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A New Fixed Point Result of Perov Type and Its Application to a Semilinear Operator System

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Abstract: In this paper, we present a new generalization of the Perov fixed point theorem on vector-valued metric space. Moreover, to show the significance of our result, we present both a nontrivial comparative example and an application to a kind of semilinear operator system about the existence of its solution.

Keywords: fixed point; vector-valued metric; eigenvalues; eigenvectors; θ-contraction

1. Introduction and Preliminaries

The well-known Banach contraction mapping principle plays a crucial role in the functional analysis and ensures the existence and uniqueness of a fixed point on a complete metric space. Many generalizations of this principle have been given either by taking into account more general contractive inequality or by changing the structure of space. In this context Perov [1] has presented this principle in vector-valued metric spaces. Many contributions in this aspect have been obtained (see, for example, Abbas et al. [2], Altun and Olgun [3], Cvetković and Rakočević [4,5], Flip and Petruşel [6], Ilić et al. [7] and Vetro and Radenović [8]). As we can see in [1,9], the results in this aspect can be used to guarantee the existence of solutions of some Cauchy problems. In order to talk about the contribution of Perov, we need to remember the following notations: Let $\mathbb{R}^m$ be the set of $m \times 1$ real matrices, X be a nonempty set and $d : X \times X \to \mathbb{R}^m$ be a function. Then d is said to be a vector-valued metric and in this case (X, d) is said to be vector-valued metric space, if the following properties are satisfied: For all $x, y, z \in X$

(i) $d(x, y) = \mathbf{0}$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \preceq d(x, z) + d(z, y)$,

where $\mathbf{0}$ is the zero $m \times 1$ matrix and $\preceq$ is the coordinate-wise ordering on $\mathbb{R}^m$, that is, for $\alpha = (\alpha_i)_{i=1}^m, \beta = (\beta_i)_{i=1}^m \in \mathbb{R}^m$

$$\alpha \preceq \beta \Leftrightarrow \alpha_i \leq \beta_i \text{ for each } i \in \{1, 2, \ldots, m\}.$$

For the rest of this paper $\alpha \preceq \beta$ and $\beta \succeq \alpha$ will be the same and $\alpha \prec \beta$ will be $\alpha_i < \beta_i$ for each $i \in \{1, 2, \ldots, m\}$. Moreover, we denote by $\mathbb{R}_+$ the set of non-negative real numbers, by $\mathbb{R}_+^m$ the set of $m \times 1$ real matrices with non-negative elements, by $\mathcal{M}_m^m(\mathbb{R}_+)$ the set of all $m \times m$ matrices with non-negative elements, by θ the zero $m \times m$ matrix, by I the identity $m \times m$ matrix. If $M \in \mathcal{M}_m^m(\mathbb{R}_+)$, then the symbol M^T stands for the transpose matrix of M. For the sake of simplicity, we will make an identification between row and column vectors in $\mathbb{R}^m$.

Notice that the convergence and Cauchyness of a sequence and completeness of the space in a vector-valued metric space are defined in a similar manner as in the usual metric space. Let $M \in \mathcal{M}_m^m(\mathbb{R}_+)$, then M is said to be convergent to zero if and only if $M^n \to \theta$ as $n \to \infty$ (See [10]).

Theorem 1 ([10])**.** *Let $M \in \mathcal{M}_m^m(\mathbb{R}_+)$. Then the following conditions are equivalent:*

1. *M is convergent to zero,*
2. *the eigenvalues of M are in the open unit disc, that is, $|\lambda| < 1$ for every $\lambda \in \mathbb{C}$ with $\det(M - \lambda I) = 0$,*
3. *the matrix $I - M$ is nonsingular and*

$$(I - M)^{-1} = I + M + \cdots + M^n + \ldots.$$

We can find some examples of matrices convergent to zero in the literature.

Example 1. *Any matrix in $\mathcal{M}_2^2(\mathbb{R}_+)$ of the form*

$$M = \begin{pmatrix} a & a \\ b & b \end{pmatrix} \text{ or } M = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$$

with $a + b < 1$, converges to zero.

Example 2. *If $\max\{a, c\} < 1$, then the matrix*

$$M = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

in $\mathcal{M}_2^2(\mathbb{R}_+)$ also converges to zero.

Example 3. *If $\max\{\gamma_i : i \in \{1, 2, \ldots, m\}\} < 1$, then the matrix*

$$M = \begin{pmatrix} \gamma_1 & 0 & \ldots & 0 \\ 0 & \gamma_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \gamma_m \end{pmatrix}_{m \times m}$$

in $\mathcal{M}_m^m(\mathbb{R}_+)$ converges to zero.

Example 4. *If $a + b \geq 1$ and $c + d \geq 1$, then the matrix*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $\mathcal{M}_2^2(\mathbb{R}_+)$ does not converges to zero.

Now we can state the contribution of Perov [1].

Theorem 2 ([1])**.** *Let (X, d) be a vector-valued metric space and $T : X \to X$ be a Perov contraction, that is, a mapping with the property that there exists a matrix $M \in \mathcal{M}_m^m(\mathbb{R}_+)$ which converges to zero such that*

$$d(Tx, Ty) \preceq Md(x, y).$$

Then

1. *T has a unique fixed point in X, say z,*
2. *for all $x_0 \in X$, the sequence of successive approximations $\{x_n\}$ defined by $x_n = T^n x_0$ is convergent to z,*
3. *one has the following estimation:*

$$d(x_n, z) \preceq M^n (I - M)^{-1} d(x_0, Tx_0).$$

Example 5. *Consider the vector-valued metric space (X, d) where $X = \left\{ x_n = \frac{1}{n^2} : n \in \{1, 2, \dots\} \right\} \cup \{0\}$ and $d : X \times X \to \mathbb{R}^2$ is given by*

$$d(x, y) = (|x - y|, |x - y|).$$

Define a mapping $T : X \to X$ by

$$Tx = \begin{cases} 0 & , \quad x = 0 \\ x_{n+1} & , \quad x = x_n \end{cases}.$$

Now we claim that T is not a Perov contraction. Assume the contrary. Then there exists a matrix $M \in \mathcal{M}_2^2(\mathbb{R}_+)$ such that M is convergent to zero and

$$d(Tx, Ty) \preceq Md(x, y) \tag{1}$$

holds for all $x, y \in X$. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then from Equation (1), for $x = x_n$ and $y = 0$, we get

$$\begin{aligned} (x_{n+1}, x_{n+1}) &= d(Tx_n, T0) \\ &\preceq Md(x_n, 0) \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_n \\ x_n \end{pmatrix} \\ &= ((a + b)x_n, (c + d)x_n). \end{aligned}$$

Therefore, since $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$, this last inequality implies that $a + b \geq 1$ and $c + d \geq 1$. On the other hand one of the eigenvalues of M is

$$\lambda = \frac{1}{2}\left(a + d + \sqrt{(a - d)^2 + 4bc}\right)$$

and the by routine calculation we can see that $\lambda \geq 1$. Therefore, since one of the eigenvalues of M does not lie in the open unit disc, then from Theorem 1, M does not converge to zero. This is a contradiction. Hence T is not a Perov contraction.

In this paper, by considering the recent technique of Jleli and Samet [11], we present a new generalization of the Perov fixed point theorem. This technique is known as θ-contraction in the literature and there are many studies using this technique (See for example [12–14]). Let $\Theta : \mathbb{R}^m_{+0} \to \mathbb{R}^m_{+1}$ be a function, where $\mathbb{R}^m_{+j}$ is the set of $m \times 1$ real matrices with every element being greater than j. For the sake of completeness, we will consider the following conditions:

(Θ1) Θ is nondecreasing in each variable, i.e., for all $\alpha = (\alpha_i)_{i=1}^m, \beta = (\beta_i)_{i=1}^m \in \mathbb{R}^m_{+0}$ such that $\alpha \preceq \beta$, then $\Theta(\alpha) \preceq \Theta(\beta)$,

(Θ2) For each sequence $\{\alpha_n\} = (\alpha_n^{(1)}, \alpha_n^{(2)}, \dots, \alpha_n^{(m)})$ of $\mathbb{R}^m_{+0}$

$$\lim_{n\to\infty} \alpha_n^{(i)} = 0^+ \text{ if and only if } \lim_{n\to\infty} \beta_n^{(i)} = 1$$

for each $i \in \{1, 2, \dots, m\}$, where

$$\Theta((\alpha_n^{(1)}, \alpha_n^{(2)}, \ldots, \alpha_n^{(m)})) = (\beta_n^{(1)}, \beta_n^{(2)}, \ldots, \beta_n^{(m)}).$$

(Θ3) There exist $r \in (0,1)$ and $l \in (0, \infty]$ such that $\lim_{\alpha_i \to 0^+} \frac{\beta_i - 1}{\alpha_i^r} = l$ for each $i \in \{1, 2, \ldots, m\}$, where

$$\Theta((\alpha_1, \alpha_2, \ldots, \alpha_m)) = (\beta_1, \beta_2, \ldots, \beta_m).$$

We denote by Ξ^m the set of all functions Θ satisfying (Θ1)–(Θ3).

Example 6. *Define* $\Theta : \mathbb{R}^m_{+0} \to \mathbb{R}^m_{+1}$ *by*

$$\Theta((\alpha_1, \alpha_2, \ldots, \alpha_m)) = (\exp \sqrt{\alpha_1}, \exp \sqrt{\alpha_2}, \ldots, \exp \sqrt{\alpha_m}),$$

then $\Theta \in \Xi^m$.

Example 7. *Define* $\Theta : \mathbb{R}^2_{+0} \to \mathbb{R}^2_{+1}$ *by*

$$\Theta((\alpha_1, \alpha_2)) = (\exp \sqrt{\alpha_1}, \exp \sqrt{\alpha_2 \exp \alpha_2}),$$

then $\Theta \in \Xi^2$.

By considering the class Ξ^m, we introduce the concept of Perov type Θ-contraction as follows: Here we use the notation $\Lambda^{[k]} := (\Lambda_i^{k_i})_{i=1}^m$ for $\Lambda = (\Lambda_i)_{i=1}^m \in \mathbb{R}^m_+$ and $k = (k_i)_{i=1}^m \in \mathbb{R}^m_+$.

Definition 1. *Let* (X, d) *be a vector-valued metric space and* $T : X \to X$ *be a map. If there exist* $\Theta \in \Xi^m$ *and* $k = (k_i)_{i=1}^m \in \mathbb{R}^m_+$ *with* $k_i < 1$ *for all* $i \in \{1, 2, \ldots, m\}$ *such that*

$$\Theta(d(Tx, Ty)) \preceq [\Theta(d(x, y))]^{[k]}, \tag{2}$$

for all $x, y \in X$ *with* $d(Tx, Ty) \succ \mathbf{0}$, *then* T *is called a Perov type* Θ*-contraction.*

If we consider $\Theta : \mathbb{R}^m_{+0} \to \mathbb{R}^m_{+1}$ by

$$\Theta((\alpha_1, \alpha_2, \ldots, \alpha_m)) = (\exp \sqrt{\alpha_1}, \exp \sqrt{\alpha_2}, \ldots, \exp \sqrt{\alpha_m}),$$

then Equation (2) turns out to be a Perov contraction. Indeed, if we represent

$$d(Tx, Ty) = (\Lambda_1, \Lambda_2, \ldots, \Lambda_m)$$

and

$$d(x, y) = (\lambda_1, \lambda_2, \ldots \lambda_m),$$

then from Equation (2) we have

$$\begin{aligned}
& \Theta(d(Tx,Ty)) \preceq [\Theta(d(x,y))]^{[k]} \\
\Leftrightarrow\ & \Theta((\Lambda_1,\Lambda_2,\ldots,\Lambda_m)) \preceq [\Theta((\lambda_1,\lambda_2,\ldots\lambda_m))]^{[k]} \\
\Leftrightarrow\ & (\exp\sqrt{\Lambda_1},\exp\sqrt{\Lambda_2},\ldots,\exp\sqrt{\Lambda_m}) \preceq (\exp\sqrt{\lambda_1},\exp\sqrt{\lambda_2},\ldots,\exp\sqrt{\lambda_m})^{[k]} \\
\Leftrightarrow\ & (\exp\sqrt{\Lambda_{i,Tx,Ty}})_{i=1}^{m} \preceq (\exp k_i\sqrt{\Lambda_{i,x,y}})_{i=1}^{m} \\
\Leftrightarrow\ & \exp\sqrt{\Lambda_i} \leq \exp k_i\sqrt{\lambda_i} \text{ for each } i \in \{1,2,\ldots m\} \\
\Leftrightarrow\ & \Lambda_i \leq k_i^2\lambda_i \text{ for each } i \in \{1,2,\ldots m\} \\
\Leftrightarrow\ & (\Lambda_1,\Lambda_2,\ldots,\Lambda_m) \preceq (k_1^2\lambda_1,k_2^2\lambda_2,\ldots,k_m^2\lambda_m) \\
\Leftrightarrow\ & (\Lambda_1,\Lambda_2,\ldots,\Lambda_m) \preceq M(\lambda_1,\lambda_2,\ldots\lambda_m) \\
\Leftrightarrow\ & d(Tx,Ty) \preceq Md(x,y),
\end{aligned}$$

where

$$M = \begin{pmatrix} k_1^2 & 0 & \ldots & 0 \\ 0 & k_2^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & k_m^2 \end{pmatrix}_{m\times m}.$$

By considering some different function Θ belonging to Ξ^m in Equation (2), we can obtain new type contractions on vector-valued metric spaces.

2. Main Result

Here we present our main result.

Theorem 3. *Let (X,d) be a complete vector-valued metric space and $T : X \to X$ be a Perov type Θ-contraction, then T has a unique fixed point.*

Proof. Let $x_0 \in X$ be an arbitrary point and define a sequence $\{x_n\}$ in X by $x_n = Tx_{n-1}$ for $n \in \{1,2,\ldots\}$. If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \{0,1,\ldots\}$, then $Tx_{n_0} = x_{n_0}$, and so T has a fixed point. Now let $x_{n+1} \neq x_n$ for every $n \in \{0,1,\ldots\}$ and let $d(x_{n+1},x_n) = (\lambda_n^{(1)},\lambda_n^{(2)},\ldots,\lambda_n^{(m)}) = \lambda_n$ for $n \in \{0,1,\ldots\}$. Then $\lambda_n^{(i)} > 0$ for all $n \in \{0,1,\ldots\}$ and for all $i \in \{1,2,\ldots,m\}$. By using the representation

$$\Theta((\lambda_n^{(1)},\lambda_n^{(2)},\ldots,\lambda_n^{(m)})) = (\Lambda_n^{(1)},\Lambda_n^{(2)},\ldots,\Lambda_n^{(m)})$$

and Equation (2), we have

$$\begin{aligned}
(\Lambda_n^{(1)},\Lambda_n^{(2)},\ldots,\Lambda_n^{(m)}) &= \Theta((\lambda_n^{(1)},\lambda_n^{(2)},\ldots,\lambda_n^{(m)}))) \\
&= \Theta(d(x_{n+1},x_n)) \\
&= \Theta(d(Tx_n,Tx_{n-1})) \\
&\preceq [\Theta(d(x_n,x_{n-1}))]^{[k]} \\
&= \left[\Theta((\lambda_{n-1}^{(1)},\lambda_{n-1}^{(2)},\ldots,\lambda_{n-1}^{(m)}))\right]^{[k]} \\
&= \left[(\Lambda_{n-1}^{(1)},\Lambda_{n-1}^{(2)},\ldots,\Lambda_{n-1}^{(m)})\right]^{[k]}.
\end{aligned}$$

Therefore we obtain

$$\Lambda_n^{(i)} \leq \left[\Lambda_{n-1}^{(i)}\right]^{k_i}$$

for all $i \in \{1,2,\ldots,m\}$ and hence

$$\Lambda_n^{(i)} \leq \left[\Lambda_0^{(i)}\right]^{k_i^n} \tag{3}$$

for all $i \in \{1,2,\ldots,m\}$. Thus from Equation (3), we get $\lim_{n\to\infty} \Lambda_n^{(i)} = 1$. Hence, from condition (Θ3), we have

$$\lim_{n\to\infty} \lambda_n^{(i)} = 0^+$$

for all $i \in \{1,2,\ldots,m\}$. From (Θ2) there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\Lambda_n^{(i)} - 1}{[\lambda_n^{(i)}]^r} = l$$

for all $i \in \{1,2,\ldots,m\}$.

Suppose that $l < \infty$. In this case, let $B = \frac{l}{2} > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$

$$\left|\frac{\Lambda_n^{(i)} - 1}{[\lambda_n^{(i)}]^r} - l\right| \leq B$$

for all $i \in \{1,2,\ldots,m\}$. This implies that, for all $n \geq n_0$,

$$\frac{\Lambda_n^{(i)} - 1}{[\lambda_n^{(i)}]^r} \geq l - B = B$$

for all $i \in \{1,2,\ldots,m\}$. Then, for all $n \geq n_0$ and for all $i \in \{1,2,\ldots,m\}$

$$Bn\left[\lambda_n^{(i)}\right]^r \leq n\left[\Lambda_n^{(i)} - 1\right].$$

Suppose now that $l = \infty$. Let $B > 0$ is an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\frac{\Lambda_n^{(i)} - 1}{[\lambda_n^{(i)}]^r} \geq B$$

or all $i \in \{1,2,\ldots,m\}$. This implies that, for all $n \geq n_0$ and for all $i \in \{1,2,\ldots,m\}$

$$Bn\left[\lambda_n^{(i)}\right]^r \leq n\left[\Lambda_n^{(i)} - 1\right].$$

Considering these two cases and Equation (3) we have

$$Bn\left[\lambda_n^{(i)}\right]^r \leq n\left[\left[\Lambda_0^{(i)}\right]^{k_i^n} - 1\right] \tag{4}$$

for all $i \in \{1,2,\ldots,m\}$ and for some $B > 0$. Letting $n \to \infty$ in Equation (4), we obtain that

$$\lim_{n\to\infty} n\left[\lambda_n^{(i)}\right]^r = 0 \tag{5}$$

for all $i \in \{1, 2, \ldots, m\}$. From Equation (5), there exists $n^{(i)} \in \{1, 2, \ldots\}$ such that $n\left[\lambda_n^{(i)}\right]^r \leq 1$ for all $n \geq n^{(i)}$. So, we have, for all $n \geq n_0 = \max\{n^{(i)} : i \in \{1, 2, \ldots, m\}\}$

$$\lambda_n^{(i)} \leq \frac{1}{n^{1/r}}. \tag{6}$$

In order to show that $\{x_n\}$ is a Cauchy sequence consider $k, l \in \mathbb{N}$ such that $k > l \geq n_0$. Using the triangular inequality for the vector-valued metric and from Equation (6), we have

$$\begin{aligned} d(x_l, x_k) &\preceq d(x_l, x_{l+1}) + d(x_{l+1}, x_{l+2}) + \cdots + d(x_{k-1}, x_k) \\ &= \lambda_l + \lambda_{l+1} + \cdots + \lambda_{k-1} \\ &= (\lambda_l^{(i)})_{i=1}^m + (\lambda_{l+1}^{(i)})_{i=1}^m + \cdots + (\lambda_{k-1}^{(i)})_{i=1}^m \\ &= \left(\sum_{j=l}^{k-1} \lambda_j^{(i)}\right)_{i=1}^m \\ &\preceq \left(\sum_{j=l}^{\infty} \lambda_j^{(i)}\right)_{i=1}^m \\ &\preceq \left(\sum_{j=q}^{\infty} \frac{1}{j^{1/r}}\right)_{i=1}^m \end{aligned}$$

By the convergence of the series $\sum_{j=1}^{\infty} \frac{1}{j^{1/r}}$, passing to limit $l \to \infty$, we get $d(x_l, x_k) \to \mathbf{0}$. This yields that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete vector-valued metric space, the sequence $\{x_n\}$ converges to some point $z \in X$, that is, $\lim_{n\to\infty} x_n = z$.

On the other hand from (Θ1) and Equation (2), we can get

$$d(Tx, Ty) \preceq d(x, y)$$

for all $x, y \in X$. Therefore, we have

$$d(Tx_n, Tz) \preceq d(x_n, z)$$

that is

$$\mathbf{0} \preceq d(x_{n+1}, Tz) \preceq d(x_n, z) \to \mathbf{0}$$

as $n \to \infty$. So we have $\lim_{n\to\infty} x_n = Tz$ and hence $Tz = z$. The uniqueness of the fixed point can be easily seen by Equation (2). $\square$

Remark 1. *By taking* $\Theta : \mathbb{R}^m_{+0} \to \mathbb{R}^m_{+1}$ *by*

$$\Theta((\alpha_1, \alpha_2, \ldots, \alpha_m)) = (\exp\sqrt{\alpha_1}, \exp\sqrt{\alpha_2}, \ldots, \exp\sqrt{\alpha_m}),$$

in Theorem 3, we obtain Theorem 2 with

$$M = \begin{pmatrix} k_1^2 & 0 & \ldots & 0 \\ 0 & k_2^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & k_m^2 \end{pmatrix}_{m\times m}.$$

Here, since $\max\{k_i : i \in \{1, 2, \ldots, m\}\} < 1$, *the matrix M is convergent to zero.*

Now we present an illustrative and at the same time comparative example.

Example 8. *Consider the complete vector-valued metric space (X, d), where $X = \{0, 1, 2, \ldots\}$ and $d : X \times X \to \mathbb{R}^2$ is given by*

$$d(x,y) = \begin{cases} (0,0) & , \quad x = y \\ \\ (x+y, x+y) & , \quad x \neq y \end{cases} .$$

Let $T : X \to X$ be defined by

$$Tx = \begin{cases} 0 & , \quad x \in \{0,1\} \\ \\ x-1 & , \quad x \geq 2 \end{cases} .$$

Then T is not a Perov contraction. Indeed, for $y \geq 2$ and $x = y+1$, then we have $d(Tx, Ty) = (2y-1, 2y-1)$ and $d(x,y) = (2y+1, 2y+1)$. Now suppose there is a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2^2(\mathbb{R}_+)$ which converges to zero satisfying

$$d(Tx, Ty) \preceq Md(x,y)$$

then we have

$$\begin{aligned} (2y-1, 2y-1) &\preceq \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2y+1 \\ 2y+1 \end{pmatrix} \\ &= ((a+b)(2y+1), (c+d)(2y+1)). \end{aligned}$$

Therefore by considering the unboundedness of y, we have $a+b \geq 1$ and $c+d \geq 1$. This shows that M does not converges to zero, which shows T is not a Perov contraction.

Now, we claim that T is a Perov type Θ-contraction with

$$\Theta(\alpha_1, \alpha_2) = (\exp\sqrt{\alpha_1 \exp \alpha_1}, \exp\sqrt{\alpha_2 \exp \alpha_2})$$

and $k = (\exp(-\frac{1}{2}), \exp(-\frac{1}{2}))$. To see this we have to show that

$$\Theta(d(Tx, Ty)) \preceq [\Theta(d(x,y))]^{[k]}$$

for all $x, y \in X$ with $d(Tx, Ty) \succ \mathbf{0}$. For this, it is sufficient to show

$$\Theta(d(Tx, Ty)) \preceq [\Theta(d(x,y))]^{[k]}$$

or equivalently

$$\frac{Tx+Ty}{x+y} \exp\{Tx + Ty - x - y\} \leq \exp(-1). \tag{7}$$

for all $x, y \in X$ with $d(Tx, Ty) \succ \mathbf{0}$. First, observe that

$$d(Tx, Ty) \succ \mathbf{0} \Leftrightarrow \textit{the set } \{x, y\} \cap \{0, 1\} \textit{ is singleton or empty.}$$

Since Equation (7) is symmetric with respect to x and y, we may assume $x > y$ in the following cases.

Case 1. Let $\{x, y\} \cap \{0, 1\}$ be singleton. Then $Tx + Ty = x - 1$ and $x + y \leq x + 1$, and so we have

$$\frac{Tx+Ty}{x+y} \exp\{Tx + Ty - x - y\} \leq \frac{x-1}{x+1} \exp(-1) \leq \exp(-1).$$

Case 2. Let $\{x,y\} \cap \{0,1\}$ be empty. Then $Tx + Ty = x + y - 2$ and so we have

$$\frac{Tx+Ty}{x+y}\exp\{Tx+Ty-x-y\} = \frac{x+y-2}{x+y}\exp(-2) \leq \exp(-1).$$

Therefore by Theorem 3, T has a unique fixed point.

3. Semilinear Operator System

Let $(B, \|\cdot\|)$ be a Banach space and $N, M : B^2 \to B$ be two nonlinear operators. In this section we will give an existence result for a semilinear operator system of the form

$$\begin{aligned} N(x,y) &= x \\ M(x,y) &= y. \end{aligned} \tag{8}$$

Since initial or boundary value problems for nonlinear differential systems can be written in the operator form of Equation (8), such systems appear in various applications of mathematics. We can see that various fixed point theorems such as Schauder, Leray–Schauder, Krasnoselskii and Perov fixed point theorems were applied in the existence of solutions of such systems in [9].

Let $X = B^2$ and define $d : X \times X \to \mathbb{R}^2$, for $u = (x_1, y_1)$, $v = (x_2, y_2) \in X$ by $d(u,v) = (\|x_1 - x_2\|, \|y_1 - y_2\|)$. Then it can be seen that (X, d) is a complete vector-valued metric space. If we define a mapping $T : X \to X$ by $Tu = (Nu, Mu)$, then Equation (8) can be written as a fixed point problem

$$Tu = u \tag{9}$$

in the space X. Therefore, we will use the Theorem 3 to investigate the sufficient conditions that guarantee the existence of a solution of the fixed point problem Equation (9).

Theorem 4. *Assume that there exists a function $\Theta \in \Xi^2$ and a constant $\gamma \in (0,1)$ such that*

$$\Theta(\|Nu - Nv\|, \|Mu - Mv\|) \leq [\Theta(\|x_1 - x_2\|, \|y_1 - y_2\|)]^{[k]}, \tag{10}$$

where $k = (\gamma, \gamma)$, for all $u = (x_1, y_1)$, $v = (x_2, y_2) \in B^2$ with $Nu \neq Nv$. Then Equation (8) has a unique solution in B^2.

Proof. By Equation (10), we have

$$\Theta\left(d(Tu, Tv)\right) \preceq \left(\Theta d(u,v)\right)^{[k]}.$$

Thus, by applying Theorem 3, T has a unique fixed point in $X = B^2$ or equivalently the semilinear operator system Equation (8) has a unique solution in B^2. □

Remark 2. *Note that, if there exists a constant $\gamma < 1$ such that*

$$\max\left\{ \begin{array}{l} \frac{\|N(x_1,y_1)-N(x_2,y_2)\|}{\|x_1-x_2\|}, \\ \frac{\|M(x_1,y_1)-M(x_2,y_2)\|}{\|y_1-y_2\|}\exp\{\|M(x_1,y_1)-M(x_2,y_2)\| - \|y_1-y_2\|\} \end{array} \right\} \leq \gamma$$

for all $u = (x_1, y_1)$, $v = (x_2, y_2) \in B^2$ with $x_1 \neq x_2$ and $y_1 \neq y_2$, then we get Equation (10) with the function

$$\Theta(\alpha_1, \alpha_2) = (\exp\{\sqrt{\alpha_1}\}, \exp\{\sqrt{\alpha_2 \exp\{\alpha_2\}}\})$$

and $k = (\sqrt{\gamma}, \sqrt{\gamma})$.

4. Conclusions

In this paper, by using the recent technique named as Θ-contraction we give a new generalization of the Perov fixed point theorem on vector-valued metric space. Then we present an existence result of solution of a kind of semilinear operator system.

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