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On a Reverse Half-Discrete Hardy-Hilbert's Inequality with Parameters

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Abstract: By means of the weight functions, the idea of introduced parameters, and the Euler-Maclaurin summation formula, a reverse half-discrete Hardy-Hilbert's inequality and the reverse equivalent forms are given. The equivalent statements of the best possible constant factor involving several parameters are considered. As applications, two results related to the case of the non-homogeneous kernel and some particular cases are obtained.

Keywords: weight function; half-discrete Hardy-Hilbert's inequality; parameter; Euler-Maclaurin summation formula; reverse inequality

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1. Introduction

If $0 < \sum_{m=1}^{\infty} a_m^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then we have the following discrete Hilbert's inequality with the best possible constant factor π (cf., [1], Theorem 315):

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \pi \left(\sum_{m=1}^{\infty} a_m^2 \sum_{n=1}^{\infty} b_n^2 \right)^{1/2} \quad (1)$$

Correspondingly, if $0 < \int_0^{\infty} f^2(x) dx < \infty$ and $0 < \int_0^{\infty} g^2(y) dy < \infty$, we still have the following Hilbert's integral inequality (cf., [1], Theorem 316):

$$\int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{x+y} dx dy < \pi \left(\int_0^{\infty} f^2(x) dx \int_0^{\infty} g^2(y) dy \right)^{1/2} \quad (2)$$

where the constant factor π is the best possible.

As is known to us, Inequalities (1) and (2) and their extensions with conjugate exponents as well as independent parameters play an important role in analysis and their applications (cf., [2–13]).

Concerning with Inequalities (1) and (2), we have the following half-discrete Hilbert-type inequality (cf., [1], Theorem 351):

If $K(x)$ ($x > 0$) is a decreasing function and $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \phi(s) = \int_0^{\infty} K(x)x^{s-1} dx < \infty$, $f(x) \geq 0$, $0 < \int_0^{\infty} f^p(x) dx < \infty$, then

$$\sum_{n=1}^{\infty} n^{p-2} \left(\int_0^{\infty} K(nx) f(x) dx \right)^p < \phi^p \left(\frac{1}{q} \right) \int_0^{\infty} f^p(x) dx. \quad (3)$$

In recent years, some new extensions of the Inequality (3) were provided in [14–19].

In 2006, with the help of the Euler-Maclaurin summation formula, Krnic et al. [20] gave an extension of (1) with the kernel $\frac{1}{(m+n)^\lambda}$ ($0 < \lambda \leq 14$). In 2019, Adiyasuren et al. [21] considered an extension of (1) with $p, q > 1$ ($\frac{1}{p} + \frac{1}{q} = 1$) involving the partial sums. In 2016–2017, by using the weight functions, Hong [22,23] considered some equivalent statements of the extensions of (1) and (2) with several parameters. Some related works can be found in [24–26].

In this paper, following the way of [20,22], by using the weight functions, the idea of introduced parameters, and the Euler-Maclaurin summation formula, a reverse half-discrete Hardy-Hilbert's inequality with the homogeneous kernel $\frac{1}{(x+n)^\lambda}$ ($0 < \lambda \leq 5$) and the reverse equivalent forms are established. The equivalent statements of the best possible constant factor related to several parameters are presented. As applications, two corollaries related to the case of the non-homogeneous kernel and some particular cases are obtained.

2. Some Lemmas

In what follows, we assume that

$$0 < p < 1 (q < 0), \frac{1}{p} + \frac{1}{q} = 1, \lambda \in (0, 5], \sigma \in (0, 2] \cap (0, \lambda), \mu \in (0, \lambda),$$

$f(x) \geq 0$ ($x \in \mathbb{R}_+ = (0, \infty)$), $a_n \geq 0$ ($n \in \mathbb{N} = \{1, 2, \dots\}$) satisfying

$$0 < \int_0^\infty x^{p[1-(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})]-1} f^p(x) dx < \infty \text{ and } 0 < \sum_{n=1}^\infty n^{q[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1} a_n^q < \infty.$$

Lemma 1. Define a weight function by

$$\omega(\sigma, x) := x^{\lambda-\sigma} \sum_{n=1}^\infty \frac{n^{\sigma-1}}{(x+n)^\lambda} \quad (x \in \mathbb{R}_+). \quad (4)$$

Then, we have

$$B(\sigma, \lambda - \sigma)(1 - \rho_\sigma(x)) < \omega(\sigma, x) < B(\sigma, \lambda - \sigma)(x \in \mathbb{R}_+), \quad (5)$$

where, $\rho_\sigma(x) := \frac{(1+\theta_x)^{-\lambda}}{\sigma B(\sigma, \lambda-\sigma)} \frac{1}{x^\sigma} = O(\frac{1}{x^\sigma}) \in (0, 1)$ ($\theta_x \in (0, \frac{1}{x}); x > 0$). $B(u, v) := \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}} dt$ ($u, v > 0$) is the beta function.

Proof. For fixed $x > 0$, we set function $g_x(t) := \frac{t^{\sigma-1}}{(x+t)^\lambda}$ ($t > 0$). Using the Euler-Maclaurin summation formula (cf., [20]), for $\rho(t) := t - [t] - \frac{1}{2}$, we have

$$\begin{aligned} \sum_{n=1}^\infty g_x(n) &= \int_1^\infty g_x(t) dt + \frac{1}{2} g_x(1) + \int_1^\infty \rho(t) g'_x(t) dt = \int_0^\infty g_x(t) dt - h(x), \\ h(x) &:= \int_0^1 g_x(t) dt - \frac{1}{2} g_x(1) - \int_1^\infty \rho(t) g'_x(t) dt. \end{aligned}$$

Thus, we obtain $-\frac{1}{2} g_x(1) = \frac{-1}{2(x+1)^\lambda}$,

$$\begin{aligned} \int_0^1 g_x(t) dt &= \int_0^1 \frac{t^{\sigma-1}}{(x+t)^\lambda} dt = \frac{1}{\sigma} \int_0^1 \frac{dt^\sigma}{(x+t)^\lambda} = \frac{1}{\sigma} \frac{t^\sigma}{(x+t)^\lambda} \Big|_0^1 + \frac{\lambda}{\sigma} \int_0^1 \frac{t^\sigma dt}{(x+t)^{\lambda+1}} \\ &= \frac{1}{\sigma} \frac{1}{(x+1)^\lambda} + \frac{\lambda}{\sigma(\sigma+1)} \int_0^1 \frac{dt^{\sigma+1}}{(x+t)^{\lambda+1}} \end{aligned}$$

$$\begin{aligned}
&> \frac{1}{\sigma} \frac{1}{(x+1)^\lambda} + \frac{\lambda}{\sigma(\sigma+1)} \left[\frac{t^{\sigma+1}}{(x+t)^{\lambda+1}} \right]_0^1 + \frac{\lambda(\lambda+1)}{\sigma(\sigma+1)(x+1)^{\lambda+2}} \int_0^1 t^{\sigma+1} dt \\
&= \frac{1}{\sigma} \frac{1}{(x+1)^\lambda} + \frac{\lambda}{\sigma(\sigma+1)} \frac{1}{(x+1)^{\lambda+1}} + \frac{\lambda(\lambda+1)}{\sigma(\sigma+1)(\sigma+2)} \frac{1}{(x+1)^{\lambda+2}}, \\
-g'_x(t) &= -\frac{(\sigma-1)t^{\sigma-2}}{(x+t)^\lambda} + \frac{\lambda t^{\sigma-1}}{(x+t)^{\lambda+1}} = \frac{(1-\sigma)t^{\sigma-2}}{(x+t)^\lambda} + \frac{\lambda t^{\sigma-2}}{(x+t)^\lambda} - \frac{\lambda x t^{\sigma-2}}{(x+t)^{\lambda+1}} \\
&= \frac{(\lambda+1-\sigma)t^{\sigma-2}}{(x+t)^\lambda} - \frac{\lambda x t^{\sigma-2}}{(x+t)^{\lambda+1}}.
\end{aligned}$$

For $0 < \sigma \leq 2, \sigma < \lambda \leq 5$, we find

$$(-1)^i \frac{d^i}{dt^i} \left[\frac{t^{\sigma-2}}{(x+t)^\lambda} \right] > 0, \quad (-1)^i \frac{d^i}{dt^i} \left[\frac{t^{\sigma-2}}{(x+t)^{\lambda+1}} \right] > 0 \quad (t > 0; i = 0, 1, 2, 3),$$

and then by using the Euler-Maclaurin summation formula (cf., [20]), we find

$$\begin{aligned}
(\lambda+1-\sigma) \int_1^\infty \rho(t) \frac{t^{\sigma-2}}{(x+t)^\lambda} dt &> -\frac{\lambda+1-\sigma}{12(x+1)^\lambda}, \\
-x\lambda \int_1^\infty \rho(t) \frac{t^{\sigma-2}}{(x+t)^{\lambda+1}} dt &> \frac{x\lambda}{12(x+1)^{\lambda+1}} - \frac{x\lambda}{720} \left[\frac{t^{\sigma-2}}{(x+t)^{\lambda+1}} \right]'' \\
&> \frac{(x+1)\lambda-\lambda}{12(x+1)^{\lambda+1}} - \frac{(x+1)\lambda}{720} \left[\frac{(\lambda+1)(\lambda+2)}{(x+1)^{\lambda+3}} + \frac{2(\lambda+1)(2-\sigma)}{(x+1)^{\lambda+2}} + \frac{(2-\sigma)(3-\sigma)}{(x+1)^{\lambda+1}} \right] \\
&= \frac{\lambda}{12(x+1)^\lambda} - \frac{\lambda}{12(x+1)^{\lambda+1}} - \frac{\lambda}{720} \left[\frac{(\lambda+1)(\lambda+2)}{(x+1)^{\lambda+2}} + \frac{2(\lambda+1)(2-\sigma)}{(x+1)^{\lambda+1}} + \frac{(2-\sigma)(3-\sigma)}{(x+1)^\lambda} \right].
\end{aligned}$$

Hence, we have

$$h(x) > \frac{h_1}{(x+1)^\lambda} + \frac{\lambda h_2}{(x+1)^{\lambda+1}} + \frac{\lambda(\lambda+1)h_3}{(x+1)^{\lambda+2}},$$

where, $h_1 := \frac{1}{\sigma} - \frac{1}{2} - \frac{1-\sigma}{12} - \frac{\lambda(2-\sigma)(3-\sigma)}{720}$, $h_2 := \frac{1}{\sigma(\sigma+1)} - \frac{1}{12} - \frac{(\lambda+1)(2-\sigma)}{720}$, and

$$h_3 := \frac{1}{\sigma(\sigma+1)(\sigma+2)} - \frac{\lambda+2}{720}.$$

For $\lambda \in (0, 5]$, $\frac{\lambda}{720} < \frac{1}{24}$, $\sigma \in (0, 2]$, it follows that

$$h_1 > \frac{1}{\sigma} - \frac{1}{2} - \frac{1-\sigma}{12} - \frac{(2-\sigma)(3-\sigma)}{24} = \frac{24-20\sigma+7\sigma^2-\sigma^3}{24\sigma} > 0.$$

In fact, setting $g(\sigma) := 24 - 20\sigma + 7\sigma^2 - \sigma^3$ ($\sigma \in (0, 2]$), we obtain

$$g'(\sigma) = -20 + 14\sigma^2 - 3\sigma^2 = -3\left(\sigma - \frac{7}{3}\right)^2 - \frac{11}{3} < 0,$$

and then we obtain $h_1 > \frac{g(\sigma)}{24\sigma} \geq \frac{g(2)}{24\sigma} = \frac{4}{24\sigma} > 0$ ($\sigma \in (0, 2]$).

We observe that $h_2 > \frac{1}{6} - \frac{1}{12} - \frac{12}{360} = \frac{1}{20} > 0$, and $h_3 \geq \frac{1}{24} - \frac{7}{720} = \frac{23}{720} > 0$. Hence, we deduce that $h(x) > 0$, and thus we have

$$\begin{aligned}
\omega(\sigma, x) &= x^{\lambda-\sigma} \sum_{n=1}^{\infty} g_x(n) < x^{\lambda-\sigma} \int_0^\infty g_x(t) dt \\
&= x^{\lambda-\sigma} \int_0^\infty \frac{t^{\sigma-1} dt}{(x+t)^\lambda} = \int_0^\infty \frac{u^{\sigma-1} du}{(1+u)^\lambda} = B(\sigma, \lambda - \sigma).
\end{aligned}$$

On the other-hand, we also have

$$\begin{aligned}\sum_{n=1}^{\infty} g_x(n) &= \int_1^{\infty} g_x(t)dt + \frac{1}{2}g_x(1) + \int_1^{\infty} \rho(t)g'_x(t)dt \\ &= \int_1^{\infty} g_x(t)dt + H(x), \\ H(x) &:= \frac{1}{2}g_x(1) + \int_1^{\infty} \rho(t)g'_x(t)dt.\end{aligned}$$

We obtain $\frac{1}{2}g_x(1) = \frac{1}{2(x+1)^{\lambda}}$ and

$$g'_x(t) = \frac{-(\lambda+1-\sigma)t^{\sigma-2}}{(x+t)^{\lambda}} + \frac{\lambda xt^{\sigma-2}}{(x+t)^{\lambda+1}}.$$

For $\sigma \in (0, 2] \cap (0, \lambda)$, $0 < \lambda \leq 5$, by the Euler-Maclaurin summation formula, we obtain

$$\begin{aligned}& -(\lambda+1-\sigma) \int_1^{\infty} \rho(t) \frac{t^{\sigma-2}}{(x+t)^{\lambda}} dt > 0, \\ & x\lambda \int_1^{\infty} \rho(t) \frac{t^{\sigma-2}}{(x+t)^{\lambda+1}} dt \\ & > -\frac{x\lambda}{12(x+1)^{\lambda+1}} = -\frac{(x+1)\lambda-\lambda}{12(x+1)^{\lambda+1}} = \frac{-\lambda}{12(x+1)^{\lambda}} + \frac{\lambda}{12(x+1)^{\lambda+1}} > \frac{-\lambda}{12(x+1)^{\lambda}}.\end{aligned}$$

Hence, we have

$$H(x) > \frac{1}{2(x+1)^{\lambda}} - \frac{\lambda}{12(x+1)^{\lambda}} = \frac{6-\lambda}{12(x+1)^{\lambda}} > 0,$$

and then

$$\begin{aligned}\omega(\sigma, x) &= x^{\lambda-\sigma} \sum_{n=1}^{\infty} g_x(n) > x^{\lambda-\sigma} \int_1^{\infty} g_x(t)dt \\ &= x^{\lambda-\sigma} \int_0^{\infty} g_x(t)dt - x^{\lambda-\sigma} \int_0^1 g_x(t)dt \\ &= B(\sigma, \lambda-\sigma) \left[1 - \frac{1}{B(\sigma, \lambda-\sigma)} \int_0^{\frac{1}{x}} \frac{u^{\sigma-1}}{(1+u)^{\lambda}} du \right] > 0.\end{aligned}$$

By the integral mid-value theorem, we find

$$\int_0^{\frac{1}{x}} \frac{u^{\sigma-1}}{(1+u)^{\lambda}} du = \frac{1}{(1+\theta_x)^{\lambda}} \int_0^{\frac{1}{x}} u^{\sigma-1} du = \frac{1}{\sigma(1+\theta_x)^{\lambda}} \frac{1}{x^{\sigma}} (\theta_x \in (0, \frac{1}{x})).$$

This proves Inequality (5). \square

Lemma 2. The following reverse inequality is valid:

$$\begin{aligned}I &= \int_0^{\infty} \sum_{n=1}^{\infty} \frac{f(x)a_n}{(x+n)^{\lambda}} dx > B^{\frac{1}{p}}(\sigma, \lambda-\sigma) B^{\frac{1}{q}}(\mu, \lambda-\mu) \\ &\quad \times \left\{ \int_0^{\infty} (1-\rho_{\sigma}(x)) x^{p[1-(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})]-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{q[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1} a_n^q \right\}^{\frac{1}{q}}.\end{aligned}\tag{6}$$

Proof. For $n \in \mathbb{N}$, setting $x = nu$, we obtain the following weight function:

$$\omega(\mu, n) := n^{\lambda-\mu} \int_0^{\infty} \frac{x^{\mu-1} dx}{(x+n)^{\lambda}} = \int_0^{\infty} \frac{u^{\mu-1} du}{(u+1)^{\lambda}} = B(\mu, \lambda-\mu).\tag{7}$$

For $0 < p < 1, q < 0$, by the reverse Hölder's inequality (cf., [27]) and the Lebesgue term by term integration theorem (cf., [28]), we obtain

$$\begin{aligned} \int_0^\infty \sum_{n=1}^\infty \frac{f(x)a_n}{(x+n)^\lambda} dx &= \int_0^\infty \sum_{n=1}^\infty \frac{1}{(x+n)^\lambda} \left[\frac{n^{(\sigma-1)/p}}{x^{(\mu-1)/q}} f(x) \right] \left[\frac{x^{(\mu-1)/q}}{n^{(\sigma-1)/p}} a_n \right] dx \\ &\geq \left\{ \int_0^\infty \left[\sum_{n=1}^\infty \frac{1}{(x+n)^\lambda} \frac{n^{\sigma-1}}{x^{(\mu-1)(p-1)}} \right] f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty \left[\int_0^\infty \frac{1}{(x+n)^\lambda} \frac{x^{\mu-1}}{n^{(\sigma-1)(q-1)}} dx \right] a_n^q \right\}^{\frac{1}{q}} \\ &= \left\{ \int_0^\infty \omega(\sigma, x) x^{p[1-(\frac{\lambda-\sigma}{p} + \frac{\mu}{q})]-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty \omega(\mu, n) n^{q[1-(\frac{\sigma}{p} + \frac{\lambda-\mu}{q})]-1} a_n^q \right\}^{\frac{1}{q}}. \end{aligned}$$

Then by (5) and (7), we obtain Inequality (6). \square

Remark 1. For $\mu + \sigma = \lambda$, we find

$$\omega(\sigma, x) = x^\mu \sum_{n=1}^\infty \frac{n^{\sigma-1}}{(x+n)^\lambda} \quad (x \in \mathbb{R}_+),$$

$$0 < \int_0^\infty x^{p(1-\mu)-1} f^p(x) dx < \infty \text{ and } 0 < \sum_{n=1}^\infty n^{q(1-\sigma)-1} a_n^q < \infty,$$

and then we reduce (6) as follows:

$$\int_0^\infty \sum_{n=1}^\infty \frac{f(x)a_n}{(x+n)^\lambda} dx > B(\mu, \sigma) \left[\int_0^\infty (1 - \rho_\sigma(x)) x^{p(1-\mu)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty n^{q(1-\sigma)-1} a_n^q \right]^{\frac{1}{q}}. \quad (8)$$

Lemma 3. The constant factor $B(\mu, \sigma)$ in (8) is the best possible.

Proof. For $0 < \varepsilon < p\mu$, we set

$$\widetilde{f}(x) := \begin{cases} 0, & 0 < x < 1, \\ x^{\mu-\frac{\varepsilon}{p}-1}, & x \geq 1 \end{cases}, \quad \widetilde{a}_n := n^{\sigma-\frac{\varepsilon}{q}-1} \quad (n \in \mathbb{N}).$$

If there exists a positive constant $M (M \geq B(\mu, \sigma))$ such that (8) is valid when replacing $B(\mu, \sigma)$ by M , then by a substitution of $f(x) = \widetilde{f}(x), a_n = \widetilde{a}_n$, we get

$$\begin{aligned} \widetilde{I} &:= \int_0^\infty \sum_{n=1}^\infty \frac{\widetilde{f}(x)\widetilde{a}_n}{(x+n)^\lambda} dx > M \\ &\times \left[\int_0^\infty (1 - \rho_\sigma(x)) x^{p(1-\mu)-1} \widetilde{f}^p(x) dx \right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty n^{q(1-\sigma)-1} \widetilde{a}_n^q \right]^{\frac{1}{q}} \\ &= M \left(\int_1^\infty (1 - O(\frac{1}{x^\sigma})) x^{-\varepsilon-1} dx \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty n^{-\varepsilon-1} \right)^{\frac{1}{q}} \\ &\geq M \left(\int_1^\infty x^{-\varepsilon-1} dx - \int_1^\infty O(\frac{1}{x^{\sigma+\varepsilon+1}}) dx \right)^{\frac{1}{p}} \left(\int_1^\infty x^{-\varepsilon-1} dx \right)^{\frac{1}{q}} \\ &= \frac{M}{\varepsilon} (1 - \varepsilon O(1))^{\frac{1}{p}}. \end{aligned}$$

For $\mu - \frac{\varepsilon}{p} > 0$ ($0 < p < 1$), by (7), we obtain

$$\begin{aligned}\widetilde{I} &= \sum_{n=1}^{\infty} n^{-\varepsilon-1} [n^{(\sigma+\frac{\varepsilon}{p})} \int_1^{\infty} \frac{x^{(\mu-\frac{\varepsilon}{p})-1}}{(x+n)^{\lambda}} dx] \leq \sum_{n=1}^{\infty} n^{-\varepsilon-1} [n^{(\sigma+\frac{\varepsilon}{p})} \int_0^{\infty} \frac{x^{(\mu-\frac{\varepsilon}{p})-1}}{(x+n)^{\lambda}} dx] \\ &= \sum_{n=1}^{\infty} n^{-\varepsilon-1} \omega(\mu - \frac{\varepsilon}{p}, n) = B(\mu - \frac{\varepsilon}{p}, \sigma + \frac{\varepsilon}{p}) (1 + \sum_{n=2}^{\infty} n^{-\varepsilon-1}) \\ &\leq B(\mu - \frac{\varepsilon}{p}, \sigma + \frac{\varepsilon}{p}) (1 + \int_1^{\infty} x^{-\varepsilon-1} dx) = \frac{\varepsilon+1}{\varepsilon} B(\mu - \frac{\varepsilon}{p}, \sigma + \frac{\varepsilon}{p}).\end{aligned}$$

Then we have

$$(\varepsilon + 1) B(\mu - \frac{\varepsilon}{p}, \sigma + \frac{\varepsilon}{p}) \geq \varepsilon \widetilde{I} \geq M(1 - \varepsilon O(1))^{\frac{1}{p}}.$$

For $\varepsilon \rightarrow 0^+$, in view of the continuity of the beta function, it follows that $B(\mu, \sigma) \geq M$. Therefore, $M = B(\mu, \sigma)$ is the best possible constant factor of (8). Lemma 3 is proved. \square

Remark 2. Setting $\hat{\mu} := \frac{\lambda-\sigma}{p} + \frac{\mu}{q}$, $\hat{\sigma} := \frac{\sigma}{p} + \frac{\lambda-\mu}{q}$, we have

$$\hat{\mu} + \hat{\sigma} = \frac{\lambda-\sigma}{p} + \frac{\mu}{q} + \frac{\sigma}{p} + \frac{\lambda-\mu}{q} = \frac{\lambda}{p} + \frac{\lambda}{q} = \lambda,$$

and for $\lambda - \mu - \sigma \in (-p\mu, p(\lambda - \mu))$, we find

$$\begin{aligned}\hat{\mu} &> \frac{(1-p)\mu}{p} + \frac{\mu}{q} = 0, \hat{\mu} < \frac{\mu+p(\lambda-\mu)}{p} + \frac{\mu}{q} = \lambda, \\ 0 &< \hat{\sigma} = \lambda - \hat{\mu} < \lambda, B(\hat{\mu}, \hat{\sigma}) \in \mathbb{R}_+.\end{aligned}$$

We can reduce (6) to the following

$$\begin{aligned}\int_0^{\infty} \sum_{n=1}^{\infty} \frac{f(x)a_n}{(x+n)^{\lambda}} dx &> B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu) \\ &\times [\int_0^{\infty} (1 - \rho_{\sigma}(x)) x^{p(1-\hat{\mu})-1} f^p(x) dx]^{\frac{1}{p}} [\sum_{n=1}^{\infty} n^{q(1-\hat{\sigma})-1} a_n^q]^{\frac{1}{q}}.\end{aligned}\tag{9}$$

Lemma 4. If $\lambda - \mu - \sigma \in (-p\mu, p(\lambda - \mu))$, the constant factor $B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu)$ in (9) is the best possible, then we have $\mu + \sigma = \lambda$.

Proof. If the constant factor $B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu)$ in (9) is the best possible, then by (8), the unique best possible constant factor must be $B(\hat{\mu}, \hat{\sigma}) \in \mathbb{R}_+$, namely,

$$B(\hat{\mu}, \hat{\sigma}) = B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu).$$

By the reverse Hölder's inequality (cf., [27]), we find

$$\begin{aligned}B(\hat{\mu}, \hat{\sigma}) &= \int_0^{\infty} \frac{t^{\hat{\mu}-1}}{(1+t)^{\lambda}} dt = \int_0^{\infty} \frac{t^{\frac{\lambda-\sigma}{p} + \frac{\mu}{q} - 1}}{(1+t)^{\lambda}} dt = \int_0^{\infty} \frac{1}{(1+t)^{\lambda}} (t^{\frac{\lambda-\sigma-1}{p}}) (t^{\frac{\mu-1}{q}}) dt \\ &\geq [\int_0^{\infty} \frac{1}{(1+t)^{\lambda}} t^{\lambda-\sigma-1} dt]^{\frac{1}{p}} [\int_0^{\infty} \frac{1}{(1+t)^{\lambda}} t^{\mu-1} dt]^{\frac{1}{q}} \\ &= B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu).\end{aligned}\tag{10}$$

We observe that (10) keeps the form of equality if and only if there exist constants A, B such that they are not all zero and

$$At^{\lambda-\sigma-1} = Bt^{\mu-1} \text{ a.e. in } \mathbb{R}_+.$$

Suppose that $A \neq 0$. We find that $t^{\lambda-\mu-\sigma} = \frac{B}{A}$ a.e. in \mathbb{R}_+ , and thus we conclude that $\lambda - \mu - \sigma = 0$, i.e., $\mu + \sigma = \lambda$. Lemma 4 is proved. \square

3. Main Results

Theorem 1. Inequality (6) is equivalent to the following inequalities:

$$J_1 := \left\{ \sum_{n=1}^{\infty} n^{p(\frac{\sigma}{p} + \frac{\lambda-\mu}{q})-1} \left[\int_0^{\infty} \frac{f(x)}{(x+n)^{\lambda}} dx \right]^p \right\}^{\frac{1}{p}} > B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu) \left\{ \int_0^{\infty} (1 - \rho_{\sigma}(x)) x^{p[1 - (\frac{\lambda-\sigma}{p} + \frac{\mu}{q})]-1} f^p(x) dx \right\}^{\frac{1}{p}}, \quad (11)$$

$$J_2 := \left\{ \int_0^{\infty} \frac{x^{q(\frac{\lambda-\sigma}{p} + \frac{\mu}{q})-1}}{(1 - \rho_{\sigma}(x))^{q-1}} \left[\sum_{n=1}^{\infty} \frac{a_n}{(x+n)^{\lambda}} \right]^q dx \right\}^{\frac{1}{q}} > B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu) \left\{ \sum_{n=1}^{\infty} n^{q[1 - (\frac{\sigma}{p} + \frac{\lambda-\mu}{q})]-1} a_n^q \right\}^{\frac{1}{q}}. \quad (12)$$

If the constant factor $B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu)$ in (6) is the best possible, then so is the constant factor in (11) and (12).

In particular, for $\mu + \sigma = \lambda$ in (6), (11) and (12), we have Inequality (8) and the following equivalent versions of reverse inequalities with the best possible constant factor $B(\mu, \sigma)$:

$$\left\{ \sum_{n=1}^{\infty} n^{p\sigma-1} \left[\int_0^{\infty} \frac{f(x)}{(x+n)^{\lambda}} dx \right]^p \right\}^{\frac{1}{p}} > B(\mu, \sigma) \left[\int_0^{\infty} (1 - \rho_{\sigma}(x)) x^{p(1-\mu)-1} f^p(x) dx \right]^{\frac{1}{p}}, \quad (13)$$

$$\left\{ \int_0^{\infty} \frac{x^{q\mu-1}}{(1 - \rho_{\sigma}(x))^{q-1}} \left[\sum_{n=1}^{\infty} \frac{a_n}{(x+n)^{\lambda}} \right]^q dx \right\}^{\frac{1}{q}} > B(\mu, \sigma) \left[\sum_{n=1}^{\infty} n^{q(1-\sigma)-1} a_n^q \right]^{\frac{1}{q}}. \quad (14)$$

Proof. Suppose that (11) is valid. By the Lebesgue term by term integration theorem and the reverse Hölder's inequality (cf., [27,28]), we have

$$I = \sum_{n=1}^{\infty} \int_0^{\infty} \frac{f(x)a_n}{(x+n)^{\lambda}} dx = \sum_{n=1}^{\infty} \left[n^{\frac{-1}{p} + (\frac{\sigma}{p} + \frac{\lambda-\mu}{q})} \int_0^{\infty} \frac{f(x)}{(x+n)^{\lambda}} dx \right] \left[n^{\frac{1}{p} - (\frac{\sigma}{p} + \frac{\lambda-\mu}{q})} a_n \right] \geq J_1 \left\{ \sum_{n=1}^{\infty} n^{q[1 - (\frac{\sigma}{p} + \frac{\lambda-\mu}{q})]-1} a_n^q \right\}^{\frac{1}{q}}. \quad (15)$$

Then by (11), we have Inequality (6). On the other-hand, assuming that Inequality (6) is valid, we set

$$a_n := n^{p(\frac{\sigma}{p} + \frac{\lambda-\mu}{q})-1} \left[\int_0^{\infty} \frac{f(x)}{(x+n)^{\lambda}} dx \right]^{p-1}, n \in \mathbb{N}.$$

If $J_1 = \infty$, then Inequality (11) is naturally valid; if $J_1 = 0$, so it is impossible to make Inequality (11) valid, namely $J_1 > 0$. Suppose that $0 < J_1 < \infty$. By (6), we have

$$\sum_{n=1}^{\infty} n^{q[1 - (\frac{\sigma}{p} + \frac{\lambda-\mu}{q})]-1} a_n^q = J_1^p = I > B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu) \times \left\{ \int_0^{\infty} (1 - \rho_{\sigma}(x)) x^{p[1 - (\frac{\lambda-\sigma}{p} + \frac{\mu}{q})]-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{q[1 - (\frac{\sigma}{p} + \frac{\lambda-\mu}{q})]-1} a_n^q \right\}^{\frac{1}{q}},$$

$$\left\{ \sum_{n=1}^{\infty} n^{q[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1} a_n^q \right\}^{\frac{1}{p}} = J_1 > B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu) \\ \times \left\{ \int_0^{\infty} (1 - \rho_{\sigma}(x)) x^{p[1-(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})]-1} f^p(x) dx \right\}^{\frac{1}{p}},$$

namely, Inequality (11) follows, which is equivalent to Inequality (6).

Suppose that Inequality (12) is valid. By the reverse Hölder's inequality, we have

$$I = \int_0^{\infty} [(1 - \rho_{\sigma}(x))^{\frac{1}{p}} x^{\frac{1}{q}-(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})} f(x)] \left[\frac{x^{-\frac{1}{q}+(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})}}{(1-\rho_{\sigma}(x))^{1/p}} \sum_{n=1}^{\infty} \frac{a_n}{(x+n)^{\lambda}} \right] dx \\ \geq \left\{ \int_0^{\infty} (1 - \rho_{\sigma}(x)) x^{p[1-(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})]-1} f^p(x) dx \right\}^{\frac{1}{p}} J_2. \quad (16)$$

Then by (12), we obtain Inequality (6). On the other-hand, assuming that Inequality (6) is valid, we set

$$f(x) := x^{q(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})-1} \left[\sum_{n=1}^{\infty} \frac{a_n}{(x+n)^{\lambda}} \right]^{q-1}, x \in \mathbb{R}_+$$

If $J_2 = \infty$, then Inequality (12) is naturally valid; if $J_2 = 0$, then it is impossible to make Inequality (12) valid, namely $J_2 > 0$. Suppose that $0 < J_2 < \infty$. By (6), we have

$$\int_0^{\infty} (1 - \rho_{\sigma}(x)) x^{p[1-(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})]-1} f^p(x) dx = J_2^q = I > B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu) \\ \times \left\{ \int_0^{\infty} (1 - \rho_{\sigma}(x)) x^{p[1-(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})]-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{q[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1} a_n^q \right\}^{\frac{1}{q}}, \\ \left\{ \int_0^{\infty} (1 - \rho_{\sigma}(x)) x^{p[1-(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})]-1} f^p(x) dx \right\}^{\frac{1}{q}} = J_2 > B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu) \\ \times \left\{ \sum_{n=1}^{\infty} n^{q[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1} a_n^q \right\}^{\frac{1}{q}},$$

namely, Inequality (12) follows, which is equivalent to Inequality (6).

Hence, Inequalities (6), (11) and (12) are equivalent.

If the constant factor $B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu)$ in (6) is the best possible, then so is the constant factor in (11) and (12). Otherwise, by (15) (or (16)), we would reach a contradiction that the constant factor in (6) is not the best possible. This completes the proof of Theorem 1. \square

Theorem 2. The following statements (i), (ii), (iii) and (iv) are equivalent.

- (i) $B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu)$ is independent of p, q ;
- (ii) $B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu)$ is expressible as a single integral;
- (iii) $B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu)$ is the best possible of (6);
- (iv) If $\lambda - \mu - \sigma \in (-p\mu, p(\lambda - \mu))$, then $\mu + \sigma = \lambda$.

Proof. (i) \Rightarrow (ii). In view of $B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu)$ is independent of p, q , we find

$$B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu) \\ = \lim_{\substack{p \rightarrow 1^-, \\ q \rightarrow -\infty}} B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu) = B(\sigma, \lambda - \sigma),$$

which is a single integral $\int_0^{\infty} \frac{t^{\sigma-1}}{(1+t)^{\lambda}} dt$.

(ii) \Rightarrow (iv). Suppose that $B^{\frac{1}{p}}(\sigma, \lambda - \sigma)B^{\frac{1}{q}}(\mu, \lambda - \mu)$ is expressible as a single integral $\int_0^\infty \frac{t^{\frac{\lambda-\sigma}{p} + \frac{\mu}{q} - 1}}{(1+t)^\lambda} dt$. Then (10) keeps the form of equality. By the proof of Lemma 4, for $\lambda - \mu - \sigma \in (-p\mu, p(\lambda - \mu))$, we have $\mu + \sigma = \lambda$.

(iv) \Rightarrow (i). If $\mu + \sigma = \lambda$, then

$$B^{\frac{1}{p}}(\sigma, \lambda - \sigma)B^{\frac{1}{q}}(\mu, \lambda - \mu) = B(\mu, \sigma),$$

which is independent of p, q .

Hence, (i) \Leftrightarrow (ii) \Leftrightarrow (iv).

(vi) \Rightarrow (iii). By Lemma 3, for $\mu + \sigma = \lambda$, $B^{\frac{1}{p}}(\sigma, \lambda - \sigma)B^{\frac{1}{q}}(\mu, \lambda - \mu)$ is the best possible of (6).

(iii) \Rightarrow (iv). By Lemma 4, we have $\mu + \sigma = \lambda$.

Therefore, we show that (iv) \Leftrightarrow (iii), and then the statements (i), (ii), (iii) and (iv) are equivalent. The proof Theorem 2 is complete. \square

4. Two Corollaries and Some Particular Inequalities

Replacing x by $\frac{1}{x}$, and then setting $F(x) = x^{\lambda-2}f(\frac{1}{x})$ in Theorems 1 and 2, we find

$$\rho_\sigma(x^{-1}) = \frac{(1 + \theta_{x^{-1}})^{-\lambda}}{\sigma B(\sigma, \lambda - \sigma)} x^\sigma = O(x^\sigma) \in (0, 1) (\theta_{x^{-1}} \in (0, x); x > 0),$$

and obtain the following corollaries:

Corollary 1. If $F(x)$, $a_n \geq 0$ such that

$$0 < \int_0^\infty x^{p[1-(\frac{\sigma}{p} + \frac{\lambda-\mu}{q})]-1} F^p(x) dx < \infty \text{ and } 0 < \sum_{n=1}^\infty n^{q[1-(\frac{\sigma}{p} + \frac{\lambda-\mu}{q})]-1} a_n^q < \infty,$$

then the following inequalities are equivalent:

$$\begin{aligned} \int_0^\infty \sum_{n=1}^\infty \frac{F(x)a_n}{(1+xn)^\lambda} dx &> B^{\frac{1}{p}}(\sigma, \lambda - \sigma)B^{\frac{1}{q}}(\mu, \lambda - \mu) \\ &\times \left\{ \int_0^\infty (1 - \rho_\sigma(x^{-1})) x^{p[1-(\frac{\sigma}{p} + \frac{\lambda-\mu}{q})]-1} F^p(x) dx \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{q[1-(\frac{\sigma}{p} + \frac{\lambda-\mu}{q})]-1} a_n^q \right\}^{\frac{1}{q}}, \end{aligned} \quad (17)$$

$$\begin{aligned} &\left\{ \sum_{n=1}^\infty n^{p(\frac{\sigma}{p} + \frac{\lambda-\mu}{q})-1} \left[\int_0^\infty \frac{F(x)}{(1+xn)^\lambda} dx \right]^p \right\}^{\frac{1}{p}} \\ &> B^{\frac{1}{p}}(\sigma, \lambda - \sigma)B^{\frac{1}{q}}(\mu, \lambda - \mu) \left\{ \int_0^\infty (1 - \rho_\sigma(x^{-1})) x^{p[1-(\frac{\sigma}{p} + \frac{\lambda-\mu}{q})]-1} F^p(x) dx \right\}^{\frac{1}{p}}, \end{aligned} \quad (18)$$

$$\begin{aligned} &\left\{ \int_0^\infty \frac{x^{q(\frac{\sigma}{p} + \frac{\lambda-\mu}{q})-1}}{(1-\rho_\sigma(x^{-1}))^{q-1}} \left[\sum_{n=1}^\infty \frac{a_n}{(1+xn)^\lambda} \right]^q dx \right\}^{\frac{1}{q}} \\ &> B^{\frac{1}{p}}(\sigma, \lambda - \sigma)B^{\frac{1}{q}}(\mu, \lambda - \mu) \left\{ \sum_{n=1}^\infty n^{q[1-(\frac{\sigma}{p} + \frac{\lambda-\mu}{q})]-1} a_n^q \right\}^{\frac{1}{q}}. \end{aligned} \quad (19)$$

If the constant factor $B^{\frac{1}{p}}(\sigma, \lambda - \sigma)B^{\frac{1}{q}}(\mu, \lambda - \mu)$ in (17) is the best possible, then so is the constant factor in (18) and (19).

In particular, for $\mu = \lambda - \sigma$ in (17), (18) and (19), we have the following equivalent inequalities with the best possible constant factor $B(\lambda - \sigma, \sigma)$:

$$\int_0^\infty \sum_{n=1}^\infty \frac{F(x)a_n}{(1+xn)^\lambda} dx > B(\lambda - \sigma, \sigma) \times \left[\int_0^\infty (1 - \rho_\sigma(x^{-1}))x^{p(1-\sigma)-1}F^p(x)dx \right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty n^{q(1-\sigma)-1}a_n^q \right]^{\frac{1}{q}}, \quad (20)$$

$$\left\{ \sum_{n=1}^\infty n^{p\sigma-1} \left[\int_0^\infty \frac{F(x)}{(1+xn)^\lambda} dx \right]^p \right\}^{\frac{1}{p}} > B(\lambda - \sigma, \sigma) \left[\int_0^\infty (1 - \rho_\sigma(x^{-1}))x^{p(1-\sigma)-1}F^p(x)dx \right]^{\frac{1}{p}}, \quad (21)$$

$$\left\{ \int_0^\infty \frac{x^{q\sigma-1}}{(1-\rho(x^{-1}))^{q-1}} \left[\sum_{n=1}^\infty \frac{a_n}{(1+xn)^\lambda} \right]^q dx \right\}^{\frac{1}{q}} > B(\lambda - \sigma, \sigma) \left[\sum_{n=1}^\infty n^{q(1-\sigma)-1}a_n^q \right]^{\frac{1}{q}}. \quad (22)$$

Corollary 2. The following statements (i), (ii), (iii) and (iv) are equivalent:

- (i) $B^{\frac{1}{p}}(\sigma, \lambda - \sigma)B^{\frac{1}{q}}(\mu, \lambda - \mu)$ is independent of p, q ;
- (ii) $B^{\frac{1}{p}}(\sigma, \lambda - \sigma)B^{\frac{1}{q}}(\mu, \lambda - \mu)$ is expressible as a single integral;
- (iii) $B^{\frac{1}{p}}(\sigma, \lambda - \sigma)B^{\frac{1}{q}}(\mu, \lambda - \mu)$ is the best possible of (17);
- (iv) If $\lambda - \mu - \sigma \in (-q\sigma, q(\lambda - \sigma))$, then we have $\mu = \lambda - \sigma$.

Remark 3. (i) For $\sigma = 2 < \lambda (\leq 5)$, $\mu = \lambda - 2$ in (8), (13) and (14), since

$$B(\lambda - 2, 2) = \frac{\Gamma(\lambda - 2)\Gamma(2)}{\Gamma(\lambda)} = \frac{\Gamma(\lambda - 2)}{(\lambda - 1)(\lambda - 2)\Gamma(\lambda - 2)} = \frac{1}{(\lambda - 1)(\lambda - 2)},$$

$$\rho_2(x) = \frac{(\lambda - 1)(\lambda - 2)}{2(1 + \theta_x)^\lambda} \frac{1}{x^2} = O\left(\frac{1}{x^2}\right) \in (0, 1) (\theta_x \in (0, \frac{1}{x}); x > 0),$$

we have the following equivalent versions of reverse inequalities with the best possible constant factor $\frac{1}{(\lambda-1)(\lambda-2)}$:

$$\int_0^\infty \sum_{n=1}^\infty \frac{f(x)a_n}{(x+n)^\lambda} dx > \frac{1}{(\lambda - 1)(\lambda - 2)} \left[\int_0^\infty (1 - \rho_2(x))x^{p(3-\lambda)-1}f^p(x)dx \right]^{\frac{1}{p}} \left(\sum_{n=1}^\infty n^{-q-1}a_n^q \right)^{\frac{1}{q}}, \quad (23)$$

$$\left\{ \sum_{n=1}^\infty n^{2p-1} \left[\int_0^\infty \frac{f(x)}{(x+n)^\lambda} dx \right]^p \right\}^{\frac{1}{p}} > \frac{1}{(\lambda - 1)(\lambda - 2)} \left[\int_0^\infty (1 - \rho_2(x))x^{p(3-\lambda)-1}f^p(x)dx \right]^{\frac{1}{p}}, \quad (24)$$

$$\left\{ \int_0^\infty \frac{x^{q(\lambda-2)-1}}{(1 - \rho_2(x))^{q-1}} \left[\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda} \right]^q dx \right\}^{\frac{1}{q}} > \frac{1}{(\lambda - 1)(\lambda - 2)} \left(\sum_{n=1}^\infty n^{-q-1}a_n^q \right)^{\frac{1}{q}}. \quad (25)$$

(ii) For $\sigma = 2 < \lambda (\leq 5)$, $\mu = \lambda - 2$ in (20), (21) and (22), we have

$$\rho_2(x^{-1}) = \frac{(\lambda - 1)(\lambda - 2)}{2(1 + \theta_{x^{-1}})^\lambda} x^2 = O(x^2) \in (0, 1) (\theta_{x^{-1}} \in (0, x); x > 0),$$

and the following equivalent versions of reverse inequalities with the best possible constant factor $\frac{1}{(\lambda-1)(\lambda-2)}$:

$$\int_0^\infty \sum_{n=1}^\infty \frac{F(x)a_n}{(1+xn)^\lambda} dx > \frac{1}{(\lambda - 1)(\lambda - 2)} \left(\int_0^\infty (1 - \rho_2(x^{-1}))x^{-p-1}F^p(x)dx \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty n^{-q-1}a_n^q \right)^{\frac{1}{q}}, \quad (26)$$

$$\left\{ \sum_{n=1}^{\infty} n^{2p-1} \left[\int_0^{\infty} \frac{F(x)}{(1+xn)^{\lambda}} dx \right]^p \right\}^{\frac{1}{p}} > \frac{1}{(\lambda-1)(\lambda-2)} \left(\int_0^{\infty} (1-\rho_2(x^{-1})) x^{-p-1} F^p(x) dx \right)^{\frac{1}{p}}, \quad (27)$$

$$\left\{ \int_0^{\infty} \frac{x^{2q-1}}{(1-\rho_2(x^{-1}))^{q-1}} \left[\sum_{n=1}^{\infty} \frac{a_n}{(1+xn)^{\lambda}} \right]^q dx \right\}^{\frac{1}{q}} \quad (28)$$

5. Conclusions

Let us give a brief summary of this paper, by the way of [20,22] and the use of the weight functions, the idea of introducing parameters and the Euler-Maclaurin summation formula, a reverse half-discrete Hardy-Hilbert's inequality and the reverse equivalent forms are given in Lemma 2 and Theorem 1. The equivalent statements of the best possible constant factor related to some parameters are proved in Theorem 2. As applications, two corollaries about the reverse cases of the non-homogeneous kernel and some particular cases are considered in Corollaries 1, 2 and Remark 3. The above-mentioned lemmas and theorems reveal some essential characters of this type of Hardy-Hilbert inequality.

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