Article

Degrees of L -Continuity for Mappings between L -Topological Spaces

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Abstract: By means of the residual implication on a frame L , a degree approach to L -continuity and L -closedness for mappings between L -cotopological spaces are defined and their properties are investigated systematically. In addition, in the situation of L -topological spaces, degrees of L -continuity and of L -openness for mappings are proposed and their connections are studied. Moreover, if L is a frame with an order-reversing involution $'$, where $b' = b \rightarrow \perp$ for $b \in L$, then degrees of L -continuity for mappings between L -cotopological spaces and degrees of L -continuity for mappings between L -topological spaces are equivalent.

Keywords: L -cotopological space; L -topological space; degree of L -continuity; degree of L -closedness; degree of L -openness

1. Introduction

Since Chang [1] introduced fuzzy set theory to topology, fuzzy topology and its related theories have been widely investigated such as [1–18]. The degree approach that equips fuzzy topology and its related structures with some degree description is also an essential character of fuzzy set theory. This approach has been developed extensively in the theory of fuzzy topology, fuzzy convergence and fuzzy convex structure. Yue and Fang [19] introduced a degree approach to T_1 and T_2 separation properties in (L, M) -fuzzy topological spaces. Shi [20,21] defined the degrees of separation axioms which are compatible with (L, M) -fuzzy metric spaces. Li and Shi [22] introduced the degree of compactness in L -fuzzy topological spaces. Pang defined the compact degree of (L, M) -fuzzy convergence spaces [23] and degrees of T_i ($i = 0, 1, 2$) separation property as well as the regular property of stratified L -generalized convergence spaces [24]. All of the above-mentioned research mainly equipped spatial properties with some of the degree descriptions.

Actually, special mappings between structured spaces and the structured space itself can also be endowed with some degrees. Xiu and his co-authors [25,26] defined degrees of fuzzy convex structures, fuzzy closure systems and fuzzy Alexandrov topologies and discussed their properties. Xiu and Pang [27] gave a degree approach to special mappings between M -fuzzifying convex spaces. In [28], Pang defined degrees of continuous mappings and open mappings between L -fuzzifying topological spaces to describe how a mapping between L -fuzzifying topological spaces becomes a continuous mapping or an open mapping in a degree sense. Liang and Shi [29] further defined the degrees of continuous mappings and open mappings between L -fuzzy

topological spaces and investigated their relationship. Li [30] defined the degrees of special mappings in the theory of L -convex spaces and investigated their properties. Xiu and Pang [27] developed the degree approach to M -fuzzifying convex spaces to define the degrees of M -CP mappings and M -CC mappings.

Following this direction, we will focus on the case of L -cotopological spaces and L -topological spaces in this paper. By means of L -closure operators and L -interior operators, we will consider degrees of L -continuity and L -closedness for mappings between L -cotopological spaces as well as degrees of L -continuity and L -openness for mappings between L -topological spaces and will investigate their properties systematically.

2. Preliminaries

In this paper, let L be a frame, X be a nonempty set and L^X be the set of all L -subsets on X . The bottom element and the top element of L are denoted by $\perp$ and $\top$, respectively. A residual implication can be defined by $a \rightarrow b = \bigvee \{\lambda \in L \mid a \wedge \lambda \leq b\}$. The operators on L can be translated onto L^X in a pointwise way. In this case, L^X is also a complete lattice. Let $\underline{\perp}$ and $\underline{\top}$ denote the smallest element and the largest element in L^X , respectively.

Let $f : X \rightarrow Y$ be a mapping. Define $f_L^- : L^Y \rightarrow L^X$ and $f_L^+ : L^X \rightarrow L^Y$ by $f_L^-(B)(y) = \bigvee_{f(x)=y} B(x)$ for $B \in L^X$ and $y \in Y$, and $f_L^+(A)(x) = A(f(x))$ for $A \in L^Y$ and $x \in X$, respectively.

Using the residual implication, the concept of fuzzy inclusion order of L -subsets is introduced.

Definition 1 ([2,31]). The fuzzy inclusion order of L -subsets is a mapping $\mathbb{S}_X(-, -) : L^X \times L^X \rightarrow L$ which satisfying for each $C, D \in L^X$,

$$\mathbb{S}_X(C, D) = \bigwedge_{x \in X} (C(x) \rightarrow D(x))$$

Lemma 1 ([2,31]). Let $f : X \rightarrow Y$ be a mapping. Then, for each $C, D, E \in L^X$ and $A, B \in L^Y$, the following statements hold:

- (1) $\mathbb{S}_X(D, E) = \top$ if and only if $D \leq E$.
- (2) $D \leq E$ implies $\mathbb{S}_X(D, C) \geq \mathbb{S}_X(E, C)$.
- (3) $D \leq E$ implies $\mathbb{S}_X(C, D) \leq \mathbb{S}_X(C, E)$.
- (4) $\mathbb{S}_X(D, E) \wedge \mathbb{S}_X(E, C) \leq \mathbb{S}_X(D, C)$.
- (5) $\mathbb{S}_X(D, E) \leq \mathbb{S}_X(f_L^-(D), f_L^-(E))$.
- (6) $\mathbb{S}_Y(A, B) \leq \mathbb{S}_Y(f_L^+(A), f_L^+(B))$.

In the following, we will only use $\mathbb{S}$ to represent the fuzzy inclusion order of L -subsets on both L^X and L^Y , not $\mathbb{S}_X$ or $\mathbb{S}_Y$. This will not lead to ambiguity in the paper.

Definition 2 ([1,18,32,33]). Let $\mathcal{C}$ be a subset of L^X . $\mathcal{C}$ is called an L -cotopology on X if it satisfies:

- (LCT1) $\underline{\perp}, \underline{\top} \in \mathcal{C}$;
- (LCT2) if $A, B \in \mathcal{C}$, then $A \vee B \in \mathcal{C}$;
- (LCT3) if $\{A_j \mid j \in J\} \subseteq \mathcal{C}$, then $\bigwedge_{j \in J} A_j \in \mathcal{C}$.

For an L -cotopology $\mathcal{C}$ on X , the pair $(X, \mathcal{C})$ is called an L -cotopological space.

If $\mathcal{C}$ also satisfies:

- (SLCT) for each $a \in L$, $\underline{a} \in \mathcal{C}$,

then it is called a stratified L -cotopology and the pair $(X, \mathcal{C})$ is called a stratified L -cotopological space.

A mapping $f : (X, \mathcal{C}_X) \rightarrow (Y, \mathcal{C}_Y)$ is called L -continuous provided that, for each $B \in L^Y$, $B \in \mathcal{C}_Y$ implies $f_L^{\leftarrow}(B) \in \mathcal{C}_X$.

A mapping $f : (X, \mathcal{C}_X) \rightarrow (Y, \mathcal{C}_Y)$ is called L -closed provided that, for each $A \in L^Y$, $A \in \mathcal{C}_X$ implies $f_L^{\rightarrow}(B) \in \mathcal{C}_Y$.

Definition 3 ([18,32,33]). An L -closure operator on X is a mapping $Cl : L^X \rightarrow L^X$ which satisfies:

$$(LCL1) \ Cl(\perp) = \perp;$$

$$(LCL2) \ A \leq Cl(A);$$

$$(LCL3) \ Cl(Cl(A)) = Cl(A);$$

$$(LCL4) \ Cl(A \vee B) = Cl(A) \vee Cl(B).$$

For an L -closure operator Cl on X , the pair (X, Cl) is called an L -closure space.

A mapping $f : (X, Cl_X) \rightarrow (Y, Cl_Y)$ is called L -continuous provided that

$$\forall A \in L^X, \ f_L^{\rightarrow}(Cl_X(A)) \leq Cl_Y(f_L^{\rightarrow}(A)).$$

It was proved in [18,32,33] that L -cotopologies and L -closure operators are conceptually equivalent with transferring process $Cl^C(A) = \bigwedge \{B \in L^X \mid A \leq B \in \mathcal{C}\}$ for each $A \in L^X$ and $\mathcal{C}^{Cl} = \{A \in L^X \mid Cl(A) = A\}$. Correspondingly, L -continuous mappings between L -cotopological spaces and L -continuous mappings between L -closure spaces are compatible. In the sequel, we treat L -cotopological spaces with their L -continuous mappings and L -closure spaces with their L -continuous mappings equivalently. We will use Cl to represent Cl^C tacitly.

Definition 4 ([1,18,32,33]). An L -topology on X is a subset $\mathcal{T} \subseteq L^X$ which satisfies:

$$(LT1) \ \perp, \top \in \mathcal{T};$$

$$(LT2) \ \text{if } A, B \in \mathcal{T}, \text{ then } A \wedge B \in \mathcal{T};$$

$$(LT3) \ \text{if } \{A_j \mid j \in J\} \subseteq \mathcal{T}, \text{ then } \bigvee_{j \in J} A_j \in \mathcal{T}.$$

For an L -topology $\mathcal{T}$ on X , the pair $(X, \mathcal{T})$ is called an L -topological space.

A mapping $f : (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$ is called L -continuous provided that for each $B \in L^Y$, $B \in \mathcal{T}_Y$ implies $f_L^{\leftarrow}(B) \in \mathcal{T}_X$.

A mapping $f : (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$ is called L -open provided that for each $A \in L^X$, $A \in \mathcal{T}_X$ implies $f_L^{\rightarrow}(A) \in \mathcal{T}_Y$.

Definition 5 ([18,33]). An L -interior operator on X is a mapping $\mathcal{N} : L^X \rightarrow L^X$ which satisfies:

$$(LN1) \ \mathcal{N}(\top) = \top;$$

$$(LN2) \ \mathcal{N}(C) \leq C;$$

$$(LN3) \ \mathcal{N}(\mathcal{N}(C)) = \mathcal{N}(C);$$

$$(LN4) \ \mathcal{N}(C \wedge D) = \mathcal{N}(C) \wedge \mathcal{N}(D).$$

For an L -interior operator $\mathcal{N}$ on X , the pair $(X, \mathcal{N})$ is called an L -interior space.

A mapping $f : (X, \mathcal{N}_X) \rightarrow (Y, \mathcal{N}_Y)$ is called L -continuous provided that

$$\forall A \in L^X, \ f_L^{\rightarrow}(\mathcal{N}_X(A)) \leq \mathcal{N}_Y(f_L^{\rightarrow}(A)).$$

It was proved in [18,33] that L -topologies and L -interior operators are conceptually equivalent with transferring process $\mathcal{N}^T(A) = \bigvee \{B \in L^X \mid B \leq A, B \in \mathcal{T}\}$ for each $A \in L^X$ and $\mathcal{T}^{\mathcal{N}} = \{A \in L^X \mid \mathcal{N}(A) = A\}$. Correspondingly, L -continuous mappings between L -topological spaces and L -continuous mappings between L -interior spaces are compatible. In the sequel, we

treat L -topological spaces with their L -continuous mappings and L -interior spaces with their L -continuous mappings equivalently. We will use $\mathcal{N}$ to represent $\mathcal{N}^T$ tacitly.

Definition 6 ([3,32]). An L -filter on X is a mapping $F : L^X \rightarrow L$ which satisfies:

- (F1) $F(\perp) = \perp, F(\top) = \top$;
- (F2) $F(C \wedge D) = F(C) \wedge F(D)$ for each $C, D \in L^X$.

Let $\mathfrak{F}_L^s(X)$ denote the family of all L -filters on X .

Definition 7 ([3]). An L -fuzzy convergence on X is a mapping $\text{Lim} : \mathfrak{F}_L^s(X) \rightarrow L^X$ which satisfies:

- (L1) $\forall x \in X, \text{Lim}x = \top$;
- (L2) $F_1 \leq F_2$ implies $\text{Lim}(F_1) \leq \text{Lim}(F_2)$.

For an L -fuzzy convergence Lim on X , the pair (X, Lim) is called an L -fuzzy convergence space.

Theorem 1 ([3]). For an L -topological space $(X, \mathcal{T})$, let $\mathcal{N}$ be its interior operator and $\mathcal{U}^x$ be the L -neighborhood filter defined by $\mathcal{U}^x(A) = \mathcal{N}(A)(x)$ for each $x \in X$. Then, the mapping $\text{Lim} : \mathfrak{F}_L^s(X) \rightarrow L^X$ defined by

$$\text{Lim}(F)(x) = \bigwedge_{A \in L^X} (\mathcal{U}^x(A) \rightarrow F(A))$$

is an L -fuzzy convergence on X .

3. Degrees of L -Continuity and L -Closedness for Mappings between L -Cotopological Spaces

In this section, we mainly define degrees of L -continuity of mappings and L -closedness of mappings to equip each mapping between L -cotopological spaces with some degree to be an L -continuous mapping and an L -closed mapping, respectively. Then, we will study their connections in a degree sense. Moreover, f always denotes a mapping from X to Y and g always denotes a mapping from Y to Z in the following sections.

Definition 8. Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be L -cotopological spaces. Then,

- (1) $D_{c_1}(f)$ defined by

$$D_{c_1}(f) = \bigwedge_{B \in L^X} \mathbb{S}(f_L^{\rightarrow}(\mathcal{C}_X(B)), \mathcal{C}_Y(f_L^{\rightarrow}(B)))$$

is called the degree of L -continuity for f .

- (2) $D_b(f)$ defined by

$$D_b(f) = \bigwedge_{A \in L^X} \mathbb{S}(\mathcal{C}_Y(f_L^{\rightarrow}(A)), f_L^{\rightarrow}(\mathcal{C}_X(A)))$$

is called the degree of L -closedness for f .

Remark 1. (1) If $D_{c_1}(f) = \top$, then $f_L^{\rightarrow}(\mathcal{C}_X(A)) \leq \mathcal{C}_Y(f_L^{\rightarrow}(A))$ for all $A \in L^X$, which is exactly the definition of L -continuous mappings between L -closure spaces. As we claimed that L -continuous mappings between L -cotopological spaces and L -continuous mappings between L -closure spaces are compatible, we don't distinguish them. Therefore, we defined the degree of L -continuity of mappings between L -cotopological spaces by using L -continuous mappings between their induced L -closure spaces.

(2) If $D_b(f) = \top$, then $\mathcal{C}_Y(f_L^{\rightarrow}(A)) \leq f_L^{\rightarrow}(\mathcal{C}_X(A))$ for all $A \in L^X$. This is exactly the equivalent form of L -closed mappings between L -cotopological spaces by means of the corresponding L -closure operators.

Lemma 2. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be mappings. Then, for each $A \in L^X$, $B \in L^Y$ and $C \in L^Z$, the following statements hold:

- (1) $A \leq f_L^{\leftarrow}(f_L^{\rightarrow}(A))$. If f is injective, then the equality holds.
- (2) $f_L^{\rightarrow}(f_L^{\leftarrow}(B)) \leq B$. If f is surjective, then the equality holds.
- (3) $(g \circ f)_L^{\rightarrow}(A) = g_L^{\rightarrow}(f_L^{\rightarrow}(A))$.
- (4) $(g \circ f)_L^{\leftarrow}(C) = f_L^{\leftarrow}(g_L^{\leftarrow}(C))$.

Proof. The proofs are routine and are omitted. $\square$

Theorem 2. Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be L -cotopological spaces. Then,

$$D_{c_1}(f) = \bigwedge_{B \in L^Y} \mathbb{S}\left(f_L^{\rightarrow}(Cl_X(f_L^{\leftarrow}(B))), Cl_Y(B)\right).$$

Proof. It follows from the definition of $D_{c_1}(f)$ that

$$\begin{aligned} D_{c_1}(f) &= \bigwedge_{A \in L^X} \mathbb{S}\left(f_L^{\rightarrow}(Cl_X(A)), Cl_Y(f_L^{\rightarrow}(A))\right) \\ &\leq \bigwedge_{B \in L^Y} \mathbb{S}\left(f_L^{\rightarrow}(Cl_X(f_L^{\leftarrow}(B))), Cl_Y(f_L^{\rightarrow}(f_L^{\leftarrow}(B)))\right) \\ &\leq \bigwedge_{B \in L^Y} \mathbb{S}\left(f_L^{\rightarrow}(Cl_X(f_L^{\leftarrow}(B))), Cl_Y(B)\right) \\ &\leq \bigwedge_{A \in L^X} \mathbb{S}\left(f_L^{\rightarrow}(Cl_X(f_L^{\leftarrow}(f_L^{\rightarrow}(A)))), Cl_Y(f_L^{\rightarrow}(A))\right) \\ &\leq \bigwedge_{A \in L^X} \mathbb{S}\left(f_L^{\rightarrow}(Cl_X(A)), Cl_Y(f_L^{\rightarrow}(A))\right) \\ &= D_{c_1}(f). \end{aligned}$$

This implies

$$D_{c_1}(f) = \bigwedge_{B \in L^Y} \mathbb{S}\left(f_L^{\rightarrow}(Cl_X(f_L^{\leftarrow}(B))), Cl_Y(B)\right),$$

as desired. $\square$

Theorem 3. (1) If $id : (X, \mathcal{C}_X) \longrightarrow (X, \mathcal{C}_X)$ is the identity mapping, then $D_{c_1}(id) = \top$ and $D_b(id) = \top$.

(2) If $y_0 : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ is a constant mapping between stratified L -cotopological spaces with the constant $y_0 \in Y$, then $D_{c_1}(y_0) = \top$.

Proof. (1) Straightforward,

(2) It follows immediately from the definition of $D_{c_1}(y_0)$ that

$$\begin{aligned} D_{c_1}(y_0) &= \bigwedge_{A \in L^X} \mathbb{S}\left(Cl_X(A), (y_0)_L^{\leftarrow}(Cl_Y((y_0)_L^{\rightarrow}(A)))\right) \\ &= \bigwedge_{A \in L^X} \bigwedge_{x \in X} \left(Cl_X(A)(x) \rightarrow Cl_Y((y_0)_L^{\rightarrow}(A))(y_0)\right). \end{aligned}$$

Since $(X, \mathcal{C}_X)$ is stratified, we know $\bigvee_{z \in X} A(z) \in \mathcal{C}_X$. Then, for each $A \in L^X$ and $x \in X$, it follows that

$$Cl_X(A)(x) = \bigwedge \{B \in L^X \mid A \leq B \in \mathcal{C}_X\} \leq \bigvee_{z \in X} A(z).$$

Furthermore, we have

$$Cl_Y((y_0)_L^{\rightarrow}(A))(y_0) \geq (y_0)_L^{\rightarrow}(A)(y_0) = \bigvee_{y_0(z)=y_0} A(z) = \bigvee_{z \in X} A(z) \geq Cl_X(A)(x).$$

This implies that $Cl_X(A)(x) \leq Cl_Y((y_0)_L^{\rightarrow}(A))(y_0)$ for each $A \in L^X$ and $x \in X$. Therefore, we have $D_{c_1}(y_0) = \top$. $\square$

Next, we give another characterizations of degrees of L -continuity for mappings between L -cotopological spaces.

Theorem 4. Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be L -cotopological spaces. Then,

$$D_{c_1}(f) = \bigwedge_{A \in L^X} \mathbb{S}\left(Cl_X(A), f_L^{\leftarrow}(Cl_Y(f_L^{\rightarrow}(A)))\right).$$

Proof. By the definition of $\mathbb{S}$, it follows that

$$\begin{aligned} & \mathbb{S}\left(f_L^{\rightarrow}(Cl_X(A)), Cl_Y(f_L^{\rightarrow}(A))\right) \\ &= \bigwedge_{y \in Y} \left(f_L^{\rightarrow}(Cl_X(A))(y) \rightarrow Cl_Y(f_L^{\rightarrow}(A))(y)\right) \\ &= \bigwedge_{y \in Y} \left(\bigvee_{f(x)=y} Cl_X(A)(x) \rightarrow Cl_Y(f_L^{\rightarrow}(A))(y)\right) \\ &= \bigwedge_{y \in Y} \bigwedge_{f(x)=y} \left(Cl_X(A)(x) \rightarrow Cl_Y(f_L^{\rightarrow}(A))(f(x))\right) \\ &= \bigwedge_{y \in Y} \bigwedge_{f(x)=y} \left(Cl_X(A)(x) \rightarrow f_L^{\leftarrow}(Cl_Y(f_L^{\rightarrow}(A)))(x)\right) \\ &= \bigwedge_{x \in X} \left(Cl_X(A)(x) \rightarrow f_L^{\leftarrow}(Cl_Y(f_L^{\rightarrow}(A)))(x)\right) \\ &= \mathbb{S}\left(Cl_X(A), f_L^{\leftarrow}(Cl_Y(f_L^{\rightarrow}(A)))\right). \end{aligned}$$

This means

$$\begin{aligned} D_{c_1}(f) &= \bigwedge_{A \in L^X} \mathbb{S}\left(f_L^{\rightarrow}(Cl_X(A)), Cl_Y(f_L^{\rightarrow}(A))\right) \\ &= \bigwedge_{A \in L^X} \mathbb{S}\left(Cl_X(A), f_L^{\leftarrow}(Cl_Y(f_L^{\rightarrow}(A)))\right), \end{aligned}$$

as desired. $\square$

Theorem 5. Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be L -cotopological spaces. Then,

$$D_{c_1}(f) = \bigwedge_{B \in L^Y} \mathbb{S}\left(Cl_X(f_L^{\leftarrow}(B)), f_L^{\leftarrow}(Cl_Y(B))\right).$$

Proof.

$$\begin{aligned} & \bigwedge_{B \in L^Y} \mathbb{S}\left(Cl_X(f_L^{\leftarrow}(B)), f_L^{\leftarrow}(Cl_Y(B))\right) \\ \leq & \bigwedge_{A \in L^X} \mathbb{S}\left(Cl_X(f_L^{\leftarrow}(f_L^{\rightarrow}(A))), f_L^{\leftarrow}(Cl_Y(f_L^{\rightarrow}(A)))\right) \\ \leq & \bigwedge_{A \in L^X} \mathbb{S}\left(Cl_X(A), f_L^{\leftarrow}(Cl_Y(f_L^{\rightarrow}(A)))\right) \\ = & D_{c_1}(f). \end{aligned}$$

By the definition of $\mathbb{S}$, it follows that

$$\begin{aligned} & \mathbb{S}\left(Cl_X(f_L^{\leftarrow}(B)), f_L^{\leftarrow}(Cl_Y(B))\right) \\ = & \bigwedge_{x \in X} \left(Cl_X(f_L^{\leftarrow}(B))(x) \rightarrow f_L^{\leftarrow}(Cl_Y(B))(x)\right) \\ = & \bigwedge_{x \in X} \left(Cl_X(f_L^{\leftarrow}(B))(x) \rightarrow (Cl_Y(B))(f(x))\right) \\ \geq & \bigwedge_{y \in Y} \left(Cl_X(f_L^{\leftarrow}(B))(f^{-1}(y)) \rightarrow (Cl_Y(B))(y)\right) \\ \geq & \bigwedge_{y \in Y} \left(\bigvee_{f(x)=y} Cl_X(f_L^{\leftarrow}(B))(x) \rightarrow (Cl_Y(B))(y)\right) \\ = & \bigwedge_{y \in Y} \left(f_L^{\rightarrow}(Cl_X(f_L^{\leftarrow}(B)))(y) \rightarrow (Cl_Y(B))(y)\right) \\ = & D_{c_1}(f). \end{aligned}$$

This implies

$$D_{c_1}(f) = \bigwedge_{B \in L^Y} \mathbb{S}\left(Cl_X(f_L^{\leftarrow}(B)), f_L^{\leftarrow}(Cl_Y(B))\right),$$

as desired. $\square$

In L -cotopological spaces, compositions of L -continuous mappings (resp. L -closed mappings) are still L -continuous mappings (resp. L -closed mappings). Now, let us give a degree representation of this result.

Theorem 6. For L -cotopological spaces $(X, \mathcal{C}_X)$, $(Y, \mathcal{C}_Y)$ and $(Z, \mathcal{C}_Z)$, the following inequalities hold:

- (1) $D_{c_1}(f) \wedge D_{c_1}(g) \leq D_{c_1}(g \circ f)$.
- (2) $D_b(f) \wedge D_b(g) \leq D_b(g \circ f)$.

Proof. (1) By Theorem 4, we have

$$\begin{aligned}
 & D_{c_1}(f) \wedge D_{c_1}(g) \\
 = & \bigwedge_{A \in L^X} \mathbb{S}(f_L^{\rightarrow}(Cl_X(A)), Cl_Y(f_L^{\rightarrow}(A))) \wedge \bigwedge_{B \in L^Y} \mathbb{S}(g_L^{\rightarrow}(Cl_Y(B)), Cl_Z(g_L^{\rightarrow}(B))) \\
 \leq & \bigwedge_{A \in L^X} \mathbb{S}(f_L^{\rightarrow}(Cl_X(A)), Cl_Y(f_L^{\rightarrow}(A))) \wedge \bigwedge_{B \in L^Y} \mathbb{S}(Cl_Y(B), g_L^{\leftarrow}(Cl_Z(g_L^{\rightarrow}(B)))) \\
 \leq & \bigwedge_{A \in L^X} \left(\mathbb{S}(f_L^{\rightarrow}(Cl_X(A)), Cl_Y(f_L^{\rightarrow}(A))) \wedge \mathbb{S}(Cl_Y(f_L^{\rightarrow}(A)), g_L^{\leftarrow}(Cl_Z(g_L^{\rightarrow}(f_L^{\rightarrow}(A))))) \right) \\
 \leq & \bigwedge_{A \in L^X} \mathbb{S}(f_L^{\rightarrow}(Cl_X(A)), g_L^{\leftarrow}(Cl_Z((g \circ f)_L^{\rightarrow}(A)))) \\
 = & \bigwedge_{A \in L^X} \mathbb{S}((g \circ f)_L^{\rightarrow}(Cl_X(A)), Cl_Z((g \circ f)_L^{\rightarrow}(A))) \\
 = & D_{c_1}(g \circ f).
 \end{aligned}$$

(2) Adopting the proof of (1), it can be verified directly. $\square$

Next, we investigate the connections between degrees of L -continuity and that of L -closedness.

Theorem 7. For L -cotopological spaces (X, C_X) , (Y, C_Y) and (Z, C_Z) , if g is injective, then the following inequality holds:

$$D_b(g \circ f) \wedge D_{c_1}(g) \leq D_b(f).$$

Proof. If g is an injective mapping, then we have $g_L^{\leftarrow}(g_L^{\rightarrow}(B)) = B$ for all $B \in L^Y$. Then, it follows that

$$\begin{aligned}
 D_b(g \circ f) \wedge D_{c_1}(g) &= \bigwedge_{A \in L^X} \mathbb{S}(Cl_Z((g \circ f)_L^{\rightarrow}(A)), (g \circ f)_L^{\rightarrow}(Cl_X(A))) \\
 &\quad \wedge \bigwedge_{B \in L^Y} \mathbb{S}(g_L^{\rightarrow}(Cl_Y(B)), Cl_Z(g_L^{\rightarrow}(B))) \\
 &\leq \bigwedge_{A \in L^X} \mathbb{S}(Cl_Z((g \circ f)_L^{\rightarrow}(A)), (g \circ f)_L^{\rightarrow}(Cl_X(A))) \\
 &\quad \wedge \bigwedge_{A \in L^X} \mathbb{S}(g_L^{\rightarrow}(Cl_Y(f_L^{\rightarrow}(A))), Cl_Z(g_L^{\rightarrow}(f_L^{\rightarrow}(A)))) \\
 &\leq \bigwedge_{A \in L^X} \mathbb{S}(Cl_Z((g \circ f)_L^{\rightarrow}(A)), (g \circ f)_L^{\rightarrow}(Cl_X(A))) \\
 &\quad \wedge \bigwedge_{A \in L^X} \mathbb{S}(g_L^{\rightarrow}(Cl_Y(f_L^{\rightarrow}(A))), Cl_Z((g \circ f)_L^{\rightarrow}(A))) \\
 &\leq \bigwedge_{A \in L^X} \mathbb{S}(g_L^{\rightarrow}(Cl_Y(f_L^{\rightarrow}(A))), (g \circ f)_L^{\rightarrow}(Cl_X(A))) \\
 &= \bigwedge_{A \in L^X} \mathbb{S}(Cl_Y(f_L^{\rightarrow}(A)), g_L^{\leftarrow}((g \circ f)_L^{\rightarrow}(Cl_X(A)))) \\
 &= \bigwedge_{A \in L^X} \mathbb{S}(Cl_Y(f_L^{\rightarrow}(A)), f_L^{\rightarrow}(Cl_X(A))) \\
 &= D_b(f).
 \end{aligned}$$

□

Theorem 8. For L -cotopological spaces $(X, \mathcal{C}_X)$, $(Y, \mathcal{C}_Y)$ and $(Z, \mathcal{C}_Z)$, if g is surjective, then the following inequality holds:

$$D_b(g \circ f) \wedge D_{c_1}(f) \leq D_b(g).$$

Proof. If f is a surjective mapping, then $f_L^{\rightarrow}(f_L^{\leftarrow}(A)) = A$ for all $A \in L^Y$. Then, it follows that

$$\begin{aligned} & \bigwedge_{B \in L^Y} \mathbb{S}(Cl_Z(g_L^{\rightarrow}(B)), g_L^{\rightarrow}(Cl_Y(B))) \\ &= \bigwedge_{B \in L^Y} \mathbb{S}(Cl_Z(g_L^{\rightarrow}(f_L^{\rightarrow}(f_L^{\leftarrow}(B)))), g_L^{\rightarrow}(Cl_Y(f_L^{\rightarrow}(f_L^{\leftarrow}(B))))) \\ &\geq \bigwedge_{A \in L^X} \mathbb{S}(Cl_Z((g \circ f)_L^{\rightarrow}(A)), g_L^{\rightarrow}(Cl_Y(f_L^{\rightarrow}(A)))) \\ &\geq \bigwedge_{B \in L^Y} \mathbb{S}(Cl_Z(g_L^{\rightarrow}(B)), g_L^{\rightarrow}(Cl_Y(B))). \end{aligned}$$

This shows

$$\bigwedge_{B \in L^Y} \mathbb{S}(Cl_Z(g_L^{\rightarrow}(B)), g_L^{\rightarrow}(Cl_Y(B))) = \bigwedge_{A \in L^X} \mathbb{S}(Cl_Z((g \circ f)_L^{\rightarrow}(A)), g_L^{\rightarrow}(Cl_Y(f_L^{\rightarrow}(A)))).$$

Then, we have

$$\begin{aligned} & D_b(g \circ f) \wedge D_{c_1}(f) \\ &= \bigwedge_{A \in L^X} \mathbb{S}(Cl_Z((g \circ f)_L^{\rightarrow}(A)), (g \circ f)_L^{\rightarrow}(Cl_X(A))) \\ &\quad \wedge \bigwedge_{A \in L^X} \mathbb{S}(f_L^{\rightarrow}(Cl_X(A)), Cl_Y(f_L^{\rightarrow}(A))) \\ &\leq \bigwedge_{A \in L^X} \mathbb{S}(Cl_Z((g \circ f)_L^{\rightarrow}(A)), (g \circ f)_L^{\rightarrow}(Cl_X(A))) \\ &\quad \wedge \bigwedge_{A \in L^X} \mathbb{S}((g \circ f)_L^{\rightarrow}(Cl_X(A)), g_L^{\rightarrow}(Cl_Y(f_L^{\rightarrow}(A)))) \\ &\leq \bigwedge_{A \in L^X} \mathbb{S}(Cl_Z((g \circ f)_L^{\rightarrow}(A)), g_L^{\rightarrow}(Cl_Y(f_L^{\rightarrow}(A)))) \\ &= \bigwedge_{B \in L^Y} \mathbb{S}(Cl_Z(g_L^{\rightarrow}(B)), g_L^{\rightarrow}(Cl_Y(B))) \\ &= D_b(g). \end{aligned}$$

□

4. Degrees of L -Continuity and L -Openness for Mappings between L -Topological Spaces

In this section, we mainly define degrees of L -continuity and L -openness to equip each mapping between L -topological spaces with some degree to be an L -continuous mapping and an L -open mapping, respectively. Then, we will study their connections in a degree sense.

Definition 9. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be L -topological spaces.

(1) $D_{c_2}(f)$ defined by

$$D_{c_2}(f) = \bigwedge_{B \in L^Y} \mathbb{S}\left(f_L^{\leftarrow}(\mathcal{N}_Y(B)), \mathcal{N}_X(f_L^{\leftarrow}(B))\right)$$

is called the degree of L -continuity for f .

(2) $D_k(f)$ defined by

$$D_k(f) = \bigwedge_{A \in L^X} \mathbb{S}\left(f_L^{\rightarrow}(\mathcal{N}_X(A)), \mathcal{N}_Y(f_L^{\rightarrow}(A))\right)$$

is called the degree of L -openness for f .

Remark 2. (1) If $D_{c_2}(f) = \top$, then $f_L^{\leftarrow}(\mathcal{N}_X(B)) \leq \mathcal{N}_Y(f_L^{\leftarrow}(B))$ for all $B \in L^Y$, which is exactly the definition of L -continuous mappings between L -interior spaces.

(2) If $D_k(f) = \top$, then $f_L^{\rightarrow}(\mathcal{N}_X(A)) \leq \mathcal{N}_Y(f_L^{\rightarrow}(A))$ for all $A \in L^X$. This is exactly the equivalent form of L -open mappings between L -topological spaces by means of the corresponding L -interior operators.

Theorem 9. Let $(X, \mathcal{T}_X)$ and $(X, \mathcal{T}_Y)$ be L -topological spaces. Then,

$$D_{c_2}(f) = \bigwedge_{x \in X} \bigwedge_{C \in L^Y} \left(\mathcal{U}_Y^{f(x)}(C) \rightarrow f(\mathcal{U}_X^x(C)) \right).$$

Proof.

$$\begin{aligned} & \bigwedge_{x \in X} \bigwedge_{C \in L^Y} \left(\mathcal{U}_Y^{f(x)}(C) \rightarrow f(\mathcal{U}_X^x(C)) \right) \\ &= \bigwedge_{x \in X} \bigwedge_{C \in L^Y} \left(\mathcal{U}_Y^{f(x)}(C) \rightarrow (\mathcal{U}_X^x)(f_L^{\leftarrow}(C)) \right) \\ &= \bigwedge_{x \in X} \bigwedge_{C \in L^Y} \left(\mathcal{N}_Y(C)(f(x)) \rightarrow \mathcal{N}_X(f_L^{\leftarrow}(C))(x) \right) \\ &= \bigwedge_{x \in X} \bigwedge_{C \in L^Y} \left(f_L^{\leftarrow}(\mathcal{N}_Y(C))(x) \rightarrow \mathcal{N}_X(f_L^{\leftarrow}(C))(x) \right) \\ &= \bigwedge_{C \in L^Y} \bigwedge_{x \in X} \left(f_L^{\leftarrow}(\mathcal{N}_Y(C))(x) \rightarrow \mathcal{N}_X(f_L^{\leftarrow}(C))(x) \right) \\ &= \bigwedge_{C \in L^Y} \mathbb{S}\left(f_L^{\leftarrow}(\mathcal{N}_Y(C)), \mathcal{N}_X(f_L^{\leftarrow}(C))\right) \\ &= D_{c_2}(f). \end{aligned}$$

□

Theorem 10. Let $(X, \mathcal{T}_X)$ and $(X, \mathcal{T}_Y)$ be L -topological spaces. Then,

$$D_{c_2}(f) = \bigwedge_{F \in \mathcal{F}_L^{\mathbb{S}}(X)} \mathbb{S}\left(\text{Lim}_X F, f_L^{\leftarrow}(\text{Lim}_Y f(F))\right).$$

Proof. On one hand,

$$\begin{aligned}
 & \bigwedge_{F \in \mathfrak{F}_L^S(X)} \mathbb{S}(\text{Lim}_X F, f_L^{\leftarrow}(\text{Lim}_Y f(F))) \\
 = & \bigwedge_{F \in \mathfrak{F}_L^S(X)} \bigwedge_{x \in X} \left(\text{Lim}_X F(x) \rightarrow f_L^{\leftarrow}(\text{Lim}_Y f(F))(x) \right) \\
 = & \bigwedge_{F \in \mathfrak{F}_L^S(X)} \bigwedge_{x \in X} \left(\text{Lim}_X F(x) \rightarrow \text{Lim}_Y f(F)(f(x)) \right) \\
 = & \bigwedge_{F \in \mathfrak{F}_L^S(X)} \bigwedge_{x \in X} \left(\bigwedge_{A \in L^X} (\mathcal{U}_X^x(A) \rightarrow F(A)) \rightarrow \bigwedge_{B \in L^Y} (\mathcal{U}_Y^{f(x)}(B) \rightarrow f(F)(B)) \right) \\
 \geq & \bigwedge_{F \in \mathfrak{F}_L^S(X)} \bigwedge_{x \in X} \left(\bigwedge_{B \in L^Y} (\mathcal{U}_X^x(f_L^{\leftarrow}(B)) \rightarrow F(f_L^{\leftarrow}(B))) \rightarrow \bigwedge_{B \in L^Y} (\mathcal{U}_Y^{f(x)}(B) \rightarrow f(F)(B)) \right) \\
 = & \bigwedge_{F \in \mathfrak{F}_L^S(X)} \bigwedge_{x \in X} \left(\bigwedge_{B \in L^Y} (\mathcal{U}_X^x(f_L^{\leftarrow}(B)) \rightarrow f(F)(B)) \rightarrow \bigwedge_{B \in L^Y} (\mathcal{U}_Y^{f(x)}(B) \rightarrow f(F)(B)) \right) \\
 \geq & \bigwedge_{x \in X} \bigwedge_{B \in L^Y} \left(\mathcal{U}_Y^{f(x)}(B) \rightarrow \mathcal{U}_X^x(f_L^{\leftarrow}(B)) \right) \\
 = & \bigwedge_{x \in X} \bigwedge_{B \in L^Y} \left(\mathcal{U}_Y^{f(x)}(B) \rightarrow f(\mathcal{U}_X^x(B)) \right) \\
 = & D_{c_2}(f).
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 & \bigwedge_{F \in \mathfrak{F}_L^S(X)} \mathbb{S}(\text{Lim}_X F, f_L^{\leftarrow}(\text{Lim}_Y f(F))) \\
 = & \bigwedge_{F \in \mathfrak{F}_L^S(X)} \bigwedge_{x \in X} \left(\text{Lim}_X F(x) \rightarrow f_L^{\leftarrow}(\text{Lim}_Y f(F))(x) \right) \\
 = & \bigwedge_{F \in \mathfrak{F}_L^S(X)} \bigwedge_{x \in X} \left(\text{Lim}_X F(x) \rightarrow \text{Lim}_Y f(F)(f(x)) \right) \\
 = & \bigwedge_{F \in \mathfrak{F}_L^S(X)} \bigwedge_{x \in X} \left(\bigwedge_{A \in L^X} (\mathcal{U}_X^x(A) \rightarrow F(A)) \rightarrow \bigwedge_{B \in L^Y} (\mathcal{U}_Y^{f(x)}(B) \rightarrow f(F)(B)) \right) \\
 \leq & \bigwedge_{x \in X} \left(\bigwedge_{A \in L^X} (\mathcal{U}_X^x(A) \rightarrow \mathcal{U}_X^x(A)) \rightarrow \bigwedge_{B \in L^Y} (\mathcal{U}_Y^{f(x)}(B) \rightarrow f(\mathcal{U}_X^x(B))) \right) \\
 = & \bigwedge_{x \in X} \bigwedge_{B \in L^Y} \left(\mathcal{U}_Y^{f(x)}(B) \rightarrow f(\mathcal{U}_X^x(B)) \right) \\
 = & D_{c_2}(f).
 \end{aligned}$$

Therefore, $D_{c_2}(f) = \bigwedge_{x \in X} \bigwedge_{B \in L^Y} (\mathcal{U}_Y^{f(x)}(B) \rightarrow f(\mathcal{U}_X^x(B)))$. $\square$

In L -topological spaces, compositions of L -continuous mappings (resp. L -open mappings) are still L -continuous mappings (resp. L -open mappings). Now, let us give a degree representation of this result.

Theorem 11. For L -topological spaces $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ and $(Z, \mathcal{T}_Z)$, the following inequalities hold:

$$(1) D_{c_2}(f) \wedge D_{c_2}(g) \leq D_{c_2}(g \circ f).$$

$$(2) D_k(f) \wedge D_k(g) \leq D_k(g \circ f).$$

Proof. (1) By Definition 9, we have

$$\begin{aligned} & D_{c_2}(f) \wedge D_{c_2}(g) \\ = & \bigwedge_{A \in L^Y} \mathbb{S}(f_L^{\leftarrow}(\mathcal{N}_Y(A)), \mathcal{N}_X(f_L^{\leftarrow}(A))) \\ & \wedge \bigwedge_{B \in L^Z} \mathbb{S}(g_L^{\leftarrow}(\mathcal{N}_Z(B)), \mathcal{N}_Y(g_L^{\leftarrow}(B))) \\ \leq & \bigwedge_{C \in L^Z} \mathbb{S}(f_L^{\leftarrow}(\mathcal{N}_X(g_L^{\leftarrow}(C))), \mathcal{N}_Y(f_L^{\leftarrow}(g_L^{\leftarrow}(C)))) \\ & \wedge \bigwedge_{B \in L^Z} \mathbb{S}(g_L^{\leftarrow}(\mathcal{N}_Z(B)), \mathcal{N}_Y(g_L^{\leftarrow}(B))) \\ \leq & \bigwedge_{C \in L^Z} \mathbb{S}(f_L^{\leftarrow}(\mathcal{N}_X(g_L^{\leftarrow}(C))), \mathcal{N}_Y(f_L^{\leftarrow}(g_L^{\leftarrow}(C)))) \\ & \wedge \bigwedge_{B \in L^Z} \mathbb{S}(f_L^{\leftarrow}(g_L^{\leftarrow}(\mathcal{N}_Z(B))), f_L^{\leftarrow}(\mathcal{N}_Y(g_L^{\leftarrow}(B)))) \\ \leq & \bigwedge_{B \in L^Z} \mathbb{S}(f_L^{\leftarrow}(\mathcal{N}_X(g_L^{\leftarrow}(B))), \mathcal{N}_Y(f_L^{\leftarrow}(g_L^{\leftarrow}(B)))) \\ & \wedge \mathbb{S}(f_L^{\leftarrow}(g_L^{\leftarrow}(\mathcal{N}_Z(B))), f_L^{\leftarrow}(\mathcal{N}_Y(g_L^{\leftarrow}(B)))) \\ \leq & \bigwedge_{B \in L^Z} \mathbb{S}(f_L^{\leftarrow}(g_L^{\leftarrow}(\mathcal{N}_Z(B))), \mathcal{N}_Y(f_L^{\leftarrow}(g_L^{\leftarrow}(B)))) \\ = & \bigwedge_{B \in L^Z} \mathbb{S}((g \circ f)_L^{\leftarrow}(\mathcal{N}_Z(B)), \mathcal{N}_X((g \circ f)_L^{\leftarrow}(B))) \\ = & D_{c_2}(g \circ f). \end{aligned}$$

(2) Adopting the proof of (1), it can be verified directly. $\square$

Next, we investigate the connections between degrees of L -continuity and that of L -openness.

Theorem 12. For L -topological spaces $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ and $(Z, \mathcal{T}_Z)$, if f is surjective, then the following inequality holds:

$$D_k(g \circ f) \wedge D_{c_2}(f) \leq D_k(g).$$

Proof. Since f is surjective, we have $f_L^{\rightarrow}(f_L^{\leftarrow}(C)) = C$ for all $C \in L^Y$. Then, it follows that

$$\begin{aligned} & \bigwedge_{B \in L^Y} \mathbb{S}(g_L^{\rightarrow}(\mathcal{N}_Y(B)), \mathcal{N}_Z(g_L^{\rightarrow}(B))) \\ = & \bigwedge_{B \in L^Y} \mathbb{S}(g_L^{\rightarrow}(\mathcal{N}_Y(f_L^{\rightarrow}(f_L^{\leftarrow}(B)))), \mathcal{N}_Z(g_L^{\rightarrow}(f_L^{\rightarrow}(f_L^{\leftarrow}(B))))) \\ \geq & \bigwedge_{A \in L^X} \mathbb{S}(g_L^{\rightarrow}(\mathcal{N}_Y(f_L^{\rightarrow}(A))), \mathcal{N}_Z((g \circ f)_L^{\rightarrow}(A))) \\ \geq & \bigwedge_{B \in L^Y} \mathbb{S}(g_L^{\rightarrow}(\mathcal{N}_Y(B)), \mathcal{N}_Z(g_L^{\rightarrow}(B))). \end{aligned}$$

This shows

$$\bigwedge_{B \in L^Y} \mathbb{S}(g_L^{\rightarrow}(\mathcal{N}_Y(B)), \mathcal{N}_Z(g_L^{\rightarrow}(B))) = \bigwedge_{A \in L^X} \mathbb{S}(g_L^{\rightarrow}(\mathcal{N}_Y(f_L^{\rightarrow}(A))), \mathcal{N}_Z((g \circ f)_L^{\rightarrow}(A))).$$

Then, we have

$$\begin{aligned} & D_k(g \circ f) \wedge D_{c_2}(f) \\ &= \bigwedge_{A \in L^X} \mathbb{S}((g \circ f)_L^{\rightarrow}(\mathcal{N}_X(A)), \mathcal{N}_Z((g \circ f)_L^{\rightarrow}(A))) \\ & \quad \wedge \bigwedge_{B \in L^Y} \mathbb{S}(f_L^{\leftarrow}(\mathcal{N}_Y(B)), \mathcal{N}_X(f_L^{\leftarrow}(B))) \\ &\leq \bigwedge_{A \in L^X} \mathbb{S}((g \circ f)_L^{\rightarrow}(\mathcal{N}_X(A)), \mathcal{N}_Z((g \circ f)_L^{\rightarrow}(A))) \\ & \quad \wedge \bigwedge_{A \in L^X} \mathbb{S}(g_L^{\rightarrow}(\mathcal{N}_Y(f_L^{\rightarrow}(A))), (g \circ f)_L^{\rightarrow}(\mathcal{N}_X(A))) \\ &\leq \bigwedge_{A \in L^X} \mathbb{S}(g_L^{\rightarrow}(\mathcal{N}_Y(f_L^{\rightarrow}(A))), \mathcal{N}_Z((g \circ f)_L^{\rightarrow}(A))) \\ &= \bigwedge_{B \in L^Y} \mathbb{S}(\mathcal{N}_Z(g_L^{\rightarrow}(\mathcal{N}_Y(B))), g_L^{\rightarrow}(B))) \\ &= D_k(g). \end{aligned}$$

□

Theorem 13. For L -topological spaces $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ and $(Z, \mathcal{T}_Z)$, if f is injective, then the following inequality holds:

$$D_k(g \circ f) \wedge D_{c_2}(g) \leq D_k(f).$$

Proof. Since g is injective, we have $g_L^{\leftarrow}(g_L^{\rightarrow}(C)) = C$ for all $C \in L^Y$. Then, it follows that

$$\begin{aligned} & D_k(g \circ f) \wedge D_{c_2}(g) \\ &= \bigwedge_{B \in L^X} \mathbb{S}((g \circ f)_L^{\rightarrow}(\mathcal{N}_X(B)), \mathcal{N}_Z((g \circ f)_L^{\rightarrow}(B))) \\ & \quad \wedge \bigwedge_{C \in L^Z} \mathbb{S}(g_L^{\leftarrow}(\mathcal{N}_Z(C)), \mathcal{N}_Y(g_L^{\leftarrow}(C))) \\ &\leq \bigwedge_{B \in L^X} \mathbb{S}(g_L^{\rightarrow}((g \circ f)_L^{\rightarrow}(\mathcal{N}_X(B))), g_L^{\rightarrow}(\mathcal{N}_Z((g \circ f)_L^{\rightarrow}(B)))) \\ & \quad \wedge \bigwedge_{B \in L^X} \mathbb{S}(g_L^{\leftarrow}(\mathcal{N}_Z((g \circ f)_L^{\rightarrow}(B))), \mathcal{N}_Y(g_L^{\leftarrow}((g \circ f)_L^{\rightarrow}(B)))) \\ &= \bigwedge_{B \in L^X} \mathbb{S}(f_L^{\rightarrow}(\mathcal{N}_X(B)), \mathcal{N}_Y(f_L^{\rightarrow}(B))) \\ &= D_k(f). \end{aligned}$$

□

Theorem 14. Suppose that L is a frame with an order-reversing involution $'$, where $a' = a \rightarrow \perp$ for $a \in L$. For an L -topological space $(X, \mathcal{T})$, $\mathcal{C} = \{B | B' \in \mathcal{T}\}$ is an L -cotopology and $Cl(B) = (\mathcal{N}(B'))'$.

Proof. It is easy to verify that $\mathcal{C}$ is an L -cotopology and $Cl(A) = (\mathcal{N}(A'))'$. $\square$

Theorem 15. Suppose that L is a frame with an order-reversing involution $'$, where $a' = a \rightarrow \perp$ for $a \in L$. For L -topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, $D_{c_1}(f) = D_{c_2}(f)$.

Proof. By Theorems 5 and 14,

$$\begin{aligned} D_{c_1}(f) &= \bigwedge_{B \in L^Y} \mathbb{S}\left(Cl_X(f_L^{\leftarrow}(B)), f_L^{\leftarrow}(Cl_Y(B))\right) \\ &= \bigwedge_{B \in L^Y} \mathbb{S}\left((\mathcal{N}_X(f_L^{\leftarrow}(B'))')', f_L^{\leftarrow}((\mathcal{N}_Y(B'))')')\right) \\ &= \bigwedge_{B \in L^Y} \mathbb{S}\left(f_L^{\leftarrow}(\mathcal{N}_Y(B')), \mathcal{N}_X(f_L^{\leftarrow}(B'))\right) \\ &= \bigwedge_{B \in L^Y} \mathbb{S}\left(f_L^{\leftarrow}(\mathcal{N}_Y(B)), \mathcal{N}_X(f_L^{\leftarrow}(B))\right) \\ &= D_{c_2}(f). \end{aligned}$$

$\square$

5. Conclusions

In this paper, we equip each mapping between L -cotopological spaces with some degree to be an L -continuous mapping and an L -closed mapping, and equip each mapping between L -topological spaces with some degree to be an L -continuous mapping and an L -open mapping. From this aspect, we could consider the degrees of L -continuity, L -closedness and L -openness for a mapping even if the mapping is not a continuous mapping, a closed mapping or an open mapping. By means of these definitions, we proved that the degrees of L -continuity, L -closedness and L -openness for mappings naturally suggest lattice-valued logical extensions of properties related to continuous mappings, closed mappings and open mappings in classical topological spaces to fuzzy topological spaces. Moreover, if L is a frame with an order-reversing involution $'$, where $b' = b \rightarrow \perp$ for $b \in L$, then degrees of L -continuity for mappings between L -cotopological spaces and degrees of L -continuity for mappings between L -topological spaces are equivalent.

As future research, we will consider the following two problems:

(1) By means of the degree method, we can also define the degrees of some topological properties. For example, we can use the convergence degree of a fuzzy ultrafilter to define the compactness degree of an L -topological space.

(2) Based on the degrees of L -continuity, L -openness and L -closedness for mappings, we can further define the degrees of L -homeomorphism in a degree. We only need to equip a bijective mapping with the degrees of L -continuity and L -openness.

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