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System of Multi-Valued Mixed Variational Inclusions with XOR-Operation in Real Ordered Uniformly Smooth Banach Spaces

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Abstract: In this paper, we consider and study a system of multi-valued mixed variational inclusions with XOR-operation $\oplus$ in real ordered uniformly smooth Banach spaces. This system consists of bimappings, multi-valued mappings and Cayley operators. An iterative algorithm is suggested to find the solution to a system of multi-valued mixed variational inclusions with XOR-operation $\oplus$ and consequently an existence and convergence result is proved. In support of our main result, an example is constructed.

Keywords: Algorithm; Cayley Operator; Existence; Variational Inclusion; Solution

MSC: 47H05; 49H10; 47J25

1. Introduction

In 1964, Stampacchia [1] investigated the theory of variational inequality which provides us a lenient way for solving perplexities occurring in industry, finance, economics, operation research, optimization, decision sciences and several other branches of pure and applied sciences, and so forth, see, for example, [2–17]. Hassouni and Moudafi [18] studied a mixed type variational inequality which involves a nonlinear term called variational inclusion. They used the resolvent operator technique in order to find the solution to their problem as the projection method does not work due to the nonlinear term.

A natural generalization of variational inequalities called the system of variational inequalities (inclusions) were considered and studied by several authors. Cohen and Chaplais [19], Ansari and Yao [20] and many more researchers considered various system of variational inequalities (inclusions), see also [21–29]. It has been shown by Pang [30] that not only the Nash equilibrium problem but also various equilibrium type problems, like the traffic equilibrium problem, spatial equilibrium problem and the general equilibrium programming problems from operation research, game theory, mathematical physics, and so forth, can be formulated as a variational inequality problem defined over a product of sets, which is equivalent to a system of variational inequalities.

Agarwal et al. [31] studied a system of generalized nonlinear mixed quasi-variational inclusions and demonstrated sensitivity analysis of their problem. Some ordered variational inclusions involving

XOR-operation $\oplus$ are studied by Li et al. [32–35], Ahmad et al. [36–39] and Ali et al. [40] and so forth. For some related work, see also [41].

In this paper, we consider and study a system of multi-valued mixed variational inclusions with XOR-operation $\oplus$ in real ordered uniformly smooth Banach spaces. We prove the existence of solutions to a system of multi-valued mixed variational inclusions with XOR-operation $\oplus$ and we discuss the convergence of the iterative sequences generated by the proposed algorithm. An example is provided.

2. Preliminaries

Let E be a real ordered uniformly smooth Banach space with norm $\|\cdot\|$ and E^* be its topological dual. We denote by d the metric induced by the norm $\|\cdot\|$ on E, by $CB(E)$ (respectively, 2^E) the family of all nonempty closed and bounded subsets (respectively, the set of all nonempty subsets) of E and by $D(\cdot,\cdot)$ the Hausdörff metric on $CB(E)$. Let $C \subseteq E$ be a cone. For arbitrary elements $x, y \in E$, $x \leq y$ holds if and only if $y - x \in C$, then the relation "$\leq$" in E is called partial order relation induced by the cone C.

Let $\langle \cdot,\cdot \rangle$ be the duality pairing between E and E^*, and $J : E \to 2^{E^*}$ be the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|\|f\|\}, \; for\ all\ x \in E.$$

We recall some well known concepts and results for the presentation of this paper.

The modulus of smoothness of a Banach space E is a function $\tau_E : [0,\infty) \to [0,\infty)$ defined by

$$\tau_E(t) = \sup\left\{\frac{\|x+y\| - \|x-y\|}{2} - 1 : \|x\| \leq 1, \|y\| \leq t\right\}.$$

A Banach space E is called uniformly smooth if

$$\lim_{t\to 0} \frac{\tau_E(t)}{t} = 0.$$

Definition 1 ([29])**.** *A mapping $g : E \to E$ is said to be*

(*i*) *accretive, if for any $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that*

$$\langle g(x) - g(y), j(x-y)\rangle \geq 0,$$

(*ii*) *strongly accretive, if for any $x, y \in E$, there exists $j(x-y) \in J(x-y)$ and a constant $\delta_g > 0$ such that*

$$\langle g(x) - g(y), j(x-y)\rangle \geq \delta_g \|x-y\|^2,$$

(*iii*) *Lipschitz continuous, if for any $x, y \in E$, there exists a constant $\lambda_g > 0$ such that*

$$\|g(x) - g(y)\| \leq \lambda_g \|x-y\|.$$

Proposition 1 ([42])**.** *Let E be a uniformly smooth Banach space and $J : E \to 2^{E^*}$ be a normalized duality mapping. Then, for any $x, y \in E$,*

(*i*) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$, *for all* $j(x+y) \in J(x+y)$,

(*ii*) $\langle x-y, j(x) - j(y)\rangle \leq 2\mathcal{C}^2\tau_E\left(4\|x-y\|/\mathcal{C}\right)$ *where* $\mathcal{C} = \sqrt{(\|x\|^2 + \|y\|^2)/2}.$

Definition 2. *A multi-valued mapping $G : E \to CB(E)$ is said to be D-Lipschitz continuous, if for any $x, y \in E$, there exists a constant $\lambda_{D_G} > 0$ such that*

$$D(G(x), G(y)) \leq \lambda_{D_G}\|x-y\|.$$

Definition 3. *A cone C is said to be normal if there exists a constant $\lambda_N > 0$ such that for $0 \leq x \leq y$, $\|x\| \leq \lambda_N \|y\|$, where λ_N is normal constant of C.*

Definition 4. *For arbitrary element $x, y \in E$, $x \leq y$ (or $y \leq x$) holds, then x and y said to be comparable to each other (denoted by $x \propto y$).*

Most of the following definitions can be found in [43].

Definition 5. *For arbitrary elements x,y of E, $lub\{x, y\}$ and $glb\{x, y\}$ mean the least upper bound and the greatest lower bound of the set $\{x, y\}$. Suppose $lub\{x, y\}$ and $glb\{x, y\}$ exist. Then some binary operations are defined as follows:*

(i) $x \vee y = lub\{x, y\}$,
(ii) $x \wedge y = glb\{x, y\}$,
(iii) $x \oplus y = (x - y) \vee (y - x)$,
(iv) $x \odot y = (x - y) \wedge (y - x)$.

The operations $\vee, \wedge, \oplus$ and $\odot$ are called OR, AND, XOR and $XNOR$ operations, respectively.

Proposition 2. *Let $\oplus$ be an XOR-operation and $\odot$ be an XNOR -operation. Then the following relations hold:*

(i) $x \odot x = 0$, $x \odot y = y \odot x = -(x \oplus y) = -(y \oplus x)$,
(ii) *if* $x \propto 0$, *then* $-x \oplus 0 \leq x \leq x \oplus 0$,
(iii) $0 \leq x \oplus y$, *if* $x \propto y$,
(iv) *if* $x \propto y$, *then* $x \oplus y = 0$, *if and only if* $x = y$.

Proposition 3 ([43])**.** *Let $C \subseteq E$ be a normal cone with normal constant λ_N. Then for each x,y of E, the following relations hold:*

(i) $\|0 \oplus 0\| = \|0\| = 0$,
(ii) $\|x \vee y\| \leq \|x\| \vee \|y\| \leq \|x\| + \|y\|$,
(iii) $\|x \oplus y| \leq \|x - y\| \leq \lambda_N \|x \oplus y\|$,
(iv) *if* $x \propto y$, *then* $\|x \oplus y\| = \|x - y\|$.

Definition 6 ([33])**.** *Let $A : E \to E$ be a single-valued mapping. Then*

(i) *A is said to be a comparison mapping, if for all $x, y \in E$, $x \propto y$ then $A(x) \propto A(y)$, $x \propto A(x)$ and $y \propto A(y)$,*
(ii) *A is said to be strongly comparison mapping, if A is a comparison mapping and $A(x) \propto A(y)$ if and only if $x \propto y$, for all $x, y \in E$,*
(iii) *A is said to be β'-ordered compression mapping, if A is a comparison mapping, and*

$$A(x) \oplus A(y) \leq \beta'(x \oplus y), \text{ for } 0 < \beta' < 1.$$

Definition 7 ([32,39])**.** *Let $M : E \to 2^E$ be a multi-valued mapping. Then*

(i) *M is said to be a comparison mapping, if for any $v_x \in M(x)$, $x \propto v_x$ and if $x \propto y$, then for any $v_x \in M(x)$ and any $v_y \in M(y)$, $v_x \propto v_y$ for all $x, y \in E$,*
(ii) *M is said to be α_M-non-ordinary difference mapping, if for all $x, y \in E$, M is a comparison mapping and $v_x \in M(x)$ and $v_y \in M(y)$ such that*

$$(v_x \oplus v_y) \oplus \alpha_M(x \oplus y) = 0,$$

(iii) *M is said to λ-XOR-ordered strongly monotone mapping, if $x \propto y$ then there exists a constant $\lambda > 0$ such that*

$$\lambda(v_x \oplus v_y) \geq x \oplus y, \text{ for all } x, y \in E, v_x \in M(x), v_y \in M(y).$$

Definition 8. *Let $A : E \to E$ be a strong comparison and β'-ordered compression mapping. Then, a comparison multi-valued mapping $M : E \to 2^E$ is said to be (α_M, λ)-XOR-NODSM , if M is α_M-non-ordinary difference mapping and λ-XOR-ordered strongly monotone mapping such that $[A \oplus \lambda M](E) = E$, for all $\alpha_M, \beta', \lambda > 0$.*

Definition 9. *Let $A : E \to E$ be a strongly comparison and β'-ordered compression mapping and let $M : E \to 2^E$ be a multi-valued, (α_M, λ)-XOR-NODSM mapping. The resolvent operator $R^M_{A,\lambda} : E \to E$ associated with A and M is defined by*

$$R^M_{A,\lambda}(x) = [A \oplus \lambda M]^{-1}(x), \quad for\ all\ x \in E, \lambda > 0. \tag{1}$$

It is proved in [39] that the resolvent operator defined by (1) is a single-valued comparison as well as θ-Lipschitz-type continuous, where $\theta = \dfrac{1}{\alpha_M \lambda \oplus \beta'}$.

Definition 10. *The Cayley operator $C^M_{A,\lambda}$ associated with M is defined as*

$$C^M_{A,\lambda}(x) = \left[2R^M_{A,\lambda}(x) - I\right](x), \quad for\ all\ x \in E, \tag{2}$$

where $R^M_{A,\lambda}$ is defined by (1) and $\lambda > 0$.

One can easily prove that the Cayley operator defined by (2) is single-valued, a comparison as well as $(2\theta + 1)$-Lipschitz-type continuous, where θ is same as in Definition 9, for more details see [40].

3. A System of Multi-Valued Mixed Variational Inclusions with XOR-Operation $\oplus$ and an Iterative Algorithm

Let E be a real ordered uniformly smooth Banach space. Let $G, F : E \to CB(E)$ be multi-valued mappings and $A, P, q : E \to E$; $S, T : E \times E \to E$ be single-valued mappings. Let $M, N : E \to 2^E$ be multi-valued mappings and $C^M_{A,\lambda}; C^N_{A,\rho} : E \to E$ be Cayley operators. We deal with the following problem.

Find $x, y \in E, u \in G(x), v \in F(y)$ such that

$$\begin{aligned} 0 &\in S(x - P(x), v) + C^M_{A,\lambda}(x) \oplus M(x), \\ 0 &\in T(u, y - q(y)) + C^N_{A,\rho}(y) \oplus N(y), \end{aligned} \tag{3}$$

where $\lambda > 0$ and $\rho > 0$ are constants. Problem (3) is called system of multi-valued mixed variational inclusions with XOR-operation $\oplus$.

If $P(x) = 0 = q(y)$, then we encounter with the following problem, that is, find $x, y \in E, u \in G(x), v \in F(y)$ such that

$$\begin{aligned} 0 &\in S(x, v) + C^M_{A,\lambda}(x) \oplus M(x), \\ 0 &\in T(u, y) + C^N_{A,\rho}(y) \oplus N(y). \end{aligned} \tag{4}$$

Problem (4) appears to be the new one.

If $C^M_{A,\lambda}(x) = 0 = C^N_{A,\rho}(y)$, and $\oplus$ is replaced by $+$, then problem (4) reduces to the problem of finding $x, y \in E, u \in G(x), v \in F(y)$ such that

$$\begin{aligned} 0 &\in S(x, v) + M(x), \\ 0 &\in T(u, y) + N(y). \end{aligned} \tag{5}$$

Problem (5) is considered in [26] in the setting of Hilbert spaces.

It is easy to check that problem (3) includes many previously studied problems related to variational inclusions.

The following Lemma is a fixed point formulation of problem (3).

Lemma 1. *$x, y \in E, u \in G(x), v \in F(y)$ is a solution to a system of multi-valued mixed variational inclusions with XOR-operation $\oplus$ (3), if and only if the following equations are satisfied:*

$$x = R^M_{A,\lambda}\left[A(x) + \lambda S(x - P(x), v) + \lambda C^M_{A,\lambda}(x)\right], \tag{6}$$

$$y = R^N_{A,\rho}\left[A(y) + \rho T(u, y - q(y)) + \rho C^N_{A,\rho}(y)\right], \tag{7}$$

where, $\lambda > 0$ and $\rho > 0$ are constants.

Proof. The proof is easy and hence omitted. □

Iterative Algorithm 1. *For any given $x_0, y_0 \in E$, we choose $u_0 \in G(x_0), v_0 \in F(y_0)$. From (6) and (7), for $0 \le \alpha, \beta < 1$ and $\lambda, \rho > 0$, let*

$$x_1 = (1-\alpha)x_0 + \alpha R^M_{A,\lambda}\left[A(x_0) + \lambda(S(x_0 - P(x_0), v_0)) + \lambda C^M_{A,\lambda}(x_0)\right],$$

and

$$y_1 = (1-\beta)y_0 + \beta R^N_{A,\rho}\left[A(y_0) + \rho(T(u_0, y_0 - q(y_0))) + \rho C^N_{A,\rho}(y_0)\right].$$

Since $u_0 \in G(x_0)$ and $v_0 \in F(y_0)$, by Nadler's theorem [44], there exist $u_1 \in G(x_1)$ and $v_1 \in F(y_1)$ such that

$$\begin{aligned} \|u_0 - u_1\| &\le (1+1)D(G(x_0), G(x_1)), \\ \|v_0 - v_1\| &\le (1+1)D(F(y_0), F(y_1)), \end{aligned}$$

where D is the Hausdörff metric on $CB(E)$. Let

$$x_2 = (1-\alpha)x_1 + \alpha R^M_{A,\lambda}\left[A(x_1) + \lambda(S(x_1 - P(x_1), v_1)) + \lambda C^M_{A,\lambda}(x_1)\right],$$

and

$$y_2 = (1-\beta)y_1 + \beta R^N_{A,\rho}\left[A(y_1) + \rho(T(u_1, y_1 - q(y_1))) + \rho C^N_{A,\rho}(y_1)\right].$$

Again by Nadler's theorem [44], there exist $u_2 \in G(x_2)$ and $v_2 \in F(y_2)$ such that

$$\begin{aligned} \|u_1 - u_2\| &\le (1+2^{-1}])D(G(x_1), G(x_2)), \\ \|v_1 - v_2\| &\le (1+2^{-1})D(F(y_1), F(y_2)). \end{aligned}$$

In a similar way, we can compute the sequences $\{x_n\}, \{y_n\}, \{u_n\}$ and $\{v_n\}$ by the following scheme:

$$x_{n+1} = (1-\alpha)x_n + \alpha R^M_{A,\lambda}\left[A(x_n) + \lambda(S(x_n - P(x_n), v_n)) + \lambda C^M_{A,\lambda}(x_n)\right], \tag{8}$$

and

$$y_{n+1} = (1-\beta)y_n + \beta R^N_{A,\rho}\left[A(y_n) + \rho(T(u_n, y_n - q(y_n))) + \rho C^N_{A,\rho}(y_n)\right], \tag{9}$$

for $n = 0, 1, 2, \cdots$.

Choose $u_{n+1} \in G(x_{n+1}), v_{n+1} \in F(y_{n+1})$ such that

$$\|u_n - u_{n+1}\| \le \left(1 + (n+1)^{-1}\right) D(G(x_{n+1}), G(x_n)), \tag{10}$$

$$\|v_n - v_{n+1}\| \le \left(1 + (n+1)^{-1}\right) D(F(y_{n+1}), G(y_n)). \tag{11}$$

4. Existence of Solutions and Convergence of Iterative Sequences

We prove the following existence and convergence result for problem (3).

Theorem 1. *Let E be a real ordered uniformly smooth Banach space with modulus of smoothness $\tau_E(t) \leq \mathcal{C}t^2$ for some $\mathcal{C} > 0$ and $C \subseteq E$ be a normal cone with normal constant λ_N. Let $A : E \to E$; $S, T : E \times E \to E$ be single-valued mappings such that A is strongly comparison and β'-ordered compression mapping; S is Lipschitz continuous in both the arguments with constant λ_{S_1} and λ_{S_2}, respectively; T is Lipschitz continuous in both the arguments with constant λ_{T_1} and λ_{T_2}, respectively. Let $F, G : E \to CB(E)$ be multi-valued mappings such that F is λ_{D_F}-D-Lipschitz continuous and G is λ_{G_D}-D-Lipschitz continuous. Suppose that $P, q : E \to E$ be single-valued mappings such that P is δ_P-strongly accretive and λ_P-Lipschitz continuous; q is δ_q-strongly accretive and λ_q-Lipschitz continuous. Let $M : E \to 2^E$ be (α_M, λ)-XOR-NODSM mapping and $N : E \to 2^E$ be (α_N, ρ)-XOR-NODSM mapping. Suppose that the resolvent operators $R^M_{A,\lambda}, R^N_{A,\rho} : E \to E$ are θ-Lipschitz-type continuous and θ'-Lipschitz-type continuous, respectively, and the Cayley operators $C^M_{A,\lambda}, C^M_{A,\rho} : E \to E$ are $(2\theta + 1)$ and $(2\theta' + 1)$-Lipschitz-type continuous, respectively. Let $x_{n+1} \propto x_n$, $y_{n+1} \propto y_n$ and for some $\lambda, \rho > 0$ the following conditions are satisfied:*

$$
\begin{aligned}
0 &< \lambda_N \left[1 - \alpha(1 - \beta'\theta) + \alpha\theta\lambda\lambda_{S_1} B(P) + \alpha\theta\lambda(2\theta + 1) + \beta\theta'\rho\lambda_{T_1}\lambda_{G_D}\right] < 1, && (12)\\
0 &< \lambda_N \left[1 - \beta(1 - \beta'\theta') + \beta\theta'\rho\lambda_{T_2} B(q) + \beta\theta'\rho(2\theta' + 1) + \alpha\theta\lambda\lambda_{S_2}\lambda_{D_F}\right] < 1, && (13)
\end{aligned}
$$

where

$$
\begin{aligned}
B(P) &= \sqrt{1 - 2\delta_P + 64\mathcal{C}{\lambda_P}^2}, && (14)\\
B(q) &= \sqrt{1 - 2\delta_q + 64\mathcal{C}{\lambda_q}^2}, && (15)\\
\theta &= \frac{1}{\alpha_M\lambda \oplus \beta'}\,, && (16)\\
\theta' &= \frac{1}{\alpha_N\rho \oplus \beta'}\,. && (17)
\end{aligned}
$$

Then, the system of multi-valued mixed variational inclusions with XOR-operation $\oplus$ (3) have a solution(x, y, u, v), where $x, y \in E, u \in G(x), v \in F(y)$ such that $x_n \to x, y_n \to y, u_n \to u$ and $v_n \to v$ strongly, where $\{x_n\}, \{y_n\}, \{u_n\}$ and $\{v_n\}$ are the sequences generated by Algorithm 1.

Proof. As $x_{n+1} \propto x_n$, using (iii) of Proposition 2 and (8) of Algorithm 1, we have

$$
\begin{aligned}
0 \leq x_{n+1} \oplus x_n &= \left[(1-\alpha)x_n + \alpha R^M_{A,\lambda}\left[A(x_n) + \lambda S(x_n - P(x_n), v_n) + \lambda C^M_{A,\lambda}(x_n)\right]\right]\\
&\quad \oplus \left[(1-\alpha)x_{n-1} + \alpha R^M_{A,\lambda}\left[A(x_{n-1}) + \lambda S(x_{n-1} - P(x_{n-1}), v_{n-1}) + \lambda C^M_{A,\lambda}(x_{n-1})\right]\right]\\
&= (1-\alpha)(x_n \oplus x_{n-1}) + \alpha R^M_{A,\lambda}\left[A(x_n) + \lambda S(x_n - P(x_n), v_n) + \lambda C^M_{A,\lambda}(x_n)\right]\\
&\quad \oplus \alpha R^M_{A,\lambda}\left[A(x_{n-1}) + \lambda S(x_{n-1} - P(x_{n-1}), v_{n-1}) + \lambda C^M_{A,\lambda}(x_{n-1})\right]. && (18)
\end{aligned}
$$

Since the resolvent operator $R^M_{A,\lambda}$ is Lipschitz-type-continuous with constant θ and A is β'-compression mapping, we evaluate

$$\begin{aligned}
& (1-\alpha)(x_n \oplus x_{n-1}) + \alpha R^M_{A,\lambda}\left[A(x_n) + \lambda S(x_n - P(x_n), v_n) + \lambda C^M_{A,\lambda}(x_n)\right] \\
& \oplus \alpha R^M_{A,\lambda}\left[A(x_{n-1}) + \lambda S(x_{n-1} - P(x_{n-1}), v_{n-1}) + \lambda C^M_{A,\lambda}(x_{n-1})\right] \\
\leq\ & (1-\alpha)(x_n \oplus x_{n-1}) + \alpha\theta\Big\{\left[A(x_n) + \lambda S(x_n - P(x_n), v_n) + \lambda C^M_{A,\lambda}(x_n)\right] \\
& \oplus \left[A(x_{n-1}) + \lambda S(x_{n-1} - P(x_{n-1}), v_{n-1}) + \lambda C^M_{A,\lambda}(x_{n-1})\right]\Big\} \\
=\ & (1-\alpha)(x_n \oplus x_{n-1}) + \alpha\theta\,[A(x_n) \oplus A(x_{n-1})] + \alpha\theta\lambda\,[S(x_n - P(x_n), v_n) \oplus S(x_{n-1} - P(x_{n-1}), v_{n-1})] \\
& + \alpha\theta\lambda\left[C^M_{I,\lambda}(x_n) \oplus C^M_{A,\lambda}(x_{n-1})\right] \\
\leq\ & (1-\alpha)(x_n \oplus x_{n-1}) + \alpha\theta\beta'\,[x_n \oplus x_{n-1}] + \alpha\theta\lambda\,[S(x_n - P(x_n), v_n) \oplus S(x_{n-1} - P(x_{n-1}), v_{n-1})] \\
& + \alpha\theta\lambda\left[C^M_{A,\lambda}(x_n) \oplus C^M_{A,\lambda}(x_{n-1})\right]. \qquad (19)
\end{aligned}$$

Combining (18) and (19), we have

$$\begin{aligned}
0 \leq x_{n+1} \oplus x_n \ \leq\ & (1-\alpha)(x_n \oplus x_{n-1}) + \alpha\theta\beta'[x_n \oplus x_{n-1}] \\
& + \alpha\theta\lambda\,[S(x_n - P(x_n), v_n) \oplus S(x_{n-1} - P(x_{n-1}), v_{n-1})] \\
& + \alpha\theta\lambda\left[C^M_{A,\lambda}(x_n) \oplus C^M_{A,\lambda}(x_{n-1})\right]. \qquad (20)
\end{aligned}$$

Using (iii) of Proposition 3 and (20), we have

$$\begin{aligned}
\|x_{n+1} \oplus x_n\| \ \leq\ & \lambda_N \Big\| [1-\alpha(1-\beta'\theta)](x_n \oplus x_{n-1}) \\
& + \alpha\theta\lambda[S(x_n - P(x_n), v_n) \oplus S(x_{n-1} - P(x_{n-1}), v_{n-1})] \\
& + \alpha\theta\lambda[C^M_{A,\lambda}(x_n) \oplus C^M_{A,\lambda}(x_{n-1})] \Big\| \\
\leq\ & \lambda_N[1-\alpha(1-\beta'\theta)]\|x_n \oplus x_{n-1}\| \\
& + \lambda_N\alpha\theta\lambda\|S(x_n - P(x_n), v_n) \oplus S(x_{n-1} - P(x_{n-1}), v_{n-1})\| \\
& + \lambda_N\alpha\theta\lambda\left\|C^M_{A,\lambda}(x_n) \oplus C^M_{A,\lambda}(x_{n-1})\right\|. \qquad (21)
\end{aligned}$$

Using the Lipschitz continuity of S in both the arguments with constants λ_{s_1} and λ_{s_2}, respectively, and using (iii) of Proposition 3, we obtain

$$\begin{aligned}
& \|S(x_n - P(x_n), v_n) \oplus S(x_{n-1} - P(x_{n-1}), v_{n-1})\| \\
=\ & \|S(x_n - P(x_n), v_n) \oplus S(x_n - P(x_n), v_{n-1}) \\
& \oplus S(x_n - P(x_n), v_{n-1}) \oplus S(x_{n-1} - P(x_{n-1}), v_{n-1})\| \\
\leq\ & \|[S(x_n - P(x_n), v_n) \oplus S(x_n - P(x_n), v_{n-1})] \\
& - [S(x_n - P(x_n), v_{n-1}) \oplus S(x_{n-1} - P(x_{n-1}), v_{n-1})]\| \\
\leq\ & \|S(x_n - P(x_n), v_n) \oplus S(x_n - P(x_n), v_{n-1})\| \\
& + \|S(x_n - P(x_n), v_{n-1}) \oplus S(x_{n-1} - P(x_{n-1}), v_{n-1})\| \\
\leq\ & \|S(x_n - P(x_n), v_n) - S(x_n - P(x_n), v_{n-1})\| \\
& + \|S(x_n - P(x_n), v_{n-1}) - S(x_{n-1} - P(x_{n-1}), v_{n-1})\| \\
\leq\ & \lambda_{S_2}\|v_n - v_{n-1}\| + \lambda_{S_1}\|x_n - x_{n-1} - (P(x_n) - P(x_{n-1}))\|. \qquad (22)
\end{aligned}$$

Using D-Lipschitz continuity of F, we have

$$\|v_n - v_{n-1}\| \leq \left(1 + n^{-1}\right) D(F(y_n), F(y_{n-1})) \leq \left(1 + n^{-1}\right) \lambda_{D_F} \|y_n - y_{n-1}\|. \tag{23}$$

Since P is strongly accretive with constant δ_p and Lipschitz continuous with constant λ_p, using the techniques of Alber and Yao [45] and Proposition 1, for $j(x_n - x_{n-1}) \in J(x_n - x_{n-1})$, we have

$$\begin{aligned}
& \|x_n - x_{n-1} - (P(x_n) - P(x_{n-1}))\|^2 \\
\leq\ & \|x_n - x_{n-1}\|^2 - 2\langle P(x_n) - P(x_{n-1}), j(x_n - x_{n-1} - (P(x_n) - P(x_{n-1}))\rangle, \\
=\ & \|x_n - x_{n-1}\|^2 - 2\langle P(x_n) - P(x_{n-1}), j(x_n - x_{n-1})\rangle \\
& -2\langle P(x_n) - P(x_{n-1}), j(x_n - x_{n-1} - (P(x_n) - P(x_{n-1})) - j(x_n - x_{n-1})\rangle, \\
\leq\ & \|x_n - x_{n-1}\|^2 - 2\delta_p\|x_n - x_{n-1}\|^2 + 4C^2\tau_E\left[\frac{4\|P(x_n) - P(x_{n-1})\|}{\mathcal{C}}\right], \\
\leq\ & \|x_n - x_{n-1}\|^2 - 2\delta_p\|x_n - x_{n-1}\|^2 + 64\mathcal{C}\|P(x_n) - P(x_{n-1}\|^2, \\
\leq\ & (1 - 2\delta_P + 64\mathcal{C}\lambda_P{}^2)\|x_n - x_{n-1}\|^2, \\
=\ & B^2(P)\|x_n - x_{n-1}\|^2,
\end{aligned} \tag{24}$$

where $B(p) = \sqrt{1 - 2\delta_P + 64\mathcal{C}\lambda_P{}^2}$.

Since the Cayley operator $C^M_{A,\lambda}$ is Lipschitz-type-continuous with constant $(2\theta + 1)$ and using (iii) of Proposition 3, we obtain

$$\left\|C^M_{A,\lambda}(x_n) \oplus C^M_{A,\lambda}(x_{n-1})\right\| \leq (2\theta + 1)\|x_n \oplus x_{n-1}\| \leq (2\theta + 1)\|x_n - x_{n-1}\|, \tag{25}$$

where $\theta = \dfrac{1}{\alpha_N\lambda \oplus \beta'}$.

As $x_{n+1} \propto x_n$ and combining (22) to (25) with (21), we obtain

$$\begin{aligned}
\|x_{n+1} - x_n\| \leq\ & \lambda_N[1 - \alpha(1 - \beta'\theta)]\|x_n - x_{n-1}\| + \lambda_N\alpha\theta\lambda\Big\{\lambda_{S_2}\left(1 + n^{-1}\right)\lambda_{D_F}\|y_n - y_{n-1}\| \\
& +\lambda_{S_1}B(P)\|x_n - x_{n-1}\|\Big\} + \lambda_N\alpha\theta\lambda(2\theta + 1)\|x_n - x_{n-1}\| \\
=\ & \lambda_N[1 - \alpha(1 - \beta'\theta) + \alpha\theta\lambda\lambda_{S_1}B(P) + \alpha\theta\lambda(2\theta + 1)]\|x_n - x_{n-1}\| \\
& +\lambda_N\alpha\theta\lambda\lambda_{S_2}\left(1 + n^{-1}\right)\lambda_{D_F}\|y_n - y_{n-1}\|.
\end{aligned} \tag{26}$$

As $y_{n+1} \propto y_n$, using (iii) of Proposition 2 and (9) of Algorithm 1, we have

$$\begin{aligned}
0 \leq y_{n+1} \oplus y_n =\ & \left[(1 - \beta)y_n + \beta R^N_{A,\rho}\left[A(y_n) + \rho T(u_n, y_n - q(y_n)) + \rho C^N_{A,\rho}(y_n)\right]\right] \\
& \oplus \left[(1 - \beta)y_{n-1} + \beta R^N_{A,\rho}\left[A(y_{n-1}) + \rho T(u_{n-1}, y_{n-1} - q(y_{n-1})) + \rho C^N_{A,\rho}(y_{n-1})\right]\right] \\
=\ & (1 - \beta)(y_n \oplus y_{n-1}) + \beta R^N_{A,\rho}\left[A(y_n) + \rho T(u_n, y_n - q(y_n)) + \rho C^N_{A,\rho}(y_n)\right] \\
& \oplus\beta R^N_{A,\rho}\left[A(y_{n-1}) + \rho T(u_{n-1}, y_{n-1} - q(y_{n-1})) + \rho C^N_{A,\rho}(y_{n-1})\right].
\end{aligned} \tag{27}$$

Since the resolvent operator $R^M_{A,\rho}$ is Lipschitz-type-continuous with constant θ' and A is β'-compression mapping, we evaluate

$$\begin{aligned}
& (1-\beta)(y_n \oplus y_{n-1}) + \beta R^N_{A,\rho}\left[A(y_n) + \rho T(u_n, y_n - q(y_n)) + \rho C^N_{A,\rho}(y_n)\right] \\
& \oplus \beta R^N_{A,\rho}\left[A(y_{n-1}) + \rho T(u_{n-1}, y_{n-1} - q(y_{n-1})) + \rho C^N_{A,\rho}(y_{n-1})\right] \\
\leq\ & (1-\beta)(y_n \oplus y_{n-1}) + \beta\theta'\Big\{\left[A(y_n) + \rho T(u_n, y_n - q(y_n)) + \rho C^N_{A,\rho}(y_n)\right] \\
& \oplus A(y_{n-1}) + \rho T(u_{n-1}, y_{n-1} - q(y_{n-1})) + \rho C^N_{A,\rho}(y_{n-1})\Big\} \\
=\ & (1-\beta)(y_n \oplus y_{n-1}) + \beta\theta'[A(y_n) \oplus A(y_{n-1})] \\
& + \beta\theta'\rho[T(u_n, y_n - q(y_n)) \oplus T(u_{n-1}, y_{n-1} - q(y_{n-1}))] + \beta\theta'\rho\left[C^N_{A,\rho}(y_n) \oplus C^N_{A,\rho}(y_{n-1})\right] \\
\leq\ & (1-\beta)(y_n \oplus y_{n-1}) + \beta\theta'\beta'[y_n \oplus y_{n-1}] \\
& + \beta\theta'\rho[T(u_n, y_n - q(y_n)) \oplus T(u_{n-1}, y_{n-1} - q(y_{n-1}))] + \beta\theta'\rho\left[C^N_{A,\rho}(y_n) \oplus C^N_{A,\rho}(y_{n-1})\right]. \quad (28)
\end{aligned}$$

Combining (27) and (28), we have

$$\begin{aligned}
0 \leq y_{n+1} \oplus y_n \leq\ & (1-\beta)(y_n \oplus y_{n-1}) + \beta\theta'\beta'[y_n \oplus y_{n-1}] \\
& + \beta\theta'\rho[T(u_n, y_n - q(y_n)) \oplus T(u_{n-1}, y_{n-1} - q(y_{n-1}))] \\
& + \beta\theta'\rho\left[C^N_{A,\rho}(y_n) \oplus C^N_{A,\rho}(y_{n-1})\right]. \quad (29)
\end{aligned}$$

Using (iii) of Proposition 3 and (29), we have

$$\begin{aligned}
\|y_{n+1} \oplus y_n\| \leq\ & \lambda_N \|[1 - \beta(1 - \beta'\theta')](y_n \oplus y_{n-1}) \\
& + \beta\theta'\rho[T(u_n, y_n - q(y_n)) \oplus T(u_{n-1}, y_{n-1} - q(y_{n-1}))] \\
& + \beta\theta'\rho\left(C^N_{A,\rho}(y_n) \oplus C^N_{A,\rho}(y_{n-1})\right)\| \\
\leq\ & \lambda_N \|[1 - \beta(1 - \beta'\theta')](y_n \oplus y_{n-1})\| \\
& + \lambda_N \beta\theta'\rho\|T(u_n, y_n - q(y_n)) \oplus T(u_{n-1}, y_{n-1} - q(y_{n-1}))\| \\
& + \lambda_N \beta\theta'\rho\left\|C^N_{A,\rho}(y_n) \oplus C^N_{A,\rho}(y_{n-1})\right\|. \quad (30)
\end{aligned}$$

Using Lipschitz continuity of T in both the arguments with constant λ_{T_1} and λ_{T_2}, respectively using (iii) of Proposition 3, we obtain

$$\begin{aligned}
& \|T(u_n, y_n - q(y_n)) \oplus T(u_{n-1}, y_{n-1} - q(y_{n-1}))\| \\
=\ & \|T(u_n, y_n - q(y_n)) \oplus T(u_{n-1}, y_n - q(y_n)) \\
& \oplus T(u_{n-1}, y_n - q(y_n)) \oplus T(u_{n-1}, y_{n-1} - q(y_{n-1}))\| \\
\leq\ & \|[T(u_n, y_n - q(y_n)) \oplus T(u_{n-1}, y_n - q(y_n))] \\
& - [T(u_{n-1}, y_n - q(y_n)) \oplus T(u_{n-1}, y_{n-1} - q(y_{n-1}))]\| \\
\leq\ & \|T(u_n, y_n - q(y_n)) \oplus T(u_{n-1}, y_n - q(y_n))\| \\
& + \|T(u_{n-1}, y_n - q(y_n)) \oplus T(u_{n-1}, y_{n-1} - q(y_{n-1}))\|, \\
\leq\ & \|T(u_n, y_n - q(y_n)) - T(u_{n-1}, y_n - q(y_n))\| \\
& + \|T(u_{n-1}, y_n - q(y_n)) - T(u_{n-1}, y_{n-1} - q(y_{n-1}))\| \\
\leq\ & \lambda_{T_1}\|u_n - u_{n-1}\| + \lambda_{T_2}\|y_n - y_{n-1} - (q(y_n) - q(y_{n-1}))\|. \quad (31)
\end{aligned}$$

Using D-Lipschitz continuity of G, we have

$$\|u_n - u_{n-1}\| \leq \left(1 + n^{-1}\right) D(G(x_n), G(x_{n-1})) \leq \left(1 + n^{-1}\right) \lambda_{G_D} \|x_n - x_{n-1}\|. \tag{32}$$

Since q is strongly accretive and Lipschitz continuous, using the same techniques as for (24), we have

$$\begin{aligned} \|y_n - y_{n-1} - (q(y_n) - q(y_{n-1}))\|^2 &\leq (1 - 2\delta_q + 64\mathcal{C}\lambda_q{}^2)\|y_n - y_{n-1}\|^2, \\ &= B^2(q)\|y_n - y_{n-1}\|^2, \end{aligned} \tag{33}$$

where $B(q) = \sqrt{1 - 2\delta_q + 64\mathcal{C}\lambda_q{}^2}$.

Since the Cayley operator $C^N_{A,\rho}$ is Lipschitz-type-continuous with constant $(2\theta' + 1)$, we obtain

$$\left\|C^N_{A,\rho}(y_n) \oplus C^N_{A,\rho}(y_{n-1})\right\| \leq (2\theta' + 1)\|y_n \oplus y_{n-1}\| \leq (2\theta' + 1)\|y_n - y_{n-1}\|, \tag{34}$$

where $\theta' = \dfrac{1}{\alpha_N \rho \oplus \beta'}$.

As $y_{n+1} \propto y_n$ and combining (31) to (34) with (30), we have

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq \lambda_N[1 - \beta(1 - \beta'\theta')]\|y_n - y_{n-1}\| + \lambda_N \beta\theta'\rho\Big\{\lambda_{T_1}\left(1 + n^{-1}\right)\lambda_{G_D}\|x_n - x_{n-1}\| \\ &\quad + \lambda_{T_2} B(q)\|y_n - y_{n-1}\|\Big\} + \lambda_N \beta\theta'\rho(2\theta' + 1)\|y_n - y_{n-1}\| \\ &= \lambda_N[[1 - \beta(1 - \beta'\theta')] + \beta\theta'\rho\lambda_{T_2} B(q) + \beta\theta'\rho(2\theta' + 1)]\|y_n - y_{n-1}\| \\ &\quad + \lambda_N \beta\theta'\rho\lambda_{T_1}\left(1 + n^{-1}\right)\lambda_{G_D}]\|x_n - x_{n-1}\|. \end{aligned} \tag{35}$$

Combining (26) and (35), we have

$$\begin{aligned} \|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| &\leq \Big\{\lambda_N[[1 - \alpha(1 - \beta'\theta)] + \alpha\theta\lambda\lambda_{S_1} B(P) + \alpha\theta\lambda(2\theta + 1) \\ &\quad + \beta\theta'\rho\lambda_{T_1}\left(1 + n^{-1}\right)\lambda_{G_D}]\Big\}\|x_n - x_{n-1}\| \\ &\quad + \Big\{\lambda_N[[1 - \beta(1 - \beta'\theta')] + \beta\theta'\rho\lambda_{T_2} B(q) + \beta\theta'\rho(2\theta' + 1) \\ &\quad + \alpha\theta\lambda\lambda_{S_2}\left(1 + n^{-1}\right)\lambda_{D_F}]\Big\}\|y_n - y_{n-1}\| \\ &\leq \xi(\theta_n, \theta'_n)\{\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|\}, \end{aligned} \tag{36}$$

where

$$\begin{aligned} \xi(\theta_n, \theta'_n) = \max\Big\{&\{\lambda_N[[1 - \alpha(1 - \beta'\theta)] + \alpha\theta\lambda\lambda_{S_1} B(P) + \alpha\theta\lambda(2\theta + 1) + \beta\theta'\rho\lambda_{T_1}(1 + n^{-1})\lambda_{G_D}]\}, \\ &\{\lambda_N[[1 - \beta(1 - \beta'\theta')] + \beta\theta'\rho\lambda_{T_2} B(q) + \beta\theta'\rho(2\theta' + 1) + \alpha\theta\lambda\lambda_{S_2}(1 + n^{-1})\lambda_{D_F}]\}\Big\}. \end{aligned}$$

Let

$$\begin{aligned} \xi(\theta, \theta') = \max\Big\{&\{\lambda_N[[1 - \alpha(1 - \beta'\theta)] + \alpha\theta\lambda\lambda_{S_1} B(P) + \alpha\theta\lambda(2\theta + 1) + \beta\theta'\rho\lambda_{T_1}\lambda_{G_D}]\}, \\ &\{\lambda_N[[1 - \beta(1 - \beta'\theta')] + \beta\theta'\rho\lambda_{T_2} B(q) + \beta\theta'\rho(2\theta' + 1) + \alpha\theta\lambda\lambda_{S_2}\lambda_{D_F}]\}\Big\}. \end{aligned}$$

Conditions (12) and (13) imply that $0 < \xi(\theta, \theta') < 1$ and so $0 < \xi(\theta_n, \theta'_n) < 1$, when n is sufficiently large. It follows from (36) that $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences. Thus, we can assume that $x_n \to x$ and $y_n \to y$, strongly.

It follows from (23) and (32), that $\{u_n\}$ and $\{v_n\}$ are also Cauchy sequences, we can assume that $u_n \to u$ and $v_n \to v$, strongly.

Now we shown that $u \in G(x)$ as $v \in F(y)$, since $u_n \in G(x_n)$, we have

$$\begin{aligned} d(u, G(x)) &\leq \|u - u_n\| + d(u_n, G(x)) \\ &\leq \|u - u_n\| + \left(1 + n^{-1}\right) D(G(x_n), G(x)) \\ &\leq \|u - u_n\| + \left(1 + n^{-1}\right) \lambda_{D_G} \|x_n - x\| \to 0, \ \ as \ \ n \to \infty. \end{aligned}$$

Hence $d(u, G(x)) \to 0$, so $u \in G(x)$ as $G(x) \in CB(E)$. Similarly, we can show that $v \in F(y)$. By Lemma 1, we conclude that (x, y, u, v) is a solution to a system of multi-valued mixed variational inclusions with XOR-operation $\oplus$ (3). □

The following example shows that all the assumptions and conditions of Theorem 1 are satisfied.

Example 1. *Let $E = \mathbb{R}^2$ with the usual inner product and $C = [0,1] \times [0,1] \subset \mathbb{R}^2$ be a normal cone with normal constant $\lambda_N = 1$. Suppose that $A : \mathbb{R}^2 \to \mathbb{R}^2$, $S, T, : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, $P, q : \mathbb{R}^2 \to \mathbb{R}^2$ are single valued mappings and $R^M_{A,\lambda}$, $R^N_{A,\rho}$, $C^M_{A,\lambda}$, $C^N_{A,\rho} : \mathbb{R}^2 \to \mathbb{R}^2$ be resolvent operators and Cayley operators, respectively, for some $\lambda, \rho > 0$.*

Let $F, G : \mathbb{R}^2 \to CB(\mathbb{R}^2)$ and $M, N : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be multi-valued mappings. Then, we define all the mappings mentioned above as:

$$\begin{aligned}
A(x) &= \left(\frac{x_1}{5}, \frac{x_2}{5}\right), \ \ for\ all \ \ x = (x_1, x_2) \in \mathbb{R}^2, \\
S(x, y) &= \left(\frac{x_1}{2} + y_1, \frac{x_2}{2} + y_2\right), \ for\ all\ x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2, \\
T(x, y) &= \left(x_1 + \frac{y_1}{3}, x_2 + \frac{y_2}{3}\right), \ for\ all\ x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2, \\
P(x) &= \left(\frac{x_1}{3}, \frac{x_2}{3}\right), \ \ for\ all \ \ x = (x_1, x_2) \in \mathbb{R}^2, \\
q(x) &= \left(\frac{x_1}{2}, \frac{x_2}{2}\right), \ \ for\ all \ \ x = (x_1, x_2) \in \mathbb{R}^2, \\
M(x) &= \left\{(2x_1, 2x_2) | (x_1, x_2) \in \mathbb{R}^2\right\}, \\
N(x) &= \left\{(3x_1, 3x_2) | (x_1, x_2) \in \mathbb{R}^2\right\}, \\
R^M_{A,\lambda}(x) &= \left(\frac{5x_1}{9}, \frac{5x_2}{9}\right), \ for\ all\ x = (x_1, x_2) \in \mathbb{R}^2, \\
C^M_{A,\lambda}(x) &= \left(\frac{x_1}{9}, \frac{x_2}{9}\right), \ for\ all\ x = (x_1, x_2) \in \mathbb{R}^2, \\
R^N_{A,\rho}(x) &= \left(\frac{10x_1}{13}, \frac{10x_2}{13}\right), \ for\ all\ x = (x_1, x_2) \in \mathbb{R}^2, \\
C^N_{A,\rho}(x) &= \left(\frac{7x_1}{13}, \frac{7x_2}{13}\right), \ for\ all\ x = (x_1, x_2) \in \mathbb{R}^2, \\
F(x) &= \left\{(x_1, 3) | x = (x_1, x_2) \in \mathbb{R}^2\ such\ that\ 0 \leq x_1 \leq 1\right\}, \\
G(x) &= \left\{(2, x_2) | x = (x_1, x_2) \in \mathbb{R}^2\ such\ that\ 0 \leq x_2 \leq 1\right\}.
\end{aligned}$$

(1) Clearly, A is strongly comparison mapping and

$$\begin{aligned} A(x) \oplus A(y) &= \left(\frac{x_1}{5}, \frac{x_2}{5}\right) \oplus \left(\frac{y_1}{5}, \frac{y_2}{5}\right) \\ &= \frac{1}{5}(x \oplus y) \\ &\leq \frac{2}{5}(x \oplus y), \ for\ all\ x = (x_1, x_2) \in \mathbb{R}^2,\ y = (y_1, y_2) \in \mathbb{R}^2. \end{aligned}$$

That is, A is $\frac{2}{5}$*-ordered compression mapping.*
(2) It is easy to check that S is Lipschitz continuous in both the arguments with constants $\frac{3}{2}$ *and* 1*, respectively and T is Lipschitz continuous in both the arguments with constants* 1 *and* $\frac{1}{3}$*, respectively.*
(3) For $j(x-y) \in J(x-y)$*, we calculate*

$$\begin{aligned} \langle P(x) - P(y), j(x-y) \rangle &= \langle P(x) - P(y), x - y \rangle \\ &= \left\langle \left(\frac{x_1 - y_1}{3}, \frac{x_2 - y_2}{3}\right), (x_1 - y_1, x_2 - y_2) \right\rangle \\ &= \frac{1}{3}\|x_1 - x_2\|^2 + \frac{1}{3}\|x_2 - y_2\|^2 = \frac{1}{3}\|x - y\|^2 \\ &\geq \frac{1}{5}\|x - y\|^2, \end{aligned}$$

and $\|P(x) - P(y)\| \leq \frac{2}{3}\|x - y\|$*. Thus, P is strongly accretive with constant* $\frac{1}{5}$ *and Lipschitz continuous with constant* $\frac{2}{3}$*.*

Similarly, we can show that q is strongly accretive with constant $\frac{1}{3}$ *and Lipschitz continuous with constant* $\frac{2}{3}$*.*
(4) One can easily show that the resolvent operators $R^M_{A,\lambda}$ *is* $\frac{5}{12}$*-Lipschitz-type-continuous,* $R^N_{A,\rho}$ *is* $\frac{10}{19}$*-Lipschitz type continuous, the Cayley operators* $C^M_{A,\lambda}$ *is* $\frac{11}{6}$*-Lipschitz-type continuous and* $C^N_{A,\rho}$ *is* $\frac{39}{19}$*-Lipschitz-type-continuous.*

Also, M is a comparison mapping and 2*-non-ordinary difference mapping, N is a comparison mapping and* 3*-non-ordinary difference mapping.*

Let $v_x = (2x_1, 2x_2) \in M(x)$ *and* $v_y = (2y_1, 2y_2) \in M(y)$*, then*

$$(v_x \oplus v_y) \oplus \alpha_M(x \oplus y) = 2[(x \oplus y) \oplus (x \oplus y)] = 0.$$

For $\lambda = 1$, $[A \oplus \lambda M](\mathbb{R}^2) = \mathbb{R}^2$ *and for* $\rho = \frac{1}{2}$, $[A \oplus \rho N](\mathbb{R}^2) = \mathbb{R}^2$*. This shows that M is* $(2,1)$*-XOR-NODSM mapping and N is* $(3, \frac{1}{2})$*-XOR-NODSM mapping.*
(5) Clearly, F and G are D-Lipschitz continuous mappings with constants 2 *and* 3 *respectively.*
(6) In order to satisfy condition (12) *and* (13) *of Theorem 1, we calculate*

$$B(p) = \sqrt{1 - 2\delta_P + 64\mathcal{C}\lambda_P^2} = \pm 1.04, \quad for\ \mathcal{C} = \frac{1}{64}$$

and

$$B(q) = \sqrt{1 - 2\delta_q + 64\mathcal{C}\lambda_q^2} = \pm 1.60, \quad for\ \mathcal{C} = \frac{1}{64}.$$

We choose $B(p) = -1.04$ *and* $B(q) = -1.60$ *and we claim that the conditions* (12) *and* (13) *are satisfied. That is,*

$$0 < \lambda_N \left[1 - \alpha(1 - \beta'\theta) + \alpha\theta\lambda\lambda_{S_1} B(P) + \alpha\theta\lambda(2\theta + 1) + \beta\theta'\rho\lambda_{T_1}\lambda_{G_D}\right] = 0.82 < 1$$

and

$$0 < \lambda_N \left[1 - \beta(1 - \beta'\theta') + \beta\theta'\rho\lambda_{T_2} B(q) + \beta\theta'\rho(2\theta' + 1) + \alpha\theta\lambda\lambda_{S_2}\lambda_{D_F}\right] = 0.97 < 1.$$

Thus, all the assumptions and conditions of Theorem 1 are satisfied.

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