


Bipolar Fuzzy Relations

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Abstract: We introduce the concepts of a bipolar fuzzy reflexive, symmetric, and transitive relation. We study bipolar fuzzy analogues of many results concerning relationships between ordinary reflexive, symmetric, and transitive relations. Next, we define the concepts of a bipolar fuzzy equivalence class and a bipolar fuzzy partition, and we prove that the set of all bipolar fuzzy equivalence classes is a bipolar fuzzy partition and that the bipolar fuzzy equivalence relation is induced by a bipolar fuzzy partition. Finally, we define an (a, b) -level set of a bipolar fuzzy relation and investigate some relationships between bipolar fuzzy relations and their (a, b) -level sets.

Keywords: bipolar fuzzy relation; bipolar fuzzy reflexive (resp., symmetric and transitive) relation; bipolar fuzzy equivalence relation; bipolar fuzzy partition; (a, b) -level set

1. Introduction

In 1965, Zadeh [1] introduced the concept of a fuzzy set as the generalization of a crisp set. In 1971, he [2] defined the notions of similarity relations and fuzzy orderings as the generalizations of crisp equivalence relations and partial orderings playing basic roles in many fields of pure and applied science. After that time, many researchers [3–11] studied fuzzy relations. Dib and Youssef [7] defined the fuzzy Cartesian product of two ordinary sets X and Y as the collection of all L -fuzzy sets of $X \times Y$, where $L = I \times I$ and I denotes the unit closed interval. In particular, Lee [10] obtained many results by using the notion of fuzzy relations introduced by Dib and Youssef.

In 1994, Zhang [12] introduced the notion of a bipolar fuzzy set (refer to [13–15]). After then, Jun and Park [16], Jun et al. [17], and Lee [18] applied bipolar fuzzy sets to BCK/BCI -algebras. Moreover, Akram and Dudek [19] studied bipolar fuzzy graph, and Majumder [20] introduced the bipolar fuzzy Γ -semigroup. Moreover, Talebi et al. [21] investigated operations on a bipolar fuzzy graph. Azhagappan and Kamaraj [22] dealt with some properties of rw -closed sets and rw -open sets in bipolar fuzzy topological spaces. Recently, Kim et al. [23] constructed the category consisting of bipolar fuzzy set and preserving mappings between them and studied this in the sense of a topological universe. Lee et al. [24] found some properties of bases, neighborhoods, and continuities in bipolar fuzzy topological spaces. In particular, Dudziak and Pekala [25] referred to an intuitionistic fuzzy relation as a bipolar fuzzy relation and investigated some properties of equivalent bipolar fuzzy relations.

In this paper, first, we introduce a bipolar fuzzy relation from a set X to Y and the composition of two bipolar fuzzy relations. Furthermore, we introduce some operations between bipolar fuzzy relations and obtain some of their properties. Second, we define a bipolar fuzzy reflexive, symmetric, and transitive relation and find bipolar fuzzy analogues of many results concerning relationships between ordinary reflexive, symmetric, and transitive relations. Third, we define the concepts of a

bipolar fuzzy equivalence class and a bipolar fuzzy partition, and we prove that the set of all bipolar fuzzy equivalence classes is a bipolar fuzzy partition and that the bipolar fuzzy equivalence relation is induced by a bipolar fuzzy partition. Finally, we define an (a, b) -level set of a bipolar fuzzy relation and investigate some relationships between bipolar fuzzy relations and their (a, b) -level sets.

2. Preliminaries

In this section, we introduce the concept of the bipolar fuzzy set, the complement of a bipolar fuzzy set, the inclusion between two bipolar fuzzy sets, and the union and the intersection of two bipolar fuzzy sets. Furthermore, we introduce the intersection and union of arbitrary bipolar fuzzy sets and list some properties.

Definition 1. ([13]). Let X be a nonempty set. Then, a pair $A = (A^-, A^+)$ is called a bipolar-valued fuzzy set (or bipolar fuzzy set) in X , if $A^+ : X \rightarrow [0, 1]$ and $A^- : X \rightarrow [-1, 0]$ are mappings.

In particular, the empty fuzzy empty set (resp. the bipolar fuzzy whole set) (see [22]), denoted by $\mathbf{0}_{bp} = (\mathbf{0}_{bp}^-, \mathbf{0}_{bp}^+)$ (resp. $\mathbf{1}_{bp} = (\mathbf{1}_{bp}^-, \mathbf{1}_{bp}^+)$), is a bipolar fuzzy set in X defined by: for each $x \in X$,

$$\mathbf{0}_{bp}^+(x) = 0 = \mathbf{0}_{bp}^-(x) \text{ (resp. } \mathbf{1}_{bp}^+(x) = 1 \text{ and } \mathbf{1}_{bp}^-(x) = -1).$$

We will denote the set of all bipolar fuzzy sets in X as $BPF(X)$.

For each $x \in X$, we use the positive membership degree $A^+(x)$ to denote the satisfaction degree of the element x to the property corresponding to the bipolar fuzzy set A and the negative membership degree $A^-(x)$ to denote the satisfaction degree of the element x to some implicit counter-property corresponding to the bipolar fuzzy set A .

If $A^+(x) \neq 0$ and $A^-(x) = 0$, then it is the situation that x is regarded as having only positive satisfaction for A . If $A^+(x) = 0$ and $A^-(x) \neq 0$, then it is the situation that x does not satisfy the property of A , but somewhat satisfies the counter-property of A . It is possible for some $x \in X$ to be such that $A^+(x) \neq 0$ and $A^-(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of X .

It is obvious that for each $A \in BPF(X)$ and $x \in X$, if $0 \leq A^+(x) - A^-(x) \leq 1$, then A is an intuitionistic fuzzy set introduced by Atanassov [26]. In fact, $A^+(x)$ (resp. $-A^-(x)$) denotes the membership degree (resp. non-membership degree) of x to A .

Definition 2. ([13]). Let X be a nonempty set, and let $A, B \in BPF(X)$.

- (i) We say that A is a subset of B , denoted by $A \subset B$, if for each $x \in X$,

$$A^+(x) \leq B^+(x) \text{ and } A^-(x) \geq B^-(x).$$

- (ii) The complement of A , denoted by $A^c = ((A^c)^-, (A^c)^+)$, is a bipolar fuzzy set in X defined as: for each $x \in X$, $A^c(x) = (-1 - A^+(x), 1 - A^-(x))$, i.e.,

$$(A^c)^+(x) = 1 - A^-(x), (A^c)^-(x) = -1 - A^+(x).$$

- (iii) The intersection of A and B , denoted by $A \cap B$, is a bipolar fuzzy set in X defined as: for each $x \in X$,

$$(A \cap B)(x) = (A^-(x) \vee B^-(x), A^+(x) \wedge B^+(x)).$$

- (iv) The union of A and B , denoted by $A \cup B$, is a bipolar fuzzy set in X defined as: for each $x \in X$,

$$(A \cup B)(x) = (A^-(x) \wedge B^-(x), A^+(x) \vee B^+(x)).$$

Definition 3. (see [13,22]). Let X be a nonempty set, and let $A, B \in \text{BPF}(X)$. We say that A is equal to B , denoted by $A = B$, if $A \subset B$ and $B \subset A$.

Result 1. ([23], Proposition 3.5). Let $A, B, C \in \text{BPF}(X)$. Then:

- (1) (Idempotent laws): $A \cup A = A, A \cap A = A$,
- (2) (Commutative laws): $A \cup B = B \cup A, A \cap B = B \cap A$,
- (3) (Associative laws): $A \cup (B \cup C) = (A \cup B) \cup C, A \cap (B \cap C) = (A \cap B) \cap C$,
- (4) (Distributive laws): $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
 $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
- (5) (Absorption laws): $A \cup (A \cap B) = A, A \cap (A \cup B) = A$.
- (6) (De Morgan's laws): $(A \cup B)^c = A^c \cap B^c, (A \cap B)^c = A^c \cup B^c$,
- (7) $(A^c)^c = A$,
- (8) $A \cap B \subset A$ and $A \cap B \subset B$,
- (9) $A \subset A \cup B$ and $B \subset A \cup B$,
- (10) if $A \subset B$ and $B \subset C$, then $A \subset C$,
- (11) if $A \subset B$, then $A \cap C \subset B \cap C$ and $A \cup C \subset B \cup C$.

Definition 4. ([23]). Let X be a nonempty set, and let $(A_j)_{j \in J} \subset \text{BPF}(X)$.

- (i) The intersection of $(A_j)_{j \in J}$, denoted by $\bigcap_{j \in J} A_j$, is a bipolar fuzzy set in X defined by: for each $x \in X$,

$$\left(\bigcap_{j \in J} A_j\right)(x) = \left(\bigvee_{j \in J} A_j^-(x), \bigwedge_{j \in J} A_j^+(x)\right).$$

- (ii) The union of $(A_j)_{j \in J}$, denoted by $\bigcup_{j \in J} A_j$, is a bipolar fuzzy set in X defined by: for each $x \in X$,

$$\left(\bigcup_{j \in J} A_j\right)(x) = \left(\bigwedge_{j \in J} A_j^-(x), \bigvee_{j \in J} A_j^+(x)\right).$$

Result 2. ([23], Proposition 3.8). Let $A \in \text{BPF}(X)$, and let $(A_j)_{j \in J} \subset \text{BPF}(X)$. Then:

- (1) (Generalized distributive laws): $A \cup \left(\bigcap_{j \in J} A_j\right) = \bigcap_{j \in J} (A \cup A_j)$,
 $A \cap \left(\bigcup_{j \in J} A_j\right) = \bigcup_{j \in J} (A \cap A_j)$,
- (2) (Generalized De Morgan's laws): $\left(\bigcup_{j \in J} A_j\right)^c = \bigcap_{j \in J} A_j^c, \left(\bigcap_{j \in J} A_j\right)^c = \bigcup_{j \in J} A_j^c$.

From Results 1 and 2, it is obvious that $(\text{BPF}(X), \cup, \cap, ^c, \mathbf{0}_{bp}, \mathbf{1}_{bp})$ is a complete distributive lattice satisfying De Morgan's laws.

3. Bipolar Fuzzy Relations

In this section, we introduce the concepts of the bipolar fuzzy relation, the composition of two bipolar fuzzy relations, and the inverse of a bipolar fuzzy relation and study some their properties.

Throughout this paper, X, Y, Z denote ordinary non-empty sets, and we define the union, the intersection, and the composition between bipolar fuzzy relations by using only the min-max operator.

Definition 5. $R = (R^-, R^+)$ is called a bipolar fuzzy relation (BPFR) from X to Y , if $R^+ : X \times Y \rightarrow [0, 1]$ and $R^- : X \times Y \rightarrow [-1, 0]$ are mappings, i.e.,

$$R \in \text{BPF}(X \times Y).$$

In particular, a BPFR from X to X is called a BPFR on X (see [21]).

The empty BPFR (resp. the whole BPFR) on X , denoted by $R_0 = (R_0^-, R_0^+)$ (resp. $R_1 = (R_1^-, R_1^+)$), is defined as follows: for each $(x, y) \in X \times X$,

$$R_0^+(x, y) = 0 = R_0^-(x, y) \text{ [resp. } R_1^+(x, y) = 1 \text{ and } R_1^-(x, y) = -1].$$

We will denote the set of all BPFRs on X (resp. from X to Y) as $BPFR(X)$ (resp. $BPFR(X \times Y)$).

It is obvious that if $R = (R^-, R^+) \in BPFR(X)$, then $-R^-$ and R^+ are fuzzy relations on X , where $(-R^{-1})(x, y) = -R^-(x, y)$ for each $(x, y) \in X \times X$.

Definition 6. Let $R \in BPFR(X \times Y)$. Then:

- (i) the inverse of R , denoted by $R^{-1} = ((R^{-1})^-, (R^{-1})^+)$, is a BPFR from Y to X defined as follows: for each $(y, x) \in Y \times X$, $R^{-1}(x, y) = R(y, x)$, i.e.,

$$(R^{-1})^+(y, x) = R^+(x, y), (R^{-1})^-(y, x) = R^-(x, y).$$

- (ii) the complement of R , denoted by $R^c = ((R^c)^-, (R^c)^+)$, is a BPFR from X to Y defined as follows: for each $(x, y) \in X \times Y$,

$$(R^c)^+(x, y) = 1 - R^+(x, y), (R^c)^-(x, y) = -1 - R^-(x, y).$$

Proposition 1. Let $R, S, P \in BPFR(X \times Y)$. Then:

- (1) $R_0 \subset R \subset R_1$,
- (2) $(R^c)^{-1} = (R^{-1})^c$,
- (3) $(R^{-1})^{-1} = R$,
- (4) $R \subset R \cup S$ and $S \subset R \cup S$,
- (5) $R \cap S \subset R$ and $R \cap S \subset S$,
- (6) if $R \subset S$, then $R^{-1} \subset S^{-1}$,
- (7) if $R \subset P$ and $S \subset P$, then $R \cup S \subset P$,
- (8) if $P \subset R$ and $P \subset S$, then $P \subset R \cap S$,
- (9) if $R \subset S$, then $R \cup S = S$ and $R \cap S = R$,
- (10) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$, $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$,

Proof. (1) The proof is obvious.

- (2) Let $(x, y) \in X \times Y$. Then

$$\begin{aligned} [(R^c)^{-1}]^-(x, y) &= (R^c)^-(y, x) = -1 - R^-(y, x) \\ &= -1 - (R^{-1})^-(x, y) \\ &= [(R^{-1})^c]^-(x, y). \end{aligned}$$

Similarly, we have $[(R^c)^{-1}]^+(x, y) = [(R^{-1})^c]^+(x, y)$. Thus, the result holds.

- (3) The proof is easy by Definition 6.

- (4) Let $(x, y) \in X \times Y$. Then,

$$(R \cup S)^-(x, y) = R^-(x, y) \wedge S^-(x, y) \leq R^-(x, y)$$

and

$$(R \cup S)^+(x, y) = R^-(x, y) \vee S^-(x, y) \geq R^+(x, y).$$

Thus, $R \subset R \cup S$. Similarly, we have $S \subset R \cup S$.

The remainder can be proven from Definitions 2, 3, and 6. \square

The following is the similar results of Results 1 and 2.

Proposition 2. Let $R, S, P \in \text{BPFR}(X \times Y)$, and let $(R_j)_{j \in I} \subset \text{BPFR}(X \times Y)$. Then:

- (1) (Idempotent laws): $R \cup R = R, R \cap R = R$,
- (2) (Commutative laws): $R \cup S = S \cup R, R \cap S = S \cap R$,
- (3) (Associative laws): $R \cup (S \cup P) = (R \cup S) \cup P, R \cap (S \cap P) = (R \cap S) \cap P$,
- (4) (Distributive laws): $R \cup (S \cap P) = (R \cup S) \cap (R \cup P), R \cap (S \cup P) = (R \cap S) \cup (R \cap P)$,
- (4)' (Generalized distributive laws): $R \cup (\bigcap_{j \in I} R_j) = \bigcap_{j \in I} (R \cup R_j), R \cap (\bigcup_{j \in I} R_j) = \bigcup_{j \in I} (R \cap R_j)$,
- (5) (Absorption laws): $R \cup (R \cap S) = R, R \cap (R \cup S) = R$,
- (6) (De Morgan's laws): $(R \cup S)^c = R^c \cap S^c, (R \cap S)^c = R^c \cup S^c$,
- (6)' (Generalized De Morgan's laws): $(\bigcup_{j \in I} R_j)^c = \bigcap_{j \in I} R_j^c, (\bigcap_{j \in I} R_j)^c = \bigcup_{j \in I} R_j^c$.
- (7) (Involution): $(R^c)^c = R$.

Example 1. Let $X = \{a, b, c\}$, and let R be a BPFR on X given by the following Table 1.

Table 1. R .

	a	b	c
a	$(-0.4, 0.6)$	$(-0.7, 0.5)$	$(-1, 0.8)$
b	$(-0.3, 0.8)$	$(-0.6, 0.3)$	$(-0.2, 0.6)$
c	$(-0.5, 0.7)$	$(-0.8, 0.3)$	$(-0.6, 0.3)$

Then, the inverse and the complement of R are given as below Tables 2–5.

Table 2. R^{-1} .

	a	b	c
a	$(-0.4, 0.6)$	$(-0.3, 0.8)$	$(-0.5, 0.7)$
b	$(-0.7, 0.5)$	$(-0.6, 0.3)$	$(-0.8, 0.3)$
c	$(-1, 0.8)$	$(-0.2, 0.6)$	$(-0.6, 0.3)$

Table 3. R^c .

	a	b	c
a	$(-0.6, 0.4)$	$(-0.3, 0.5)$	$(0, 0.2)$
b	$(-0.7, 0.2)$	$(-0.4, 0.7)$	$(-0.8, 0.4)$
c	$(-0.5, 0.3)$	$(-0.2, 0.7)$	$(-0.4, 0.7)$

Table 4. $R \cap R^c$.

	a	b	c
a	$(-0.4, 0.4)$	$(-0.3, 0.5)$	$(0, 0.2)$
b	$(-0.3, 0.2)$	$(-0.4, 0.3)$	$(-0.2, 0.4)$
c	$(-0.5, 0.3)$	$(-0.2, 0.3)$	$(-0.4, 0.3)$

Table 5. $R \cup R^c$.

	a	b	c
a	$(-0.6, 0.6)$	$(-0.7, 0.5)$	$(-1, 0.8)$
b	$(-0.7, 0.8)$	$(-0.6, 0.7)$	$(-0.8, 0.6)$
c	$(-0.5, 0.7)$	$(-0.8, 0.7)$	$(-0.6, 0.7)$

Remark 1. For each $R \in \text{BPFR}(X)$, $R \cap R^c = R_0$ and $R \cup R^c = R_1$ do not hold, in general.

Consider the BPFR in Example 1. Then, $R \cap R^c \neq R_0$ and $R \cup R^c \neq R_1$.

Definition 7. Let $R \in \text{BPFR}(X \times Y)$, and let $S \in \text{BPFR}(Y \times Z)$. Then, the composition of R and S , denoted by $S \circ R = ((S \circ R)^+, (S \circ R)^-)$, is a BPFR from X to Z defined as: for each $(x, z) \in X \times Z$,

$$(S \circ R)^+(x, z) = (S^+ \circ R^+)(x, z) = \bigvee_{y \in Y} [R^+(x, y) \wedge S^+(y, z)],$$

$$(S \circ R)^-(x, z) = (S^- \circ R^-)(x, z) = \bigwedge_{y \in Y} [R^-(x, y) \vee S^-(y, z)].$$

We can easily see that $(-S^- \circ -R^-)(x, y) = \bigvee_{y \in Y} [-R^+(x, y) \wedge -S^-(x, y)]$.

Proposition 3.

- (1) $P \circ (S \circ R) = (P \circ S) \circ R$, where $R \in \text{BPFR}(X \times Y)$, $S \in \text{BPFR}(Y \times Z)$, and $P \in \text{BPFR}(Z \times W)$.
- (2) $P \circ (R \cup S) = (P \circ R) \cup (P \circ S)$, $P \circ (R \cap S) = (P \circ R) \cap (P \circ S)$, where $R, S \in \text{BPFR}(X \times Y)$ and $P \in \text{BPFR}(Y \times Z)$.
- (3) If $R \subset S$, then $P \circ R \subset P \circ S$, where $R, S \in \text{BPFR}(X \times Y)$ and $P \in \text{BPFR}(Y \times Z)$.
- (4) $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$, where $R \in \text{BPFR}(X \times Y)$ and $S \in \text{BPFR}(Y \times Z)$.

Proof.

- (1) The proof is straightforward.
- (2) The proof is straightforward.
- (3) Let $R, S \in \text{BPFR}(X \times Y)$ and $P \in \text{BPFR}(Y \times Z)$. Suppose $R \subset S$, and let $(x, z) \in X \times Z$. Then

$$\begin{aligned} (P \circ R)^-(x, z) &= \bigwedge_{y \in Y} [R^-(x, y) \vee P^-(y, z)] \\ &\geq \bigwedge_{y \in Y} [S^-(x, y) \vee P^-(y, z)] \\ &[\text{Since } R \subset S, R^-(x, y) \geq S^-(x, y)] \\ &= (P \circ S)^-(x, z). \end{aligned}$$

Similarly, we can prove that $(P \circ R)^+(x, z) \leq (P \circ S)^+(x, z)$. Furthermore, the proof of the second part is similar to the first part. Thus, the result holds.

- (4) Let $R \in \text{BPFR}(X \times Y)$ and $S \in \text{BPFR}(Y \times Z)$, and let $(x, z) \in X \times Z$. Then

$$\begin{aligned} [(S \circ R)^{-1}]^-(z, x) &= (S \circ R)(x, z) \\ &= \bigwedge_{y \in Y} [R^-(x, y) \vee S^-(y, z)] \\ &= \bigwedge_{y \in Y} [(S^{-1})^-(z, y) \vee (R^{-1})^-(y, x)] \\ &= (R^{-1} \circ S^{-1})^-(z, x). \end{aligned}$$

Similarly, we can see that $[(S \circ R)^{-1}]^+(z, x) = (R^{-1} \circ S^{-1})^+(z, x)$. Thus, the result holds. \square

Remark 2. For any BPFRs R and S , $S \circ R \neq R \circ S$, in general.

Let $\text{IFR}(X)$ be the set of all intuitionistic fuzzy relations on a set X introduced by Bustince and Burillo [27]. Then, we have the following result.

Proposition 4. Let $BPFR_b(X) = \{R \in BPFR(X) : R^+(x) - R^-(x) \leq 1, \forall (x, y) \in X \times X\} \cup \{R_{0_b}, R_{1_b}\}$, where R_{0_b} (resp. R_{1_b}) denotes the bipolar fuzzy empty (resp. whole) relation on $BPFR_b(X)$ defined by: for each $x \in X$,

$$R_{0_b}(x) = (-1, 0) \text{ (resp. } R_{1_b} = (0, 1)).$$

We define two mappings $f : BPFR_b(X) \rightarrow IFR(X)$ and $g : IFR(X) \rightarrow BPFR_b(X)$ as follows, respectively:

$$[f(R)](x, y) = (R^+(x, y), -R^-(x, y)), \forall R \in BPFR_b(X), \forall (x, y) \in X \times X,$$

$$[g(S)](x, y) = (-\nu_S(x, y), \mu_S(x, y)), \forall S \in IFR(X), \forall (x, y) \in X \times X.$$

$$\text{Then } g \circ f = 1_{BPFR_b(X)}, f \circ g = 1_{IFR(X)}.$$

Proof. The proof is similar to Proposition 3.14 in [23]. \square

Let $IVR(X)$ be the set of all interval-valued fuzzy relations on a set X (see [28]). Then, we have the following result.

Proposition 5. We define two mappings $f : IVR(X) \rightarrow IFR(X)$ and $g : IFR(X) \rightarrow IVR(X)$ as follows, respectively:

$$[f(R)](x, y) = (R^L(x, y), 1 - R^U(x, y)), \forall R \in IVR(X), \forall (x, y) \in X \times X,$$

$$[g(S)](x, y) = [\mu_S(x), 1 - \nu_S(x)], \forall S \in IFS(X), \forall (x, y) \in X \times X.$$

$$\text{Then } g \circ f = 1_{IVR(X)}, f \circ g = 1_{IFR(X)}.$$

Proof. The proof is similar to Lemma 1 in [29]. \square

From Propositions 4 and 5, we have the following.

Corollary 1. We define two mappings $f : BPFR_b(X) \rightarrow IVR(X)$ and $g : IVR(X) \rightarrow BPFR_b(X)$ as follows, respectively:

$$[f(R)](x, y) = [R^+(x, y), 1 + R^-(x, y)], \forall R \in BPFR_b(X), \forall (x, y) \in X \times X,$$

$$[g(S)](x, y) = (-1 + S^U(x, y), S^L(x, y)), \forall S \in IVR(X), \forall (x, y) \in X \times X.$$

$$\text{Then, } g \circ f = 1_{BPFR_b(X)}, f \circ g = 1_{IVR(X)}.$$

4. Bipolar Fuzzy Reflexive, Symmetric, and Transitive Relations

In this section, we introduce bipolar fuzzy reflexive, symmetric, and transitive relations and obtain some properties related to them.

Definition 8. The bipolar fuzzy identity relation on X , denoted by I_X (simply, I), is a BPFR on X defined as: for each $(x, y) \in X \times X$,

$$I_X^-(x, y) = \begin{cases} -1 & \text{if } x = y \\ 0 & \text{if } x \neq y, \end{cases} \quad I_X^+(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

It is clear that $I = I^{-1}$ and $I^c = (I^c)^{-1}$. Moreover, it is obvious that if I_X is the bipolar fuzzy identity relation on X , then $-I_X^{-1}$ and I_X^+ are fuzzy identity relations on X .

Definition 9. $R \in BPFR(X)$ is said to be:

- (i) reflexive, if for each $x \in X$, $R(x, x) = (-1, 1)$, i.e.,

$$R^-(x, x) = -1, R^+(x, x) = 1,$$

- (ii) anti-reflexive, if for each $x \in X$, $R(x, x) = (0, 0)$.

From Definitions 8 and 9, it is obvious that R is bipolar fuzzy reflexive if and only if $I \subset R$.

The following is the immediate results of the above definition.

It is clear that $R = (R^-, R^+)$ is a bipolar fuzzy reflexive (resp. anti-reflexive) relation on X , then $-R^-$ and R^+ are fuzzy reflexive (resp. anti-reflexive) relations on X . Thus, R and S are fuzzy reflexive (resp. anti-reflexive) relations on X iff $(-R, S)$ or $(-S, R)$ are bipolar fuzzy reflexive (resp. anti-reflexive) relations on X .

Proposition 6. Let $R \in BPFR(X)$.

- (1) R is reflexive if and only if R^{-1} is reflexive.
- (2) If R is reflexive, then $R \cup S$ is reflexive, for each $S \in BPFR(X)$.
- (3) If R is reflexive, then $R \cap S$ is reflexive if and only if $S \in BPFR(X)$ is reflexive.

The following is the immediate result of Definitions 2, 3, 6, and 9.

Proposition 7. Let $R \in BPFR(X)$.

- (1) R is anti-reflexive if and only if R^{-1} is anti-reflexive.
- (2) If R is anti-reflexive, then $R \cup S$ is anti-reflexive if and only if $S \in BPFR(X)$ is anti-reflexive.
- (3) If R is anti-reflexive, then $R \cap S$ is anti-reflexive, for each $S \in BPFR(X)$.

Proposition 8. Let $R, S \in BPFR(X)$. If R and S are reflexive, then $S \circ R$ is reflexive.

Proof. Let $x \in X$. Since R and S are reflexive,

$$R^-(x, x) = -1, R^+(x, x) = 1 \text{ and } S^-(x, x) = -1, S^+(x, x) = 1.$$

Thus

$$\begin{aligned} (S \circ R)^-(x, x) &= \bigwedge_{y \in X} [R^-(x, y) \vee S^-(y, x)] \\ &= [\bigwedge_{x \neq y \in X} (R^-(x, y) \vee S^-(y, x))] \wedge [R^-(x, x) \vee S^-(x, x)] \\ &= [\bigwedge_{x \neq y \in X} (R^-(x, y) \vee S^-(y, x))] \wedge (-1 \wedge -1) \\ &= -1, \end{aligned}$$

$$\begin{aligned} (S \circ R)^+(x, x) &= \bigvee_{y \in X} [R^+(x, y) \wedge S^+(y, x)] \\ &= [\bigvee_{x \neq y \in X} (R^+(x, y) \wedge S^+(y, x))] \vee [R^+(x, x) \wedge S^+(x, x)] \\ &= [\bigvee_{x \neq y \in X} (T_R(x, y) \wedge T_S(y, x))] \vee (1 \wedge 1) \\ &= 1. \end{aligned}$$

Therefore, $S \circ R$ is reflexive. \square

Definition 10. Let $R \in BPFR(X)$. Then:

- (i) R is said to be symmetric, if for each $x, y \in X$,

$$R(x, y) = R(y, x), \text{ i.e., } R^-(x, y) = R^-(y, x) \text{ and } R^+(x, y) = R^+(y, x),$$

- (ii) R is said to be anti-symmetric, if for each $(x, y) \in X \times X$ with $x \neq y$,

$$R(x, y) \neq R(y, x), \text{ i.e., } R^-(x, y) \neq R^-(y, x) \text{ and } R^+(x, y) \neq R^+(y, x).$$

From Definitions 9 and 10, it is obvious that R_0 is a symmetric and anti-reflexive BPFR, R_1 and I are symmetric and reflexive BPFRs, and I^c is an anti-reflexive BPFR.

The following is the immediate result of Definitions 6 and 10.

Proposition 9. Let $R \in \text{BPFR}(X)$. Then, R is symmetric iff $R = R^{-1}$.

Proposition 10. Let $R, S \in \text{BPFR}(X)$. If R and S are symmetric, then $R \cup S$ and $R \cap S$ are symmetric.

Proof. Let $(x, y) \in X \times X$. Then, since R and S are symmetric,

$$(R \cup S)^-(x, y) = R^-(x, y) \wedge S^-(x, y) = R^-(y, x) \wedge S^-(y, x) = (R \cup S)^-(y, x)$$

and:

$$(R \cup S)^+(x, y) = R^+(x, y) \vee S^+(x, y) = R^+(y, x) \vee S^+(y, x) = (R \cup S)^+(y, x).$$

Thus, $R \cup S$ is symmetric.

Similarly, we can prove that $R \cap S$ is symmetric. \square

Remark 3. R and S are symmetric, but $S \circ R$ is not symmetric, in general.

Example 2. Let $X = \{a, b, c\}$, and consider two BPFRs R and S on X given by the following Tables 6 and 7.

Table 6. R .

	a	b	c
a	$(-0.4, 0.5)$	$(-0.3, 0.6)$	$(-0.6, 0.3)$
b	$(-0.3, 0.6)$	$(-0.6, 0.7)$	$(-0.2, 0.4)$
c	$(-0.6, 0.3)$	$(-0.2, 0.4)$	$(-0.4, 0.7)$

Table 7. S .

	a	b	c
a	$(-0.6, 0.7)$	$(-0.7, 0.5)$	$(-1, 0.8)$
b	$(-0.7, 0.5)$	$(-0.4, 0.3)$	$(-0.8, 0.6)$
c	$(-1, 0.8)$	$(-0.8, 0.6)$	$(-0.6, 0.7)$

Then, clearly, R and S are symmetric. However:

$$(S \circ R)^+(a, b) = 0.5 \neq 0.6 = (S \circ R)^+(b, a).$$

Thus, $S \circ R$ is not symmetric.

The following gives the condition for its being symmetric.

Proposition 11. Let $R, S \in \text{SVNR}(X)$. Let R and S be symmetric. Then, $S \circ R$ is symmetric if and only if $S \circ R = R \circ S$.

Proof. Suppose $S \circ R$ is symmetric. Then

$$\begin{aligned} S \circ R &= S^{-1} \circ R^{-1} \text{ (by Proposition 9)} \\ &= (R \circ S)^{-1} \text{ (by Proposition 3 (4))} \\ &= R \circ S \text{ (by the hypothesis).} \end{aligned}$$

Conversely, suppose $S \circ R = R \circ S$. Then

$$(S \circ R)^{-1} = R^{-1} \circ S^{-1} \text{ (by Proposition 3 (4))}$$

$$\begin{aligned}
&= R \circ S \text{ (since } R \text{ and } S \text{ are symmetric)} \\
&= S \circ R \text{ (by the hypothesis).}
\end{aligned}$$

This completes the proof. \square

The following is the immediate result of Proposition 11.

Corollary 2. *If R is symmetric, then R^n is symmetric, for all positive integers n , where $R^n = R \circ R \circ \dots$ n times.*

Definition 11. $R \in \text{BPFR}(X)$ is said to be transitive, if $R \circ R \subset R$, i.e., $R^2 \subset R$.

It is clear that if $R = (R^-, R^+)$ is a bipolar fuzzy transitive relation on X , then $-R^-$ and R^+ are fuzzy transitive relations on X . Thus, R and S are fuzzy transitive relations on X iff $(-R, S)$ and $(-S, R)$ are bipolar fuzzy transitive relations on X .

Proposition 12. *Let $R \in \text{BPFR}(X)$. If R is transitive, then R^{-1} is also.*

Proof. Let $(x, y) \in X \times X$. Then

$$\begin{aligned}
(R^{-1})^-(x, y) &= R^-(y, x) \\
&\leq (R \circ R)^-(y, x) \\
&= \bigwedge_{z \in X} [R^-(y, z) \vee R^-(z, x)] \\
&= \bigwedge_{z \in X} [(R^{-1})^-(z, y) \vee (R^{-1})^-(x, z)] \\
&= \bigwedge_{z \in X} [(R^{-1})^-(x, z) \vee (R^{-1})^-(z, y)] \\
&= (R^{-1} \circ R^{-1})^-(x, y).
\end{aligned}$$

Similarly, we can prove that:

$$(R^{-1})^+(x, y) \geq (R^{-1} \circ R^{-1})^+(x, y).$$

Thus, the result holds. \square

Proposition 13. *Let $R \in \text{BPFR}(X)$. If R is transitive, then so is R^2 .*

Proof. Let $(x, y) \in X \times X$. Then

$$\begin{aligned}
(R^2)^-(x, y) &= \bigwedge_{z \in X} [R^-(x, z) \vee R^-(z, y)] \\
&\leq \bigvee_{z \in X} [(R^2)^-(x, z) \vee (R^2)^-(z, y)] \\
&= [(R^2)^- \circ (R^2)^-](x, y).
\end{aligned}$$

Similarly, we can see that $(R^2)^-(x, y) \geq [(R^2)^- \circ (R^2)^-](x, y)$. Thus, the result holds. \square

Proposition 14. *Let $R, S \in \text{BPFR}(X)$. If R and S are transitive, then $R \cap S$ is transitive.*

Proof. Let $(x, y) \in X \times X$. Then

$$\begin{aligned}
&[(R \cap S) \circ (R \cap S)]^-(x, y) \\
&= \bigwedge_{z \in X} [(R \cap S)^-(y, z) \vee (R \cap S)^-(z, x)] \\
&= \bigwedge_{z \in X} [(R^-(x, z) \vee S^-(x, z)) \vee (R^-(z, y) \vee S^-(z, y))] \\
&= \bigwedge_{z \in X} [(R^-(x, z) \vee R^-(z, y)) \vee (S^-(x, z) \vee S^-(z, y))] \\
&= (\bigwedge_{z \in X} [R^-(x, z) \wedge R^-(z, y)]) \vee (\bigwedge_{z \in X} [S^-(x, z) \wedge S^-(z, y)]) \\
&= (R \circ R)^-(x, y) \vee (S \circ S)^-(x, y) \\
&\geq R^-(x, y) \vee S^-(x, y) \text{ (since } R \text{ and } S \text{ are transitive)} \\
&= (R \cap S)^-(x, y).
\end{aligned}$$

Similarly, we can prove that:

$$[(R \cap S) \circ (R \cap S)]^+(x, y) \leq (R \cap S)^+(x, y).$$

Thus, $R \cap S$ is transitive. \square

Remark 4. For two bipolar fuzzy transitive relation R and S in X , $R \cup S$ is not transitive, in general.

Example 3. Let $X = \{a, b, c\}$, and consider two BPFRs R and S in X given by the following Tables 8 and 9.

Table 8. R .

	a	b	c
a	$(-0.4, 0.5)$	$(-0.3, 0.6)$	$(-0.3, 0.7)$
b	$(-0.3, 0.6)$	$(-0.6, 0.7)$	$(-0.5, 0.4)$
c	$(-0.3, 0.7)$	$(-0.5, 0.4)$	$(-0.6, 0.8)$

Table 9. S .

	a	b	c
a	$(-0.3, 0.7)$	$(-0.2, 0.5)$	$(-0.2, 0.6)$
b	$(-0.7, 0.5)$	$(-1, 0.3)$	$(-0.2, 0.6)$
c	$(-0.2, 0.6)$	$(-0.2, 0.6)$	$(-0.8, 0.7)$

Then, we can easily see that R and S are transitive. Moreover, we have Table 10 as $R \cup S$.

Table 10. $R \cup S$.

	a	b	c
a	$(-0.4, 0.7)$	$(-0.3, 0.6)$	$(-0.3, 0.7)$
b	$(-0.7, 0.6)$	$(-1, 0.7)$	$(-0.5, 0.6)$
c	$(-0.3, 0.7)$	$(-0.5, 0.6)$	$(-0.8, 0.8)$

Thus, $[(R \cup S) \circ (R \cup S)]^-(c, a) = -0.5 \not\geq -0.3 = (R \cup S)^-(c, a)$. Therefore, $R \cup S$ is not transitive.

5. Bipolar Fuzzy Equivalence Relation

In this section, we define the concept of a bipolar fuzzy equivalence class and a bipolar fuzzy partition, and we prove that the set of all bipolar fuzzy equivalence classes is a bipolar fuzzy partition and induce the bipolar fuzzy equivalence relation from a bipolar fuzzy partition.

Definition 12. $R \in \text{BPFR}(X)$ is called a:

- (i) tolerance relation on X , if it is reflexive and symmetric,
- (ii) similarity (or equivalence) relation on X , if it is reflexive, symmetric, and transitive.
- (iii) partial order relation on X , if it is reflexive, anti-symmetric, and transitive.

We will denote the set of all tolerance (resp., equivalence and order) relations on X as $\text{BPFT}(X)$ (resp. $\text{BPFE}(X)$ and $\text{BPFO}(X)$).

We can easily see that $R = (R^-, R^+)$ is a bipolar fuzzy tolerance (resp. similarity and partial order) relation on X , then $-R^-$ and R^+ are fuzzy tolerance (resp. similarity and partial order) relations on X . Furthermore, R and S are fuzzy tolerance (resp. similarity and partial order) relations on X iff $(-R, S)$ and $(-S, R)$ are bipolar fuzzy tolerance (resp. similarity and partial order) relations on X .

The following is the immediate result of Propositions 6, 10, and 14.

Proposition 15. Let $(R_j)_{j \in J} \subset \text{BPFT}(X)$ (resp., $\text{BPFE}(X)$ and $\text{BPFO}(X)$). Then, $\bigcap R_j \in \text{BPFT}(X)$ (resp., $\text{BPFE}(X)$ and $\text{BPFO}(X)$).

Proposition 16. Let $R \in \text{BPFE}(X)$. Then, $R = R \circ R$.

Proof. From Definition 11, it is clear that $R \circ R \subset R$.

Let $(x, y) \in X \times X$. Then

$$\begin{aligned}(R \circ R)^-(x, y) &= \bigwedge_{z \in X} [R^-(x, z) \vee R^-(z, y)] \\ &\leq R^-(x, x) \vee R^-(x, y) \\ &= -1 \vee R^-(x, y) \text{ (since } R \text{ is reflexive)} \\ &= R^-(x, y),\end{aligned}$$

$$\begin{aligned}(R \circ R)^+(x, y) &= \bigvee_{z \in X} [R^+(x, z) \wedge R^+(z, y)] \\ &\geq R^+(x, x) \wedge R^+(x, y) \\ &= 1 \wedge R^+(x, y) \text{ (since } R \text{ is reflexive)} \\ &= R^+(x, y).\end{aligned}$$

Thus, $R \circ R \supset R$. Therefore, $R \circ R = R$. \square

Definition 13. Let $A \in \text{BPF}(X)$. Then, A is said to be normal, if:

$$\bigwedge_{x \in X} A^-(x) = -1, \quad \bigvee_{x \in X} A^+(x) = 1.$$

Definition 14. Let $R \in \text{BPFE}(X)$, and let $x \in X$. Then, the bipolar fuzzy equivalence class of x by R , denoted by R_x , is a BPFS in X defined as:

$$R_x = (R_x^-, R_x^+),$$

where $R_x^+ : X \rightarrow [0, 1]$ and $R_x^- : X \rightarrow [-1, 0]$ are mappings defined as: for each $y \in X$,

$$R_x^+(y) = R^+(x, y) \text{ and } R_x^-(y) = R^-(x, y).$$

We will denote the set of all bipolar fuzzy equivalence classes by R as X/R , and it will be called the bipolar fuzzy quotient set of X by R .

Proposition 17. Let $R \in \text{BPFE}(X)$, and let $x, y \in X$. Then:

- (1) R_x is normal; in fact, $R_x \neq \mathbf{0}_{bp}$,
- (2) $R_x \cap R_y = \mathbf{0}_{bp}$ iff $R(x, y) = (0, 0)$,
- (3) $R_x = R_y$ iff $R(x, y) = (-1, 1)$,
- (4) $\bigcup_{x \in X} R_x = \mathbf{1}_{bp}$.

Proof. (1) Since R is reflexive,

$$R_x^+(x) = R^+(x, x) = 1 \text{ and } R_x^-(x) = R^-(x, x) = -1.$$

Then, $\bigvee_{y \in X} R_x^+(y) = 1$ and $\bigwedge_{y \in X} R_x^-(y) = -1$. Therefore, R_x is normal. Moreover, $R_x = (-1, 1) \neq (0, 0) = \mathbf{0}_{bp}(x)$. Hence, $R_x \neq \mathbf{0}_{bp}$.

(2) Suppose $R_x \cap R_y = \mathbf{0}_{bp}$, and let $z \in X$. Then

$$\begin{aligned}0 &= (R_x \cap R_y)^-(z) \\ &= R_x^-(z) \vee R_y^-(z) \\ &= R^-(x, z) \vee R^-(y, z) \text{ (by Definition 14)} \\ &= R^-(x, z) \vee R^-(z, y) \text{ (since } R \text{ is symmetric),}\end{aligned}$$

$$\begin{aligned}
 0 &= (R_x \cap R_y)^+(z) \\
 &= R_x^+(z) \wedge R_y^+(z) \\
 &= R^+(x, z) \wedge R^+(y, z) \text{ (by Definition 14)} \\
 &= R^-(x, z) \wedge R^-(z, y) \text{ (since } R \text{ is symmetric).}
 \end{aligned}$$

Thus

$$\begin{aligned}
 0 &= \bigwedge_{z \in X} [R^-(x, z) \vee R^-(z, y)] \\
 &= (R \circ R)^-(x, y) \\
 &= R^-(x, y) \text{ (by Proposition 16),}
 \end{aligned}$$

$$\begin{aligned}
 0 &= \bigvee_{z \in X} [(R^+(x, z) \wedge R^+(z, y))] \\
 &= (R \circ R)^+(x, y) \\
 &= R^+(x, y) \text{ (by Proposition 16).}
 \end{aligned}$$

Therefore, $R(x, y) = (0, 0)$.

The sufficient condition is easily proven.

(3) Suppose $R_x = R_y$, and let $z \in X$. Then, $R(x, z) = R(y, z)$. In particular, $R(x, y) = R(y, y)$. Since R is reflexive, $R(x, y) = (-1, 1)$.

Conversely, suppose $R(x, y) = (-1, 1)$, and let $z \in X$. Since R is transitive, $R \circ R \subset R$. Then:

$$R^-(x, y) \vee R^-(y, z) \geq R^-(x, z), \quad R^+(x, y) \wedge R^+(y, z) \leq R^+(x, z).$$

Since $R(x, y) = (-1, 1)$, $R^+(x, y) = 1$ and $R^-(x, y) = -1$. Thus:

$$R^-(y, z) \geq R^-(x, z), \quad R^+(y, z) \leq R^+(x, z).$$

Therefore, $R_y^-(z) \geq R_x^-(z)$, $R_y^+(z) \leq R_x^+(z)$. Hence, $R_y \subset R_x$.

Similarly, we can see that $R_x \subset R_y$. Therefore, $R_x = R_y$.

(4) Let $y \in X$. Then

$$\begin{aligned}
 [\bigcup_{x \in X} R_x](y) &= (\bigvee_{x \in X} R_x^+(y), \bigwedge_{x \in X} R_x^-(y)) \\
 &= (\bigvee_{x \in X} R^+(x, y), \bigwedge_{x \in X} R^-(x, y)) \\
 &= (R^+(y, y), R^-(y, y)) = (-1, 1) \\
 &= \mathbf{1}_{bp}(y).
 \end{aligned}$$

Thus, the result holds. \square

Definition 15. Let $\Sigma = (A_j)_{j \in J} \subset BPF(X)$. Then, Σ is called a bipolar fuzzy partition of X , if it satisfies the following:

- (i) A_j is normal, for each $j \in J$,
- (ii) either $A_j = A_k$ or $A_j \neq A_k$, for any $j, k \in J$,
- (iii) $\bigcup_{j \in J} A_j = \mathbf{1}_{bp}$.

The following is the immediate result of Proposition 17 and Definition 15.

Corollary 3. Let $R \in BPFE(X)$. Then, X/R is a bipolar fuzzy partition of X .

Proposition 18. Let Σ be a bipolar fuzzy partition of X . We define

$R(\Sigma) = (R(\Sigma)^+, R(\Sigma)^-)$ as: for each $(x, y) \in X \times X$,

$$R(\Sigma)^+(x, y) = \bigvee_{A \in \Sigma} [A^+(x) \wedge A^+(y)], \quad R(\Sigma)^-(x, y) = \bigwedge_{A \in \Sigma} [A^-(x) \vee A^-(y)],$$

where $R(\Sigma)^+ : X \times X \rightarrow [0, 1]$ and $R(\Sigma)^- : X \times X \rightarrow [-1, 0]$ are mappings.

Then, $R(\Sigma) \in BPFE(X)$.

Proof. Let $x \in X$. Then, by Definition 15 (iii),

$$R(\Sigma)^-(x, x) = \bigwedge_{A \in \Sigma} [A^-(x) \vee A^-(x)] = \bigwedge_{A \in \Sigma} A^-(x) = -1$$

and

$$R(\Sigma)^+(x, x) = \bigvee_{A \in \Sigma} [A^+(x) \wedge A^-(x)] = \bigvee_{A \in \Sigma} A^+(x) = 1.$$

Thus, $R(\Sigma)$ is reflexive.

From the definition of $R(\Sigma)$, it is clear that $R(\Sigma)$ is symmetric.

Let $(x, y) \in X \times X$. Then

$$\begin{aligned} & [R(\Sigma) \circ R(\Sigma)]^-(x, y) \\ &= \bigwedge_{z \in X} [R(\Sigma)^-(x, z) \vee R(\Sigma)^-(z, y)] \\ &= \bigwedge_{z \in X} [\bigwedge_{A \in \Sigma} (A^-(x) \vee A^-(z)) \vee \bigwedge_{B \in \Sigma} (B^-(z) \vee B^-(y))] \\ &= \bigwedge_{z \in X} [(\bigwedge_{A \in \Sigma} A^-(z) \vee \bigwedge_{B \in \Sigma} B^-(z)) \vee (A^-(x) \vee B^-(y))] \\ &= \bigwedge_{z \in X} [(-1 \vee -1) \vee (A^-(x) \vee B^-(y))] \text{ (since } A \text{ and } B \text{ are normal)} \\ &= \bigwedge_{z \in X} [A^-(x) \vee B^-(y)] \\ &= R(\Sigma)^-(x, y). \end{aligned}$$

Similarly, we can prove that $[R(\Sigma) \circ R(\Sigma)]^+(x, y) = R(\Sigma)^+(x, y)$. Thus, $R(\Sigma)$ is transitive. Therefore, $R(\Sigma) \in BPFE(X)$. \square

Proposition 19. Let $R, S \in BPFE(X)$. Then, $R \subset S$ iff $R_x \subset S_x$, for each $x \in X$.

Proof. Suppose $R \subset S$, and let $x \in X$. Let $y \in X$. Then, by the hypothesis,

$$R_x^-(y) = R^-(x, y) \geq S^-(x, y) = S_x^-(y),$$

$$R_x^+(y) = R^+(x, y) \leq S^+(x, y) = S_x^+(y).$$

Thus, $R_x \subset S_x$.

The converse can be easily proven. \square

Proposition 20. Let $R, S \in BPFE(X)$. Then, $S \circ R \in BPFE(X)$ iff $S \circ R = R \circ S$.

Proof. Suppose $S \circ R = R \circ S$. Since R and S are reflexive, by Proposition 8, $S \circ R$ is reflexive. Since R and S are symmetric, by the hypothesis and Proposition 11, $S \circ R$ is symmetric. Then, it is sufficient to show that $S \circ R$ is transitive:

$$\begin{aligned} (S \circ R) \circ (S \circ R) &= S \circ (R \circ S) \circ R \text{ (by Proposition 3 (1))} \\ &= S \circ (S \circ R) \circ R \\ &= (S \circ S) \circ (R \circ R) \\ &\subset S \circ R. \end{aligned}$$

Thus, $S \circ R$ is transitive. Therefore, $S \circ R \in BPFE(X)$.

The converse can be easily proven. \square

Proposition 21. Let $R, S \in BPFE(X)$. If $R \cup S = S \circ R$, then $R \cup S \in BPFE(X)$.

Proof. Suppose $R \cup S = S \circ R$. Since R and S are reflexive, by Proposition 6 (2), $R \cup S$ is reflexive. Since R and S are symmetric, by the hypothesis and Proposition 10, $R \cup S$ is symmetric. Then, by the hypothesis, $S \circ R$ is symmetric. Thus, by Proposition 11, $S \circ R = R \circ S$. Therefore, by Proposition 20, $S \circ R \in BPFE(X)$. Hence, $R \cup S \in BPFE(X)$. \square

6. Relationships between a Bipolar Fuzzy Relation and Its Level Set

Each member of $[-1, 0] \times [0, 1]$ will be called a bipolar point. We define the order \leq and the equality $=$ between two bipolar points as follows: for any $(a, b), (c, d) \in [-1, 0] \times [0, 1]$,

- (i) $(a, b) \leq (c, d)$ iff $a \geq c$ and $b \leq d$,
- (ii) $(a, b) = (c, d)$ iff $a = c$ and $b = d$.

Definition 16. Let $R \in \text{BPFR}(X \times Y)$, and let $(a, b) \in [-1, 0] \times [0, 1]$.

- (i) The strong (a, b) -level subset or strong (a, b) -cut of R , denoted by $[R]_{(a,b)}^*$, is an ordinary relation from X to Y defined as:

$$[R]_{(a,b)}^* = \{(x, y) \in X \times Y : R^-(x, y) < a, R^+(x, y) > b\}.$$

- (ii) The (a, b) -level subset or (a, b) -cut of R , denoted by $[R]_{(a,b)}$, is an ordinary relation from X to Y defined as:

$$[R]_{(a,b)} = \{(x, y) \in X \times Y : R^-(x, y) \leq a, R^+(x, y) \geq b\}.$$

Example 4. Consider the BPFR R in Example 1. Then

$$\begin{aligned} [R]_{(-0.3, 0.8)} &= \{(x, y) \in X \times X : R^+(x, y) \geq 0.8, R^-(x, y) \leq -0.3\} = \{(a, c), (b, a)\}, \\ [R]_{(-0.3, 0.8)}^* &= \{(x, y) \in X \times X : R^+(x, y) > 0.8, R^-(x, y) < -0.3\} = \emptyset, \\ [R]_{(-0.4, 0.5)} &= \{(a, a), (a, b), (a, c), (c, a)\}, \\ [R]_{(-0.4, 0.5)}^* &= \{(a, c), (c, a)\}. \end{aligned}$$

Proposition 22. Let $R, S \in \text{BPFR}(X \times Y)$, and let $(a, b), (a_1, b_1), (a_2, b_2) \in [-1, 0] \times [0, 1]$.

- (1) If $R \subset S$, then $[R]_{(a,b)} \subset [S]_{(a,b)}$ and $[R]_{(a,b)}^* \subset [S]_{(a,b)}^*$.
- (2) If $(a_1, b_1) \leq (a_2, b_2)$, then $[R]_{(a_2, b_2)} \subset [R]_{(a_1, b_1)}$ and $[R]_{(a_2, b_2)}^* \subset [R]_{(a_1, b_1)}^*$.

Proof. The proofs are straightforward. \square

Proposition 23. Let $R \in \text{BPFR}(X \times Y)$. Then:

- (1) $[R]_{(a,b)}$ is an ordinary relation from X to Y , for each $(a, b) \in [-1, 0] \times [0, 1]$,
- (2) $[R]_{(a,b)}^*$ is an ordinary relation from X to Y , for each $(a, b) \in (-1, 0] \times [0, 1]$,
- (3) $[R]_{(a,b)} = \bigcap_{(c,d) < (a,b)} [R]_{(c,d)}$, for each $(a, b) \in [-1, 0] \times (0, 1]$,
- (4) $[R]_{(a,b)}^* = \bigcup_{(c,d) > (a,b)} [R]_{(c,d)}^*$, for each $(a, b) \in (-1, 0] \times [0, 1]$.

Proof. The proofs of (1) and (2) are clear from Definition 16.

(3) From Proposition 22, it is obvious that $\{[R]_{(a,b)} : (a, b) \in [-1, 0] \times [0, 1]\}$ is a descending family of ordinary relations from X to Y . Let $(a, b) \in (0, 1] \times [-1, 0]$. Then, clearly, $[R]_{(a,b)} \subset \bigcap_{(c,d) < (a,b)} [R]_{(c,d)}$. Assume that $(x, y) \notin [R]_{(a,b)}$. Then, $R^+(x, y) < b$ or $R^-(x, y) > a$. Thus, there is $(c, d) \in [-1, 0] \times (0, 1]$ such that:

$$R^+(x, y) < d < b \text{ or } R^-(x, y) > c > a.$$

Therefore, $(x, y) \notin [R]_{(c,d)}$, i.e., $(x, y) \notin \bigcap_{(c,d) < (a,b)} [R]_{(c,d)}$. Hence:

$$\bigcap_{(c,d) < (a,b)} [R]_{(c,d)} \subset [R]_{(a,b)}.$$

Therefore, the result holds.

(4) Furthermore, from Proposition 22, it is obvious that $\{[R]_{(a,b)}^* : (a, b) \in [-1, 0] \times [0, 1]\}$ is a descending family of ordinary relations from X to Y . Let $(a, b) \in (-1, 0] \times [0, 1]$. Then, clearly,

$[R]_{(a,b)}^* \subset \bigcup_{(c,d) > (a,b)} [R]_{(c,d)}^*$. Assume that $(x, y) \notin [R]_{(a,b)}^*$. Then, $R^+(x, y) \leq b$ or $R^-(x, y) \geq a$. Thus, there exists $(c, d) \in (-1, 0] \times [0, 1]$ such that:

$$R^+(x, y) \leq b < c \text{ or } R^-(x, y) \geq a > d.$$

Thus, $(x, y) \notin [R]_{(c,d)}$, i.e., $(x, y) \notin \bigcup_{(c,d) > (a,b)} [R]_{(c,d)}^*$. Therefore,

$$\bigcup_{(c,d) > (a,b)} [R]_{(c,d)}^* \subset [R]_{(a,b)}^*.$$

Hence, $[R]_{(a,b)}^* = \bigcup_{(c,d) > (a,b)} [R]_{(c,d)}^*$. \square

Definition 17. Let X, Y be non-empty sets; let R be an ordinary relation from X to Y ; and let $R_B \in \text{BPFR}(X \times Y)$. Then, R_B is said to be compatible with R , if $R = S(R_B)$, where $S(R_B) = \{(x, y) : R_B^+(x, y) > 0, R_B^-(x, y) < 0\}$.

Example 5. (1) Let X, Y be non-empty sets, and let $\phi_{X \times Y}$ be the ordinary empty relation from X to Y . Then, clearly, $S(R_0) = \phi_{X \times Y}$. Thus, R_0 is compatible with $\phi_{X \times Y}$.

(2) Let X, Y be non-empty sets, and let $X \times Y$ be the whole ordinary relation from X to Y . Then, clearly, $S(R_1) = X \times Y$. Thus, R_1 is compatible with $X \times Y$.

From Definitions 9, 10, and 16, it is clear that $R \in \text{BPFR}(X)$ is reflexive (resp. symmetric), then $[R]_{(a,b)}$ and $[R]_{(a,b)}^*$ are ordinary reflexive (resp. symmetric) on X , for each $(a, b) \in [-1, 0] \times [0, 1]$.

Proposition 24. Let $R \in \text{BPFR}(X)$, and let $(a, b) \in [-1, 0] \times [0, 1]$. If R is transitive, then $[R]_{(a,b)}$ and $[R]_{(a,b)}^*$ are ordinary transitive on X .

Proof. Suppose R is transitive. Then, $R \circ R \subset R$, and let $(x, z) \in [R]_{(a,b)} \circ [R]_{(a,b)}$. Then, there exists $y \in X$ such that $(x, z), (z, y) \in [R]_{(a,b)}$. Thus:

$$R^+(x, z) \geq b, R^-(x, z) \leq a \text{ and } R^+(z, y) \geq b, R^-(z, y) \leq a.$$

Therefore, $R^+(x, z) \wedge R^+(z, y) \geq b, R^-(x, z) \vee R^-(z, y) \leq a$. Since $R \circ R \subset R$,

$$R^+(x, y) \geq R^+(x, z) \wedge R^+(z, y), R^-(x, y) \leq R^-(x, z) \vee R^-(z, y).$$

Hence, $R^+(x, y) \geq b, R^-(x, y) \leq a$, i.e., $(x, y) \in [R]_{(a,b)}$. Therefore, $[R]_{(a,b)}$ is ordinary transitive. The proof of the second part is similar. \square

The following is the immediate results of Definitions 9, 10, and 16 and Proposition 24.

Corollary 4. Let $R \in \text{BPFE}(X)$, and let $(a, b) \in [-1, 0] \times [0, 1]$. Then, $[R]_{(a,b)}$ and $[R]_{(a,b)}^*$ are the ordinary equivalence relation on X .

7. Conclusions

This paper dealt with the properties of bipolar fuzzy reflexive, symmetric, and transitive relations and bipolar fuzzy equivalence relations. In particular, we defined a bipolar fuzzy equivalence class of a point in a set X modulo a bipolar fuzzy equivalence relation R and a bipolar fuzzy partition of a set X . In addition, we proved that the set of all bipolar fuzzy equivalence classes is a bipolar fuzzy partition and induced the bipolar fuzzy equivalence relation by a bipolar fuzzy partition. Furthermore, we defined the (a, b) -level set of a BPFR and investigated some relationships between BPFRs and their (a, b) -level set. Then, we could see that bipolar fuzzy relations generalized fuzzy relations.

In the future, we expect that one will study bipolar fuzzy relations on a fixed BPFS A and deal with a decomposition of a mapping $f : X \rightarrow Y$ by bipolar fuzzy relations. Furthermore, we think that the bipolar fuzzy relation can be applied to congruences in a semigroup, algebras, topologies, etc.

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