(n+1)}{2}-\frac{m(m+1)}{2} = \frac{(n-m)(n+m+1)}{2}.$$

Hence, we have

$$\frac{\mathfrak{H}(\Upsilon\kappa_m,\Gamma\kappa_n)}{M(\kappa_n,\kappa_m)}e^{\mathfrak{H}(\Upsilon\kappa_m,\Gamma\kappa_n)-M(\kappa_m,\kappa_n)} \leq \frac{n+m-1}{n+m+1}e^{-(n-m)} < e^{-1}.$$

On the other hand, it is easy to see that (Υ,Γ) *is a symmetric* α*-admissible pair. In addition, if we take* $\eta_0 = \kappa_2, \eta_1 = \kappa_1$, *then* $\eta_1 \in \Upsilon\eta_0$ *and* $\alpha(\eta_0,\eta_1) \geq 1$. *Thus, by Corollary 2,* Υ *and* Γ *have a common* *fixed point. Here, 0 is a common fixed point of* Υ *and* Γ*. Note that* Υ *and* Γ *are not a generalized contraction. Since*

$$\sup_{n\geq 3}\frac{\mathfrak{H}(\Upsilon\kappa_n,\Gamma\kappa_1)}{M(\kappa_n,\kappa_1)}=\sup_{n\geq 3}\frac{\kappa_{n-1}-1}{\kappa_n-1}=\sup_{n\geq 3}\frac{\frac{n(n-1)}{2}-1}{\frac{n(n+1)}{2}-1}=1,$$

Theorem 2.2 in [9] *can not apply to this example.*

Defining $\alpha:\Lambda\times\Lambda\to[0,\infty)$ by $\alpha(\eta,\varrho)=1$, for all $\eta,\varrho\in\Lambda$ in Theorem 3, we have the following result.

Theorem 4. *Let* (Λ,d) *be a complete metric space and* $\Upsilon,\Gamma:\Lambda\to\mathcal{K}(\Lambda)$ *be two mappings satisfying*

$$\Omega(\mathfrak{H}(\Upsilon\eta,\Gamma\varrho))\leq\varphi(\Omega(M(\eta,\varrho))) \tag{13}$$

for all $\eta,\varrho\in\Lambda$ *with* $\mathfrak{H}(\Upsilon\eta,\Gamma\varrho)>0$, *where* $\varphi\in\Phi$ *and* $\Omega\in\Xi$*. If* Υ,Γ *or* Ω *be continuous, then* Υ *and* Γ *have a common fixed point.*

Taking $\varphi(t)=t-\tau$ in Theorem 4, we obtain the following corollary.

Corollary 3. *Let* (Λ,d) *be a complete metric space and* $\Upsilon,\Gamma:\Lambda\to\mathcal{K}(\Lambda)$ *be two mappings satisfying*

$$\tau+\Omega(\mathfrak{H}(\Upsilon\eta,\Gamma\varrho))\leq\Omega(M(\eta,\varrho))$$

for all $\eta,\varrho\in\Lambda$ *with* $\mathfrak{H}(\Upsilon\eta,\Gamma\varrho)>0$, *where* $\tau>0$, $\Omega\in\Xi$ *and*

$$M(\eta,\varrho)=\max\{d(\eta,\varrho),d(\eta,\Upsilon\eta),d(\varrho,\Gamma\varrho),\frac{d(\eta,\Gamma\varrho)+d(\varrho,\Upsilon\eta)}{2}\}.$$

If Υ,Γ *or* Ω *is continuous, then* Υ *and* Γ *have a common fixed point.*

Taking $\Omega(t)=\ln t+t$ in the Corollary 3, we obtain the following corollary.

Corollary 4. *Let* (Λ,d) *be a complete metric space and* $\Upsilon,\Gamma:\Lambda\to\mathcal{K}(\Lambda)$ *be two mappings satisfying*

$$\frac{\mathfrak{H}(\Upsilon\eta,\Gamma\varrho)}{M(\eta,\varrho)}e^{\mathfrak{H}(\Upsilon\eta,\Gamma\varrho)-M(\eta,\varrho)}\leq e^{-1}$$

for all $\eta,\varrho\in\Lambda$ *with* $\mathfrak{H}(\Upsilon\eta,\Gamma\varrho)>0$, *where*

$$M(\eta,\varrho)=\max\{d(\eta,\varrho),d(\eta,\Upsilon\eta),d(\varrho,\Gamma\varrho),\frac{d(\eta,\Gamma\varrho)+d(\varrho,\Upsilon\eta)}{2}\}.$$

If Υ,Γ *are continuous, then* Υ *and* Γ *have a common fixed point.*

4. An Application to Volterra-Type Integral Inclusions

Let $\Lambda:=\mathcal{C}(\mathcal{J},\mathbb{R})$ ($\mathcal{J}=[a,b]$) be the set of all real valued continuous functions with domain $\mathcal{J}$ and let

$$d(\eta,\varrho)=\sup_{t\in\mathcal{J}}(|\eta(t)-\varrho(t)|)=||\eta-\varrho||,\quad \text{for all } \eta,\varrho\in\Lambda.$$

Consider the system of Volterra-type integral inclusions:

$$\begin{cases} \eta(t) \in p(t) + \int\limits_a^t K(t,s,\eta(s))ds, & t \in [a,b], \\ \eta(t) \in p(t) + \int\limits_a^t G(t,s,\eta(s))ds, & t \in [a,b], \end{cases} \tag{14}$$

where $p : \mathcal{J} \to \mathbb{R}$ and $K, G : \mathcal{J} \times \mathcal{J} \times \mathbb{R} \to \mathcal{CB}(\mathbb{R})$ are continuous.

Theorem 5. *Assume that there exist $\tau > 0$ and a continuous function $q : \mathcal{J} \to \mathbb{R}^+$ with $\int_a^b q(t)dt \le 1$ such that*

$$\mathfrak{H}(K(t,s,\eta(s)) - G(t,s,\varrho(s))) \le \frac{q(s)|\eta(s) - \varrho(s)|}{\tau^2||\eta - \varrho|| + 2\tau\sqrt{||\eta - \varrho||} + 1}, \tag{15}$$

for each $s, t \in \mathcal{J}$ and $\eta, \varrho \in \Lambda$. Then, the system of integral inclusions (14) has a solution in Λ.

Proof. Define $\Upsilon, \Gamma : \Lambda \to \mathcal{K}(\Lambda)$ as

$$\Upsilon\eta(t) = \{u \in \Lambda \colon u \in p(t) + \int_a^t K(t,s,\eta(s))ds\}, \quad t \in [a,b]$$

and

$$\Gamma\eta(t) = \{u \in \Lambda \colon u \in p(t) + \int_a^t G(t,s,\eta(s))ds\}, \quad t \in [a,b].$$

As in [12], it is easy to show that $\Upsilon\eta$ and $\Gamma\eta$ are nonempty, for all $\eta \in \Lambda$. Now, let $\eta, \varrho \in \Lambda$ and $u \in \Upsilon\eta$. Then, there exists $k_\eta(t,s) \in K_\eta(t,s)$ for each $t, s \in \mathcal{J}$ such that $u(t) = p(t) + \int_a^t k_\eta(t,s)ds$, for all $t \in \mathcal{J}$. From (15) and as in [12], it is easily seen that there exists $v(t,s) = g_\varrho(t,s) \in G_\varrho(t,s)$ satisfying

$$|k_\eta(t,s) - v(t,s)| \le \frac{q(s)|\eta(s) - \varrho(s)|}{\tau^2||\eta - \varrho|| + 2\tau\sqrt{||\eta - \varrho||} + 1}. \tag{16}$$

Taking $r(t) = p(t) + \int_a^t g_\varrho(t,s)ds$, we have $r(t) \in \Gamma\varrho$ and

$$\begin{aligned} |u(t) - r(t)| &= |\int\limits_a^t k_\eta(t,s)ds - \int\limits_a^t g_\varrho(t,s)ds| \\ &\le \int\limits_a^t |k_\eta(t,s) - g_\varrho(t,s)|ds \\ &\le \int\limits_a^b \frac{q(s)|\eta(s) - \varrho(s)|}{\tau^2||\eta - \varrho|| + 2\tau\sqrt{||\eta - \varrho||} + 1} \\ &\le (\int\limits_a^b q(s)ds) \frac{||\eta - \varrho||}{\tau^2||\eta - \varrho|| + 2\tau\sqrt{||\eta - \varrho||} + 1} \\ &\le \frac{d(\eta,\varrho)}{\tau^2 d(\eta,\varrho) + 2\tau\sqrt{d(\eta,\varrho)} + 1}. \end{aligned} \tag{17}$$

Taking sup as $t \in \mathcal{J}$, we obtain

$$||u - r|| \le \frac{d(\eta,\varrho)}{\tau^2 d(\eta,\varrho) + 2\tau\sqrt{d(\eta,\varrho)} + 1}.$$

Interchanging the rule of η, ϱ in the above argument yields that

$$\mathfrak{H}(\Upsilon\eta, \Gamma\varrho) \leq \frac{d(\eta, \varrho)}{\tau^2 d(\eta, \varrho) + 2\tau\sqrt{d(\eta, \varrho)} + 1}. \tag{18}$$

Therefore,

$$\sqrt{\mathfrak{H}(\Upsilon\eta, \Gamma\varrho)} \leq \frac{\sqrt{d(\eta, \varrho)}}{\tau\sqrt{d(\eta, \varrho)} + 1} \tag{19}$$

for all $\eta, \varrho \in \Lambda$ with $\Upsilon\eta \neq \Gamma\varrho$ (and subsequently $\eta \neq \varrho$). Inverting the above inequality and performing some algebra actions, we get

$$\tau + \frac{-1}{\sqrt{\mathfrak{H}(\Upsilon\eta, \Gamma\varrho)}} \leq \frac{-1}{\sqrt{d(\eta, \varrho)}} \tag{20}$$

for all $\eta, \varrho \in \Lambda$ with $\Upsilon\eta \neq \Gamma\varrho$. Now taking $\Omega(t) = \frac{-1}{\sqrt{t}}$, we obtain

$$\tau + \Omega(\mathfrak{H}(\Upsilon\eta, \Gamma\varrho)) \leq \Omega(d(\eta, \varrho)) \leq \Omega(M(\eta, \varrho)), \tag{21}$$

for all $\eta, \varrho \in \Lambda$ with $\Upsilon\eta \neq \Gamma\varrho$, where $M(\eta, \varrho)$ is as in Corollary 3. We see that the conditions of Corollary 3 are satisfied. Thus, Υ and Γ have a common fixed point. Hence, there is a solution for (14). □

5. Conclusions

In this paper, we introduced a new generalization of Wardowski type contractions and established common fixed point theorems for such multi-valued contractions. The new contraction will be a powerful tool for the existence solution of the systems of integral inclusions and fractional differential inclusions. We think that different versions of this new contraction can be considered in abstract spaces.

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