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Feng-Liu Type Fixed Point Results for Multivalued Mappings in GMMS

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Abstract: We present the concept of multivalued mappings in generalized modular metric spaces (GMMS). In addition, we give Caristi and Feng-Liu fixed point results for this type of mappings in GMMS. Then, we obtain an application for final outcomes in the sense of Jleli and Samet.

Keywords: fixed-point theorems; set-valued maps; set-valued functions; generalized modular metric space; multivalued mapping; Feng-Liu type; Caristi-type

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1. Introduction

Multivalued mappings have many applications in pure and applied mathematics. Topology, theory of functions of a real variable, nonlinear functional analysis, the theory of games, and mathematical economics are some examples for those mentioned areas.

We picture the basics of the theory of multivalued mappings. We describe two topologies and linked open balls in the generalized modular metric spaces (GMMS). After that we compare some other topologies in modular spaces, modular metric spaces, and JS(Jleli-Samet)-metric spaces defined by Jleli and Samet in [1]. For this purpose, we give some useful definitions and clarify why we need them. Then, we add the definition of generalized Hausdorff modular distance.

We investigate a new type of definitions in GMMS, such as multivalued Lipschitzian mapping and D-multivalued contraction. In addition, we focus connection between those definitions. We give a generalization of the Banach principle of contraction mappings and we explain how we find a fixed point if we have a multivalued contraction mapping of a GMMS X_D into the nonempty D-closed and bounded subsets of X_D.

Nadler initiated the fixed point theory of set-valued contractions. After that it is developed by many authors in different directions. Some fixed point theorems for multivalued contraction mapping are proved, as well as a theorem on the behaviour of fixed points as the mappings vary. We choose two of them: Caristi and Feng-Liu type approaches for the existence of a fixed point in GMMS.

Finally, we give an application for a non-homogeneous linear parabolic partial differential equation and an initial value problem in GMMS to make our results clear.

Definition 1. *Let X be an abstract set. A function $D : (0, \infty) \times X \times X \to [0, \infty]$ is said to be a generalized modular metric(GMM) on X, if it satisfies the following three axioms:*

(GMM_1) if $D_\lambda(x, y) = 0$, for all $\lambda > 0$, then $x = y$, for all $x, y \in X$;
(GMM_2) $D_\lambda(x, y) = D_\lambda(y, x)$, for all $\lambda > 0$ and $x, y \in X$;

(GMM_3) there exists $C > 0$ such that, if $(x, y) \in X \times X$, $\{x_n\} \subset X$ with $\lim_{n\to\infty} D_\lambda(x_n, x) = 0$, for some $\lambda > 0$, then

$$D_\lambda(x, y) \leq C \limsup_{n\to\infty} D_\lambda(x_n, y).$$

The pair (X, D) is said to be a generalized modular metric space(GMMS).

Example 1. *Let $X = \{1, 2, 3\}$ and define $D : (0, \infty) \times X \times X \to [0, \infty]$ as $D_\lambda(1, 1) = D_\lambda(2, 2) = D_\lambda(3, 3) = 0$, $D_\lambda(1, 2) = D_\lambda(2, 1) = 2$, $D_\lambda(1, 3) = D_\lambda(3, 1) = 6$, $D_\lambda(2, 3) = D_\lambda(3, 2) = 2$ for $\lambda > 0$. Then, GMM_1 and GMM_2 are obvious. $D_{\lambda+\mu}(1, 3) \leq D_\lambda(1, 2) + D_\mu(2, 3)$ gives us $6 \leq 4$, is not true, so (X, D) is not a modular metric space. For GMM_3, we have $\lim_{n\to\infty} D_\lambda(x_n, x) = 0$, while we have $\lim_{n\to\infty} x_n = x$. It is clear that if we take $x = y$ then GMM_3 is satisfied, if we choose $x \neq y$ it is easy to show that $D_\lambda(x, y) \leq C \limsup_{n\to\infty} D_\lambda(x_n, y)$ is true.*

It is easy to check that if there exist $x, y \in X$ and $\{x_n\} \subset X$ with $\lim_{n\to\infty} D_\lambda(x_n, x) = 0$, for some $\lambda > 0$, while $D_\lambda(x, y) < \infty$, then we must have $C \geq 1$. In fact, throughout this work, we assume $C \geq 1$.

Let D be a GMM on X. Fix $x_0 \in X$. The set

$$X_D = X_D(x_0) = \{x \in X : D_\lambda(x, x_0) \to 0 \text{ as } \lambda \to \infty\}$$

is called *generalized modular set*.

Definition 2. *Let (X_D, D) be a GMMS.*

(1) The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_D is said to be D-convergent to $x \in X_D$ if and only if $D_\lambda(x_n, x) \to 0$, as $n \to \infty$, for some $\lambda > 0$.

(2) The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_D is said to be D-Cauchy if $D_\lambda(x_m, x_n) \to 0$, as $m, n \to \infty$, for some $\lambda > 0$.

(3) A subset A of X_D is said to be D-closed if for any $\{x_n\}$ from A which D-converges to x, where $x \in A$. The set $\mathcal{C}(X_D)$ is defined as all the nonemty D-closed subsets of X_D.

(4) A subset A of X_D is said to be D-complete if for any $\{x_n\}$ D-Cauchy sequence in A such that $\lim_{n,m\to\infty} D_\lambda(x_n, x_m) = 0$ for some λ, there exists a point $x \in A$ such that $\lim_{n\to\infty} D_\lambda(x_n, x) = 0$.

Now, we give a definition of multivalued mappings and show some results related to fixed points in GMMS.

2. Results

2.1. Multivalued Mappings in GMMS

Two fixed point theorems for multivalued contraction mapping are proved in [2] by Nadler. The first, a generalization of the contraction mapping principle of Banach, states as a multivalued contraction mapping of a complete metric space into the nonempty closed and bounded subsets of same metric space has a fixed point. The second, a generalization of a result of Edelstein, is a fixed point theorem for compact set-valued local contractions. Nadler's study is applied through other metric spaces, such as in [3–19].

Feng and Liu [20] gave one of the most important generalization of Nadler's result without using Pompei-Hausdorff distance. Then many studies focused on those results and applied them in different metric spaces; for example in [21]. We consider Feng-Liu theorem in GMMS.

Let (X_D, D) be a GMMS and $B \subset X_D$ $D-$sequentially open subset of X_D, while each sequence of X_D has $\lim_{n\to\infty} D_\lambda(x_n, x) = 0$ for some λ, and there exists a point $x \in B$ such that most part of the sequence included in B.

Let τ_{X_D} be a family of all sequentially open subsets of X_D. Any convergent sequence in X_D is convergent in a topological space (X_D, τ_{X_D}). When we take $\mathcal{C}(X_D)$ as a family of all nonempty closed subsets of (X_D, τ_{X_D}) and $\mathcal{N}$ a family of all nonempty subsets A of X_D, we have the following property. Then we threat these two subsets as they are equal. If $D_\lambda(x, A) = 0$, then $x \in A$ for all $x \in X_D$, while $D_\lambda(x, A) = \inf\{D_\lambda(x, y) : y \in A\}$. If the property is satisfied for any $B \subset \mathcal{C}(X_D)$ and $x \in X_D$, then there exists a sequence in B such that $\lim_{n\to\infty} D_\lambda(x_n, x) = 0$. In a topological space (X_D, τ_{X_D}), we have $x \in B$ such that most part of the sequence included in B, which means $B \cap A \neq \emptyset$, so $x \in A = \bar{A}$. As a result $\mathcal{C}(X_D) \subset \mathcal{N}$. If we have $A \subset \mathcal{N}$, $x \in X_D - A$ and a sequence in X_D such that $\lim_{n\to\infty} D_\lambda(x_n, x) = 0$, then no subsequence in A satisfy $D_\lambda(x, A) = 0$ for any $x \in A$. So $X_D - A \in \tau_{X_D}$ is found. We have $A \in \mathcal{C}(X_D)$. The result gives us $\mathcal{C}(X_D) = \mathcal{N}$. In addition, the definition of an open subset is given by using open balls in GMMS as the following. If A is a subset of X_D for any $x \in X_D$, there exists $\epsilon > 0$ such that $B(x, \epsilon) := \{y \in X_D : D_\lambda(x, y) < \epsilon\} \subseteq A$.

In our related paper [22], τ_{X_D} meets properties of usual topology. For example, if we take modular vector spaces as in [23], the ρ-ball $B_\rho(x, r)$, where $x \in X_\rho$ and $r \geq 0$, is defined by $B_\rho(x, r) = \{y \in X_\rho; \rho(x - y) < r\}$. B_ρ is an open ball and a subset of A in vector space X_ρ. In the example of the topology (τ_ρ) for all ρ-open subsets of X_ρ is similar for open subsets of τ_{X_ρ} in a modular space X_ρ.

Chistyakov [24] defined modular open balls and gave their topological structure as: A nonempty set in X is said to ω-open if for every $x \in A$ and $\lambda > 0$ there exists $\mu > 0$ such that $B(x)_{\lambda,\mu} \subset A$ by using ω as a modular metric . Denoted by $\tau(\omega)$ for all ω-open subsets of X_ω we have a ω-topology (modular topology) on X_ω, which is similar to τ_{X_D} in a modular metric space.

When we take a JS-metric space and the topology on JS-metric space, as in [21], we find the usual topology on JS-metric space is, again, equal to τ_{X_D}.

Now we can begin with the definition of generalized Hausdorff modular. Next, we interpret some material and produce their relation in the following section.

Let (X_D, D) be a GMMS. For all nonempty $A, B \subset X_D$, the generalized Hausdorff modular is defined by:

$$H_D(\lambda, A, B) = \max\{\sup_{a\in A} D_\lambda(a, B), \sup_{b\in B} D_\lambda(b, A)\}$$

on $\mathcal{C}(X_D)$—D-strongly complete version of X_D is defined in the next section—where $D_\lambda(a, B) = \inf_{b\in B} D_\lambda(a, b)$.

If $\lambda = 1$, we have:

$$H_D(A, B) = \max\{\sup_{a\in A} D_1(a, B), \sup_{b\in B} D_1(b, A)\}$$

on $\mathcal{C}(X_D)$, where $D_1(a, B) = \inf_{b\in B} D_1(a, b)$.

Example 2. *If we use the GMMS which is given in the first example, for $A = \{1, 2\}, B = \{3\} \subset X$, we have*

$$H_D(\lambda, \{1, 2\}, \{3\}) = \max\{\sup_{a\in\{1,2\}} D_\lambda(a, \{3\}), \sup_{b\in\{3\}} D_\lambda(b, \{1, 2\})\},$$

where $D_\lambda(a, \{3\}) = \inf_{b\in\{3\}} D_\lambda(a, b)$ and $D_\lambda(\{1, 2\}, b) = \inf_{a\in\{1,2\}} D_\lambda(a, b)$. All possible results can be calculated easily.

2.1.1. Fixed Point Results for Multivalued Mappings

Abdou and Khamsi searched the existence of fixed point for contractive-type multivalued map in the setting of modular metric spaces in their study, and they investigated the existence of fixed point of multivalued modular contractive mappings in modular metric spaces in [25]. They claimed that their results generalize or improve the fixed point result of Nadler in [2] and Edelstein. Their study inspired us to work on similar ideas and generalize their results for GMMS.

In a primary sense, we define Lipschitzian mapping, fixed point and D-multivalued contraction in GMMS for more generalized form of Lipschitzian maps we take $\lambda = 1$ such as in [25]. Then we give some essential definitions, such as D-strongly Cauchy sequence in GMMS. Afterwards, we show relations between these definitions and generalized Hausdorff modular metric. At the end of this section, we give A linked fixed point theorem for D-multivalued contraction mapping in GMMS.

Definition 3. *Let (X_D, D) be a GMMS. A mapping $f : X_D \to \mathcal{C}(X_D)$ is called a multivalued Lipschitzian mapping, if there exists a constant $k \geq 0$ such that for any $x, y \in X_D$, for every $a \in f(x)$ there exists $b \in f(y)$, such that*

$$D_1(a, b) \leq k\, D_1(x, y).$$

A point $x \in X_D$ is called a fixed point of f whenever $x \in f(x)$. The set of fixed points of f will be denoted by $Fix(f)$.

The mapping f is called as D-multivalued contraction, if the constant $k < 1$.

Example 3. *If we take the same example as before, a mapping $f : X \to \mathcal{C}(X)$ such that $f(1) = f(2) = 1$ and $f(3) = 2$ for every $a \in f(x)$ there exists $b \in f(y)$ the inequality $D_1(a, b) \leq k\, D_1(x, y)$ is verified in X.*

Definition 4. *Let X_D be a GMMS and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of X_D.*

(1) The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_D is said to be D-strongly Cauchy if $\sum\limits_{n=1}^{\infty} D_\lambda(x_n, x_{n+1}) < \infty$, for some $\lambda > 0$.

(2) A subset M of X_D is said to be D-strongly complete if for any $\{x_n\}$ D-strongly Cauchy sequence in M such that $\sum\limits_{n=1}^{\infty} D_\lambda(x_n, x_{n+1}) < \infty$ for some λ, there exists a point $x \in M$ such that $\lim\limits_{n\to\infty} D_\lambda(x_n, x) = 0$.

(3) D is said to satisfy 1-Fatou property if for any convergent sequence $\{x_n\}_{n\in\mathbb{N}} \subset X_D$ and $x \in X_D$, such that $\lim\limits_{n\to\infty} D_1(x_n, x) = 0$, we have

$$D_1(x, y) \leq \liminf_{n\to\infty} D_1(x_n, y),$$

for any $y \in X_D$.

Let (X_D, D) be a GMMS. Let $f : X_D \to \mathcal{C}(X_D)$ be a multivalued map and $x, y \in X_D$. Assume that $H_D(f(x), f(y)) \leq k\, D_1(x, y)$ with $0 < D_1(x, y) < \infty$ for some constant $k \geq 0$ and for every $a \in f(x)$ there exists $b \in f(y)$, such that

$$\begin{aligned} D_1(a, b) &\leq H_D(f(x), f(y)) + \varepsilon \\ &\leq k\, D_1(x, y) + \varepsilon \\ &= k\, D_1(x, y) + \tfrac{1-k}{2} D_1(x, y) \\ &= \tfrac{1+k}{2} D_1(x, y), \end{aligned}$$

where $\varepsilon = \frac{1-k}{2} D_1(x, y) > 0$. If we say $\frac{1+k}{2} = k'$ then $D_1(a, b) \leq k'\, D_1(x, y)$.

At this point, we explain that D-multivalued contraction mapping f has a fixed point in particular space X_D.

Theorem 1. *Let (X_D, D) be a GMMS. Assume that X_D is D-strongly complete and D satisfy 1-Fatou property. Let $f : X_D \to \mathcal{C}(X_D)$ be a D-multivalued contraction mapping. Assume that $D_1(x_0, x)$ is finite for some $x_0 \in X_D$ and $x \in f(x_0)$. Then f has a fixed point.*

Proof. Fix $x_0 \in X_D$ such that $D_1(x_0, f(x)) < \infty$ for some $x_1 \in f(x_0)$ then there exists $x_2 \in f(x_1)$ such that

$$D_1(x_1, x_2) \leq k\, D_1(x_0, x_1),$$

where $D_1(x_1, x_2) < \infty$.

$$D_1(x_2, x_3) \leq k^2\, D_1(x_0, x_1),$$

where $D_1(x_2, x_3) < \infty$. By induction, we build a sequence $\{x_n\}$ there is $x_1 \in f(x_{n+1})$, for every $x_0 \in f(x_n)$, then there exists $x_{n+1} \in f(x_n)$, since f is a D-multivalued contraction:

$$D_1(x_n, x_{n+1}) \leq k^n\, D_1(x_0, x_1),$$

where $D_1(x_n, x_{n+1}) < \infty$, for every $n \geq 0$. Since $k < 1$, $\sum_{n=1}^{\infty} D_1(x_n, x_{n+1})$ is convergent, i.e., $\{x_n\}$ is D-strongly Cauchy. Since X_D is D-strogly complete, there exists a point $x \in X_D$ such that $\lim_{n\to\infty} D_1(x_n, x) = 0$. Since there is $x_0 \in f(x)$, for every $x_1 \in f(x_n)$,

$$D_1(x_0, x_1) \leq k\, D_1(x_n, x),$$

and D_1 has 1-Fatou property,

$$D_1(x_0, x_1) \leq k\, D_1(x, x) \leq k\, \liminf_{n\to\infty} D_1(x_n, x),$$

we conclude that $\lim_{n\to\infty} D_1(x_0, x_1) = 0$, then x is fixed point of f. □

2.1.2. From Caristi-Type to Feng-Liu-Type Fixed Point Results for Multivalued Mappings

Caristi proved a general fixed point theorem and applied it to derive a generalization of the Contraction Mapping Principle in a complete metric space, then gave an application together with the characterization of weakly inward mappings to obtain some fixed point theorems in Banach spaces [26]. Following that, many authors expanded his approach through different metric spaces; for example in [27]. In addition, there exist an application of Caristi-type mappings in [28]. We examine Caristi-type mappings and state Feng-Liu-type results in GMMS in this section.

Theorem 2. *Let X_D be a D-complete GMMS and $f : X_D \to CB(X_D)$ be a nonexpansive mapping such that for each $x \in X_D$ and $y \in f(x)$ we have*

$$D_1(x, y) \leq \Theta_D(x, y) - \Theta_D(y, z)$$

for $z \in f(y)$, while $CB(X_D)$ is D-closed and bounded subsets of X_D and the function $\Theta_D : X_D \times X_D \to [0, \infty]$ is lower semicontinuous with its first variable. Then $D_1(x_n, x_{n+1}) < \infty$, so f has a fixed point.

Proof. Let $x_0 \in X_D$ and $x_1 \in f(x_0)$. If $x_1 = x_0$, then proof is completed. Let $x_0 \neq x_1$. Using the above inequality of the theorem, then

$$D_1(x_0, x_1) \leq \Theta_D(x_0, x_1) - \Theta_D(x_1, x_2),$$

for $x_2 \in f(x_1)$. When we continue the process, we have $x_n \in f(x_n)$ while $x_n \neq x_{n+1}$, then we have

$$0 < D_1(x_{n-1}, x_n) \leq \Theta_D(x_{n-1}, x_n) - \Theta_D(x_n, x_{n+1}),$$

for $x_{n+1} \in f(x_n)$. We have $\Theta_D(x_{n-1}, x_n)_{n\in\mathbb{N}}$ nonincreasing sequence and converges to $\omega > 0$. If we take limit for the above inequality, we have

$$\begin{aligned}
\lim_{n\to\infty} D_1(x_{n-1}, x_n) &\leq \lim_{n\to\infty} \{\Theta_D(x_{n-1}, x_n) - \Theta_D(x_n, x_{n+1})\}, \\
\lim_{n\to\infty} D_1(x_{n-1}, x_n) &\leq \lim_{n\to\infty} \Theta_D(x_{n-1}, x_n) - \lim_{n\to\infty} \Theta_D(x_n, x_{n+1}), \\
\lim_{n\to\infty} D_1(x_{n-1}, x_n) &\leq \omega - \omega = 0
\end{aligned}$$

for $n \in \mathbb{N}$. It is the same way to show $\{x_n\}_{n\in\mathbb{N}}$ is D-Cauchy sequence. Then we assume ω is a fixed point of f:

$$\begin{aligned}
D_1(\omega, f(\omega)) &\leq D_{\frac{1}{2}}(\omega, x_{n+1}) - D_{\frac{1}{2}}(f(\omega), x_{n+1}), \\
&\leq D_{\frac{1}{2}}(\omega, x_{n+1}) - H_D(f(\omega), f(x_{n+1})), \\
&\leq D_{\frac{1}{2}}(\omega, x_{n+1}) - D_{\frac{1}{2}}(\omega, x_n),
\end{aligned}$$

for the last equality we pass the limit, and then we have

$$\begin{aligned}
\lim_{n\to\infty} D_1(\omega, f(\omega)) &\leq \lim_{n\to\infty} D_{\frac{1}{2}}(\omega, x_{n+1}) - \lim_{n\to\infty} D_{\frac{1}{2}}(\omega, x_n), \\
\lim_{n\to\infty} D_1(\omega, f(\omega)) &\leq D_{\frac{1}{2}}(\omega, \omega) - D_{\frac{1}{2}}(\omega, \omega) = 0.
\end{aligned}$$

Then ω is a fixed point of f. □

Next, it is available to generalize even more as below.

Theorem 3. *Let X_D be a D-complete GMMS and $f : X_D \to CB(X_D)$ be a multivalued mapping*

$$H_D(f(x), f(y)) \leq v(D_1(x, y))$$

for all $x, y \in X_D$ and $v : [0, \infty] \to [0, \infty]$ is a lower semicontunious map defined as $v(t) < t$ for $t \in [0, \infty]$ and satisfied that $\frac{v(t)}{t}$ is nondecreasing. Then $D_1(x_n, x_{n+1}) < \infty$, so f has a fixed point.

Feng and Liu [20] gave the following theorem without using Hausdorff distance. To state their result, we use the following notation for a multivalued mapping f on X_D, let and we define

$$I^x_{D,\beta}(f) = \{a \in f(x); \beta\, D_\lambda(x, a) \leq D_\lambda(x, f(x))\}.$$

The function f is called D-lower semicontinuous, and for any sequence $\{x_n\} \in X_D$ is convergent to $x \in X_D$, if $D_1(x, f(x)) \leq \liminf_{n\to\infty} D_1(x_n, f(x_n))$.

Example 4. *If we take the same example as before, a mapping $f : X \to \mathcal{C}(X)$ such that $f(x) = 3$, $\beta = \frac{1}{2}$, $x \in X$, we can show for any calculation of $\beta\, D_1(x, a) \leq D_1(x, f(x))$, where $a \in f(x)$, it is satisfied. Then f is called D-lower semicontinuous for any sequence $\{x_n\} \in X$ is convergent to $x \in X$, if $D_1(x, f(x)) \leq \liminf_{n\to\infty} D_1(x_n, f(x_n))$.*

Theorem 4. *Let (X_D, D) be a complete GMMS and f be D-multivalued mapping on X_D. Suppose there exists a constant $K > 0$ such that $\frac{K}{\beta} < 1$ for any $x \in X_D$ there is $y \in I^x_{D,\beta}(f)$ satisfying*

$$D_1(y, f(y)) \leq K\, D_1(x, y).$$

If there exists $x_0 \in X_D$ such that $D_1(x_0, f(x_0)) < \infty$. Assume there exists a sequence $\{x_n\}$ in X_D such that $\beta\, D_1(x_{n+1}, x_{n+2}) \leq K\, D_1(x_n, x_{n+1})$ and $\beta\, D_1(x_{n+1}, f(x_{n+1})) \leq K\, D_1(x_n, f(x_n))$; while $x_{n+1} \in f(x_n)$ and $D_1(x_n, x_{n+1}) < \infty$ for any $n \in \mathbb{N}$.

The sequence is D-strongly Cauchy, and if we assume $D_1(x, f(x))$ is D-lower semicontinuous, then f has a fixed point.

Proof. Since $f(x) \in X_D$ for all $x \in X_D$, then $I^x_{D,b}(f)$ is nonempty. Let us start choosing $x_0 \in X_D$ such that $D_1(x_0, f(x_0)) < \infty$. From $D_1(y, f(y)) \leq K\, D_1(x, y)$, there exists $x_1 \in I^{x_0}_{D,\beta}(f)$ such that,

$$D_1(x_1, f(x_1)) \leq K\, D_1(x_0, x_1).$$

Since $x_1 \in I^{x_0}_{D,\beta}(f)$, then $x_1 \in f(x_0)$ and

$$\beta\, D_1(x_0, x_1) \leq D_1(x_0, f(x_0)) < \infty.$$

Choosing $x_1 \in X_D$ such that $D_1(x_1, f(x_1)) < \infty$. From $D_1(y, f(y)) \leq K\, D_1(x, y)$, there exists $x_2 \in I^{x_1}_{D,\beta}(f)$ such that,

$$D_1(x_2, f(x_2)) \leq K\, D_1(x_1, x_2).$$

Since $x_2 \in I^{x_1}_{D,\beta}(f)$, then $x_2 \in f(x_1)$ and

$$b\, D_1(x_1, x_2) \leq D_1(x_1, f(x_1)) < \infty.$$

By choosing $x_{n+1} \in M$ such that $D_1(x_{n+1}, f(x_{n+1})) < \infty$. From $D_1(y, f(y)) \leq K\, D_1(x, y)$, there exists $x_{n+1} \in I^{x_n}_{D,\beta}(f)$ such that,

$$D_1(x_n, f(x_n)) \leq K\, D_1(x_n, x_{n+1}).$$

Since $x_{n+1} \in I^{x_n}_{D,\beta}(f)$, then $x_{n+1} \in f(x_n)$ and

$$\beta\, D_1(x_n, x_{n+1}) \leq D_1(x_n, f(x_n)) < \infty.$$

Then, we have,

$$\beta\, D_1(x_{n+1}, x_{n+2}) \leq D_1(x_n, f(x_{n+1})) \leq K\, D_1(x_n, x_{n+1}),$$

which give us, while $x_{n+1} \in f(x_n)$,

$$\beta\, D_1(x_{n+1}, f(x_{n+1})) \leq D_1(x_n, f(x_{n+1})) \leq K\, D_1(x_n, f(x_n)).$$

Then, we have

$$D_1(x_{n+1}, f(x_{n+1})) \leq \frac{K}{\beta}\, D_1(x_n, f(x_n))$$

for $\frac{K}{\beta} < 1$ for any $x \in X_D$,

$$D_1(x_{n+1}, f(x_{n+1})) \leq \left(\frac{K}{\beta}\right)^n D_1(x_0, x_1),$$

while $\sum_{n=1}^{\infty} D_1(x_n, x_{n+1}) < \infty$ and $\{x_n\}$ is D-strongly Cauchy and X_D is D-strongly complete; then

$$0 = \lim_{n\to\infty} D_1(x_n, x_{n+1}) = \lim_{n\to\infty} D_1(x_n, f(x_n)).$$

$D_1(x, f(x))$ is D-lower semicontinuous,

$$0 \leq D_1(z, f(z)) \leq \liminf_{n\to\infty} D_1(x_n, f(x_n));$$

since $f(z) \in X_D$, then we have $z \in f(z)$. □

2.1.3. An Application

When we mention applications of multivalued mappings, one of them is given by Khamsi et al. in [23] for modular vector spaces. They pointed out a fixed point theorem for uniformly Lipschitzian mappings in modular vector spaces which has the uniform normal structure property in the modular sense. They expanded their results in the variable exponent space. Another application of them is given by Borisut et al. in [29]. They proved some fixed point theorems in generalized metric spaces by using the generalized contraction and they applied the fixed point theorems to show the existence and uniqueness of solution to the ordinary difference equation (ODE), partial difference equation (PDEs) and fractional boundary value problem.

For a non-homogeneous linear parabolic partial differential equation, initial value problem is given in [6], such as

$$\begin{aligned} f_t(x,t) &= f_{xx}(x,t) + S(x,t,f(x,t),f_x(x,t)), -\infty < x < \infty, 0 < t \leq T, \\ f(x,0) &= \phi(x) \geq 0 \end{aligned}$$

for same valued $x \in X_D$, where S is continuous and ϕ assumed to be continuously differentiable such that ϕ and ϕ' are bounded. By a solution of this problem, a function $f = f(x,t)$ defined on $\mathbb{R} \times I = [0,T]$, where I satisfying the following conditions:

(i) $f, f_t, f_x, f_{xx} \in C(\mathbb{R} \times I)$ while it denotes the space of all continuous real valued functions,
(ii) f, f_x are bounded $\in \mathbb{R} \times I$,
(iii) $f_t(x,t) = f_{xx}(x,t) + S(x,t,f(x,t),f_x(x,t)), (x,t) \in \mathbb{R} \times I$,
(iv) $f(x,0) = \phi(x) \geq 0$ for all $x \in \mathbb{R}$,

The differential equation problem below, is equivalent to the following integral equation problem:

$$f(x,t) = \int_{-\infty}^{\infty} K(x-\delta,t)\phi(\delta)d\delta + \int_0^t \int_{-\infty}^{\infty} K(x-\delta,t-u)S(\delta,u,f(\delta,u),f_x(\delta,u))d\delta du$$

for all $x \in \mathbb{R}$ and $0 < t \leq T$ where

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}}.$$

This problem admits a solution if and only if the corresponding problem just below has a solution. Let

$$B := \{f(x,t) : f, f_x \in C(\mathbb{R} \times I), ||f|| < \infty\}$$

where

$$||f|| := \sup_{x\in\mathbb{R},t\in I} |f(x,t)| + \sup_{x\in\mathbb{R},t\in I} |f_x(x,t)|.$$

Now, we take a function D_1 as

$$D_1(x,y) := \frac{1}{\lambda}\omega_1(x,y) = \frac{1}{\lambda^2}|x-y|$$

is a GMM on B. Obviously, the GMMS B_ω is a $D-$complete and independent of generators.

While D_1 is a GMMS, lower semicontinous it is easy to proof for Feng-Liu-type.

Theorem 5. *Let the problem*

$$\begin{aligned} f_t(x,t) &= f_{xx}(x,t) + S(x,t,f(x,t),f_x(x,t)), -\infty < x < \infty, 0 < t \leq T, \\ f(x,0) &= \phi(x) \geq 0. \end{aligned}$$

and assume the following:

(i) *For $c > 0$ with $|s| < c$ and $|p| < c$ the function $S(x,t,s,p)$ is uniformly Hölder continuous in x and t for each compact subset of $\mathbb{R} \times I$,*

(ii) *There exists a constant $c_S \leq T + 2\pi^{-\frac{1}{2}} T^{\frac{1}{2}} \leq q$, where $q \in (0,1)$ such that*

$$\begin{array}{rcl} 0 & \leq & \frac{1}{\lambda} S[(x,t,s_2,p_2) - S(x,t,s_1,p_1)] \\ c_S & \leq & [\frac{s_2 - s_1 + p_2 - p_1}{\lambda}] \end{array}$$

for all $(s_1,p_1),(s_2,p_2) \in \mathbb{R} \times \mathbb{R}$ with $s_1 \leq s_2$ and $p_1 \leq p_2$,

(iii) *S is bounded for bounded s and p;*

Then the problem has a solution.

Proof. Let choose $x \in B_\omega$ is a solution of the problem below, if and only if $x \in B_\omega$ is a solution integral equivalent.

When we take the graph G with $V(G) = B_\omega$ and $E(G) = \{(z,v) \in B_\omega \times B_\omega : z(x,t) \leq v(x,t)$ and $z_x(x,t) \leq v_x(x,t)$ for each $(x,t) \in \mathbb{R} \times I\}.E(G)$ is partially ordered and $(B_\omega, E(G))$ satisfy property (A).

The mapping $\Omega : B_\omega \to B_\omega$ defined as

$$f(u(x,t)) := \int_{-\infty}^{\infty} K(x-\delta,t)\phi(\delta)d\delta + \int_0^t \int_{-\infty}^{\infty} K(x-\delta,t-u)S(\delta,u,f(\delta,u),f_x(\delta,u))d\delta du$$

for all $x \in \mathbb{R}$ and when we solve the problem, the solution gives us the existence of fixed point of f.

Since $(z,v),(z_x,v_x),(f(z),f(v)),(f(z_x),f(v_x)) \in E(G)$ and from the definition of f and (ii)

$$\frac{1}{\lambda}|f(v(x,t)) - f(z(x,t))| \leq c_S D_1(z,v).$$

Then, we have

$$\begin{array}{rl} \frac{1}{\lambda}|f_x(v(x,t)) - f_x(z(x,t))| & \leq c_S D_1(z,v) \int_{-\infty}^{\infty} K(x-\delta,t)\phi(\delta)d\delta \\ & \leq 2\pi^{-\frac{1}{2}} T^{\frac{1}{2}} c_S D_1(z,v). \end{array}$$

When we take the solutions together:

$$\begin{array}{rl} D_1(f(z),f(v)) & \leq (T + 2\pi^{-\frac{1}{2}} T^{\frac{1}{2}})\ c_S\ D_1(z,v) \\ D_1(f(z),f(v)) & \leq c\ D_1(z,v),\ c \in (0,1) \\ |f(z) - f(v)| & \leq \lambda^2\ |z-v|,\ \lambda \in (0,1). \end{array}$$

From Feng-Liu's perspective, we have

$$\begin{array}{lll} \lambda^2|v - f(v)| & \leq \lambda^2 H_d(f(z),f(v)) & \leq \lambda^2\ |z-v| \\ d(v,f(v)) & \leq H_d(f(z),f(v)) & \leq d(z,v) \\ D_1(v,f(v)) & \leq H_D(f(z),f(v)) & \leq D_1(z,v) \end{array}$$

since we have $b\ D_1(z,v) \leq D_1(z,f(z))$, while $b \in (0,1)$. Then, there exists a $z^\star \in B_\omega$ such that $z^\star = f(z^\star)$, which is the solution of the problem. □

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