Article

On Two Open Problems on Double Vertex-Edge Domination in Graphs

Fang Miao ^{1,2}, Wenjie Fan ^{1,2}, Mustapha Chellali ³, Rana Khoeilar ⁴,
Seyed Mahmoud Sheikholeslami ^{4,*} and Marzieh Soroudi ⁴

¹ The Big Data Research Institute, Chengdu University, Chengdu 610106, China; miaoofang@126.com (F.M.); fanwenjie@cdu.edu.cn (W.F.)

² Key Laboratory of Pattern Recognition and Intelligent Information Processing, Chengdu 610106, China

³ LAMDA-RO Laboratory, Department of Mathematics, University of Blida, B.P. 270 Blida, Algeria; m_chellali@yahoo.com

⁴ Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz 51368, Iran; khoeilar@azaruniv.ac.ir (R.K.); m.soroudi@azaruniv.ac.ir (M.S.)

* Correspondence: s.m.sheikholeslami@azaruniv.ac.ir

Received: 13 September 2019; Accepted: 21 October 2019; Published: 24 October 2019

Abstract: A vertex v of a graph $G = (V, E)$, ve -dominates every edge incident to v , as well as every edge adjacent to these incident edges. A set $S \subseteq V$ is a double vertex-edge dominating set if every edge of E is ve -dominated by at least two vertices of S . The double vertex-edge domination number $\gamma_{dve}(G)$ is the minimum cardinality of a double vertex-edge dominating set in G . A subset $S \subseteq V$ is a total dominating set (respectively, a 2-dominating set) if every vertex in V has a neighbor in S (respectively, every vertex in $V - S$ has at least two neighbors in S). The total domination number $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G , and the 2-domination number $\gamma_2(G)$ is the minimum cardinality of a 2-dominating set of G . Krishnakumari et al. (2017) showed that for every triangle-free graph G , $\gamma_{dve}(G) \leq \gamma_2(G)$, and in addition, if G has no isolated vertices, then $\gamma_{dve}(G) \leq \gamma_t(G)$. Moreover, they posed the problem of characterizing those graphs attaining the equality in the previous bounds. In this paper, we characterize all trees T with $\gamma_{dve}(T) = \gamma_t(T)$ or $\gamma_{dve}(T) = \gamma_2(T)$.

Keywords: vertex-edge dominating set; double vertex-edge dominating set; total dominating set; 2-dominating set

1. Introduction

In this paper, G is a simple nontrivial connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For a vertex $v \in V$, the “open neighborhood” of v is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the “closed neighborhood” of v is the set $N[v] = N(v) \cup \{v\}$. The “degree” of a vertex $v \in V$ is $\deg_G(v) = |N(v)|$. A vertex of degree one is called a “pendant vertex” or a “leaf” and its neighbor is called a “support vertex”. A “strong support vertex” is a support vertex adjacent to at least two leaves and an “end support vertex” is a support vertex having at most one non-leaf neighbor. If v is a support vertex, then L_v denotes the set of leaves adjacent to v . A “pendant path” P of a graph G is an induced path starting from a leaf and containing only vertices of degree two in G as inner vertices. The “distance” between two vertices u and v in a connected graph G is the length of a shortest uv -path in G . The “diameter” of connected graph G , denoted by $\text{diam}(G)$, is the maximum value among minimum distances between all pairs of vertices of G . For a vertex v in a rooted tree T , let $C(v)$ and $D(v)$ denote the set of children and descendants of v , respectively, and let $D[v] = D(v) \cup \{v\}$. Also, the “depth” of v , $\text{depth}(v)$, is the largest distance

from v to a vertex in $D(v)$. The “maximal subtree” at v is the subtree of T induced by $D[v]$, and is denoted by T_v . We write P_n for a path of order n . A “double star” $DS_{p,q}$ is a tree containing exactly two non-pendant vertices, one of which is adjacent to p leaves and the other is adjacent to q leaves. If $A \subseteq V(G)$ and f is a mapping from $V(G)$ into some set of numbers, then $f(A) = \sum_{x \in A} f(x)$, and the sum $f(V(G))$ is called the “weight” $\omega(f)$ of f .

A subset $S \subseteq V$ is a “2-dominating set”, abbreviated 2d-set, of G if every vertex in $V \setminus S$ is adjacent to at least two vertices in S and it is a “total dominating set” (td-set), if every vertex in V is adjacent to a vertex in S . The “2-domination number”, $\gamma_2(G)$ (respectively, “total domination number”, $\gamma_t(G)$) of G , is the minimum cardinality of a 2d-set (respectively, td-set) of G . A 2d-set (respectively, td-set) of G with minimum cardinality is called a $\gamma_2(G)$ -set (respectively, $\gamma_t(G)$ -set). The literature on the subject of domination and total domination in graphs has been surveyed and detailed in two books [1,2].

A vertex v is said to “ve-dominate” every edge incident to any vertex in $N[v]$. A set $S \subseteq V$ is a “vertex-edge dominating set” (or simply, a ve-dominating set) if for every edge $e \in E$, there exists a vertex $v \in S$ that ve-dominates e . The minimum cardinality of a ve-dominating set of G , $\gamma_{ve}(G)$, is called the “vertex-edge domination number”. Vertex-edge domination was introduced by Peters [3] in his 1986 PhD thesis and studied further in [4–10].

A vertex-edge dominating set $D \subseteq V$ is called a “double vertex-edge dominating set (or simply, a dve-dominating set) of G , if every edge of E is ve-dominated by at least two vertices of D . The “double vertex-edge domination number” of G , $\gamma_{dve}(G)$, is the minimum cardinality of a double ve-dominating set. The concept of double vertex-edge domination in graphs was introduced by Krishnakumari, Chellali, and Venkatakrishnan in [11], where they proved the following results.

Proposition 1 ([11]). *For every triangle-free graph G without isolated vertices,*

$$\gamma_{dve}(G) \leq \gamma_t(G). \quad (1)$$

Proposition 2 ([11]). *For every graph G ,*

$$\gamma_{dve}(G) \leq \gamma_2(G). \quad (2)$$

Moreover, the authors [11] posed these two problems.

Problem 1. Characterize all connected graphs G with $\gamma_{dve}(G) = \gamma_2(G)$.

Problem 2. Characterize all nontrivial trees T with $\gamma_{dve}(T) = \gamma_t(T)$.

In this paper, we settle the above open problems for trees by providing a constructive characterization of all trees T with $\gamma_{dve}(T) = \gamma_2(T)$ or $\gamma_{dve}(T) = \gamma_t(T)$.

2. Preliminaries

In this section, we provide some definitions and observations that will be useful throughout the paper. We introduce some sets of vertices satisfying special condition.

Definition 1. Let u be a vertex of a graph G . A subset S of vertices is said to be an almost dve-dominating set with respect to u if the following conditions are fulfilled: (i) any edge not incident to u , is ve-dominated by at least two vertices in S , and (ii) any edge incident to u is ve-dominated by at least one vertex in S . Define

$$\gamma_{dve}(G; u) = \min\{|S| : S \text{ is an almost dve-dominating set with respect to } u\}.$$

Clearly, any dve-dominating set on G is an almost dve-dominating set with respect to any vertex of G and so $\gamma_{dve}(G; u) \leq \gamma_{dve}(G)$ for each $u \in V(G)$. Define

$$W_G^1 = \{u \in V(G) | \gamma_{dve}(G; u) = \gamma_{dve}(G)\}.$$

Definition 2. For a graph G , define

$$\begin{aligned} W_G^2 &= \{v \in V(G) \mid \text{there exists a } \gamma_{dve}(G)\text{-set } S \text{ such that } |S \cap N_G[v]| \geq 1\}, \\ W_G^3 &= \{v \in V(G) \mid \text{there exists a } \gamma_2(G)\text{-set } S \text{ such that } v \in S\}, \\ W_G^4 &= \{v \in V(G) \mid \text{there exists no dve-dominating set } S \text{ of } G \text{ such that } v \in S\}. \end{aligned}$$

Observation 1. All leaves of a graph G are in every 2-dominating set of G .

Observation 2. For any connected graph G with diameter at least three, there exists a $\gamma_t(G)$ -set (respectively, $\gamma_{dve}(G)$ -set) that contains no leaf of G .

Proof. Among all $\gamma_t(G)$ -sets, let D be one containing the minimum number of leaves of G . Assume that there exists a leaf of G , say x with $x \in D$. Let y be the support vertex adjacent to x . Clearly, $y \in D$ (else x will be without neighbors in D), and $A = N(y) - L_v \neq \emptyset$ (as G has diameter at least three). If $A \cap D \neq \emptyset$, then $D - \{x\}$ is a td-set of G , a contradiction. Therefore, $A \cap D = \emptyset$. Then, replacing x in D by any vertex of A , provides a $\gamma_t(G)$ -set containing fewer leaves than D , contradicting our choice of D .

A similar argument can be used to prove also that G has a $\gamma_{dve}(G)$ -set that contains no leaf. $\square$

Proposition 3. Let G be a nontrivial connected graph and $u \in V(G)$ a vertex of degree at least two which is a support vertex or adjacent to an end-support vertex. If G' is the graph obtained from G by adding a new vertex v connected by an edge to u , then $\gamma_{dve}(G) = \gamma_{dve}(G')$ and $\gamma_t(G) = \gamma_t(G')$.

Proof. If G' is a star, then the results are immediate. Suppose G' is not a star. Thus, both G' and G have diameter at least three. By Observation 2, any $\gamma_t(G)$ -set (respectively, $\gamma_t(G')$ -set) containing no leaves contains u , and so is a td-set of G' (respectively, G) yielding $\gamma_t(G') \leq \gamma_t(G)$ (respectively, $\gamma_t(G) \leq \gamma_t(G')$). Therefore, $\gamma_t(G') = \gamma_t(G)$.

Assume now that D is a $\gamma_{dve}(G')$ -set containing no leaves. Then to ve-dominate the edge uv , we may assume that $D \cap (N[u] - \{v\}) \neq \emptyset$ and D is clearly a dve-dominating set of G implying that $\gamma_{dve}(G') \geq \gamma_{dve}(G)$. On the other hand, clearly any $\gamma_{dve}(G)$ -set containing no leaves is a double ve-dominating set of G' and so $\gamma_{dve}(G') \leq \gamma_{dve}(G)$. Thus, $\gamma_{dve}(G') = \gamma_{dve}(G)$. $\square$

Proposition 4. Let G be a nontrivial connected graph and $u \in V(G)$ such that ux_1x_2 is a path in G in which $\deg_G(x_1) = 2$ and $\deg_G(x_2) = 1$. If G' is a graph obtained from G by attaching a path $P_2 = y_1y_2$ and joining y_1 to u , then $\gamma_t(G') = \gamma_t(G) + 1$, $\gamma_{dve}(G') = \gamma_{dve}(G) + 1$ and $\gamma_2(G') = \gamma_2(G) + 1$.

Proof. Clearly, any $\gamma_t(G)$ -set (respectively, $\gamma_t(G')$ -set) containing no leaves, contains u, x_1 (respectively, u, x_1, y_1) and for which adding (respectively, removing) vertex y_1 yields a td-set of G' (respectively, G). Therefore, $\gamma_t(G') \leq \gamma_t(G) + 1$ (respectively, $\gamma_t(G) \leq \gamma_t(G') - 1$), and so $\gamma_t(G') = \gamma_t(G) + 1$. Similarly, we can see that $\gamma_{dve}(G') = \gamma_{dve}(G) + 1$ and $\gamma_2(G') = \gamma_2(G) + 1$. $\square$

Proposition 5. Let G be a nontrivial connected graph and let u be a vertex of G . If G' is the graph obtained from G by adding a path $P_4 = x_1x_2x_3x_4$ and joining u to x_1 , then $\gamma_t(G') = \gamma_t(G) + 2$ and $\gamma_{dve}(G') \geq \gamma_{dve}(G) + 2$. Moreover, if $u \in W_G^2$, then $\gamma_{dve}(G') = \gamma_{dve}(G) + 2$.

Proof. Clearly, any $\gamma_t(G)$ -set can be extended to a td-set of G' by adding x_2, x_3 , and so $\gamma_t(G') \leq \gamma_t(G) + 2$.

To prove the inverse inequality, let $v \in N_G(u)$ and let D be a $\gamma_t(G')$ -set containing no leaves. Then, $x_2, x_3 \in D$. If $v \in D$ or $x_1 \notin D$, then $D \setminus \{x_2, x_3\}$ is a td-set of G yielding $\gamma_t(G') \geq \gamma_t(G) + 2$. Thus, assume that $v \notin D$ and $x_1 \in D$. Then, $(D \setminus \{x_1, x_2, x_3\}) \cup \{v\}$ is a total dominating set of G yielding $\gamma_t(G') \geq \gamma_t(G) + 2$. Therefore, $\gamma_t(G') = \gamma_t(G) + 2$.

Similarly, we can see that $\gamma_{dve}(G') \geq \gamma_{dve}(G) + 2$. Now, if $u \in W_G^2$, then clearly some $\gamma_{dve}(G)$ -set contains a vertex $N_G[u]$, and so it can be extended to a dve-dominating set of G' by adding x_2, x_3 and thus $\gamma_{dve}(G') \leq \gamma_{dve}(G) + 2$. Therefore, $\gamma_{dve}(G') = \gamma_{dve}(G) + 2$ when $u \in W_G^2$. $\square$

Proposition 6. Let G be a nontrivial connected graph and let u be a vertex of G , such that ux_1x_2 is a pendant path in G with $\deg_G(x_2) = 1$. If G' is the graph obtained from G by adding a path $P_3 = y_1y_2y_3$ and joining u to y_1 , then $\gamma_t(G') = \gamma_t(G) + 2$, $\gamma_2(G') = \gamma_2(G) + 2$ and $\gamma_{dve}(G') = \gamma_{dve}(G) + 2$.

Proof. Clearly, any $\gamma_t(G)$ -set (respectively, $\gamma_t(G')$ -set) D containing no leaves contains u, x_1 (respectively, u, x_1, y_1, y_2) and so $D \cup \{y_1, y_2\}$ (respectively, $D - \{y_1, y_2\}$) is a td-set of G' (respectively, G) yielding $\gamma_t(G') = \gamma_t(G) + 2$. Similarly, we can easily see that $\gamma_2(G') = \gamma_2(G) + 2$.

Assume now that D is a $\gamma_{dve}(G')$ -set containing no leaves. Obviously, to dve-dominate the edges x_1x_2, y_2y_3 , we must have $u, x_1, y_1, y_2 \in D$ and thus $D - \{y_1, y_2\}$ is a dve-dominating of G . Thus, $\gamma_{dve}(G') \geq \gamma_{dve}(G) + 2$. The equality is obtained from the fact that any $\gamma_{dve}(G)$ -set can be extended to a dve-dominating set of G' by adding y_1, y_2 . $\square$

Proposition 7. Let G be a nontrivial connected graph and let u be a vertex of G such that $ux_1x_2x_3$ is a pendant path in G with $\deg_G(x_3) = 1$. If G' is the graph obtained from G by adding a path $P_3 = y_1y_2y_3$ and joining u to y_1 , then $\gamma_t(G') = \gamma_t(G) + 2$ and $\gamma_{dve}(G') \leq \gamma_{dve}(G) + 2$. Furthermore, if $u \in W_G^1$, then $\gamma_{dve}(G') = \gamma_{dve}(G) + 2$.

Proof. As in the proof of Proposition 6, we can see that $\gamma_t(G') = \gamma_t(G) + 2$. The inequality $\gamma_{dve}(G') \leq \gamma_{dve}(G) + 2$ follows from the fact that any $\gamma_{dve}(G)$ -set can be extended to a dve-dominating set of G' by adding y_1, y_2 . Assume now that $u \in W_G^1$ and let D be a $\gamma_{dve}(G')$ -set containing no leaves. Then we must have $x_1, x_2, y_1, y_2 \in D$, and thus $D - \{y_1, y_2\}$ is an almost dve-dominating set of G with respect to u . We deduce from $u \in W_G^1$ that $|D - \{y_1, y_2\}| \geq \gamma_{dve}(G; u) = \gamma_{dve}(G)$ and so $\gamma_{dve}(G') \geq \gamma_{dve}(G) + 2$. Hence $\gamma_{dve}(G') = \gamma_{dve}(G) + 2$ when $u \in W_G^1$. $\square$

Let $k \geq 2$ be an integer and H_k be the graph obtained from the complete bipartite graph $K_{1,k}$ (also known by a star) by subdividing every edge twice (see Figure 1).

Figure 1. The graph H_k ($k \geq 2$) which is defined above.

Proposition 8. Let G be a nontrivial connected graph and let u be a vertex of G . If G' is the graph obtained from G by adding the graph H_k and joining u to the center vertex of H_k . Then, $\gamma_2(G') = \gamma_2(G) + 2k$ and $\gamma_{dve}(G') = \gamma_{dve}(G) + 2k$.

Proof. Clearly, any $\gamma_2(G)$ -set can be extended to a 2d-set of G' by adding a_i, c_i for $1 \leq i \leq k$, implying that $\gamma_2(G') \leq \gamma_2(G) + 2k$.

To prove the inverse inequality, let D be $\gamma_2(G')$ -set. Obviously, $c_i \in D$ for each i . Now to 2-dominate the vertex b_i we must have $|D \cap \{a_i, b_i\}| \geq 1$ for each i . If $a \notin D$, then $D - V(H_k)$ is a 2d-set of G and if $a \in D$, then $(D - V(H_k)) \cup \{u\}$ is a 2d-set of G . In either case, $\gamma_2(G') \geq \gamma_2(G) + 2k$, and thus $\gamma_2(G') = \gamma_2(G) + 2k$. Similarly, we can easily see that $\gamma_{dve}(G') = \gamma_{dve}(G) + 2k$. $\square$

Observation 3. Let G be a triangle-free graph without isolated vertices, and let H be a subgraph of G .

1. If $\gamma_{dve}(H) = \gamma_t(H)$, $\gamma_t(G) \leq \gamma_t(H) + s$ and $\gamma_{dve}(G) \geq \gamma_{dve}(H) + s$ for some non-negative integer s , then $\gamma_{dve}(G) = \gamma_t(G)$.
2. If $\gamma_{dve}(G) = \gamma_t(G)$, $\gamma_{dve}(G) \leq \gamma_{dve}(H) + s$ and $\gamma_t(G) \geq \gamma_t(H) + s$ for some non-negative integer s , then $\gamma_{dve}(H) = \gamma_t(H)$.

Proof. 1. We conclude from the assumptions that $\gamma_{dve}(G) \geq \gamma_{dve}(H) + s = \gamma_t(H) + s \geq \gamma_t(G)$ and this leads to the result in (1).

2. By inequality (1) and the assumptions, we have

$$\gamma_t(G) = \gamma_{dve}(G) \leq \gamma_{dve}(H) + s \leq \gamma_t(H) + s \leq \gamma_t(G).$$

Thus, all inequalities occurring in the above chain become equalities. In particular, $\gamma_{dve}(H) = \gamma_t(H)$.

□

The proof of next result is similar to the proof of Observation 3, and therefore we omit the details.

Observation 4. Let H be a subgraph of a graph G .

1. If $\gamma_2(H) = \gamma_{dve}(H)$, $\gamma_2(G) \leq \gamma_2(H) + s$ and $\gamma_{dve}(G) \geq \gamma_{dve}(H) + s$ for some non-negative integer s , then $\gamma_2(G) = \gamma_{dve}(G)$.
2. If $\gamma_2(G) = \gamma_{dve}(G)$, $\gamma_{dve}(G) \leq \gamma_{dve}(H) + s$ and $\gamma_2(G) \geq \gamma_2(H) + s$ for some non-negative integer s , then $\gamma_2(H) = \gamma_{dve}(H)$.

We close this section with the following simple observation.

Observation 5. If T is a tree of order $n \geq 4$ with $2 \leq \text{diam}(T) \leq 3$, then $\gamma_t(T) = \gamma_{dve}(T) = 2$ and $\gamma_{dve}(T) \neq \gamma_2(T)$.

3. Trees T with $\gamma_{dve}(T) = \gamma_2(T)$

In this section, we provide a constructive characterization of all trees T with $\gamma_{dve}(T) = \gamma_2(T)$. For this purpose, we define the family $\mathcal{F}$ of unlabeled trees T that can be obtained from a sequence $T_1, T_2, \dots, T_k$ ($k \geq 1$) of trees such that $T_1 \in \{P_2, P_3\}$ and $T = T_k$. If $k \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the following operations.

Operation $\mathcal{T}_1$: If $u \in V(T_i)$ and there is a pendant path ux_1x_2 such that $\deg_{T_i}(x_1) = 2$ and $\deg_{T_i}(x_2) = 1$, then $\mathcal{T}_1$ adds a path $P_2 = y_1y_2$ and an edge uy_1 to obtain T_{i+1} .

Operation $\mathcal{T}_2$: If $u \in V(T_i)$ and there is a pendant path ux_1x_2 such that $\deg_{T_i}(x_1) = 2$ and $\deg_{T_i}(x_2) = 1$, then $\mathcal{T}_2$ adds a path $P_3 = y_1y_2y_3$ and an edge uy_1 to obtain T_{i+1} .

Operation $\mathcal{T}_3$: If $u \in V(T_i)$, then $\mathcal{T}_3$ adds a copy of the graph H_k for some $k \geq 2$, connected by an edge from its center vertex a to u to obtain T_{i+1} .

Operation $\mathcal{T}_4$: If $u \in W_{T_i}^1 \cap W_{T_i}^4$, then $\mathcal{T}_4$ adds a path $P_5 = x_1x_2x_3x_4x_5$ and an edge ux_2 to obtain T_{i+1} .

Operation $\mathcal{T}_5$: If $u \in W_{T_i}^3$, then $\mathcal{T}_5$ adds a path $P_4 = x_1x_2x_3x_4$ and an edge ux_1 to obtain T_{i+1} .

Operation $\mathcal{T}_6$: If $u \in W_{T_i}^4$ and there is a pendant path $ux_1x_2x_3$, then $\mathcal{T}_6$ adds a path $P_3 = y_1y_2y_3$ and an edge uy_1 to obtain T_{i+1} .

Lemma 1. If T_i is a tree with $\gamma_{dve}(T_i) = \gamma_2(T_i)$ and T_{i+1} is a tree obtained from T_i by one of the operations $\mathcal{T}_i$ ($1 \leq i \leq 6$), then $\gamma_{dve}(T_{i+1}) = \gamma_2(T_{i+1})$.

Proof. If T_{i+1} is a tree obtained from T_i by Operation $\mathcal{T}_i$ ($1 \leq i \leq 3$), then the result follows by Propositions 4, 6, and 8.

Let T_{i+1} be a tree obtained from T_i by Operation $\mathcal{T}_4$. Clearly, any $\gamma_2(T_i)$ -set can be extended to a 2d-set of T_{i+1} by adding x_1, x_3, x_5 and so $\gamma_2(T_{i+1}) \leq \gamma_2(T_i) + 3$. Now, let D be a $\gamma_{dve}(T_{i+1})$ -set containing no leaves. Clearly, to dve-dominate the edge x_4x_5 , we must have $x_3, x_4 \in D$. If $D \cap \{x_1, x_2\} = \emptyset$, then to dve-dominate the edge x_1x_2 we must have $u \in D$ and as $u \in W_{T_i}^4$, we have $|D \setminus \{x_3, x_4\}| \geq \gamma_{dve}(T_i) + 1$. If $D \cap \{x_1, x_2\} \neq \emptyset$, then $(D \setminus \{x_1, x_2, x_3, x_4\}) \cup \{u\}$ is an almost dve-dominating set of T_i and, as above, we have $|(D \setminus \{x_1, x_2, x_3, x_4\}) \cup \{u\}| \geq \gamma_{dve}(T_i) + 1$. In either case, $\gamma_{dve}(T_{i+1}) \geq \gamma_{dve}(T_i) + 3$. Therefore we conclude from Observation 4 (Item 1) that $\gamma_2(T_{i+1}) = \gamma_{dve}(T_{i+1})$.

Next, let T_{i+1} be a tree obtained from T_i by Operation $\mathcal{T}_5$. As $u \in W_{T_i}^3$, any $\gamma_2(T_i)$ -set containing u can be extended to a 2d-set of T_{i+1} by adding x_2, x_4 , and thus $\gamma_2(T_{i+1}) \leq \gamma_2(T_i) + 2$. On the other hand, by Proposition 5, we have $\gamma_{dve}(T_{i+1}) \geq \gamma_{dve}(T_i) + 2$. Therefore, by Observation 4 (Item 1), we obtain $\gamma_2(T_{i+1}) = \gamma_{dve}(T_{i+1})$.

Finally, let T_{i+1} be a tree obtained from T_i by Operation $\mathcal{T}_6$. Clearly, any $\gamma_2(T_i)$ -set can be extended to a 2d-set of T_{i+1} by adding y_1, y_2 , and so $\gamma_2(T_{i+1}) \leq \gamma_2(T_i) + 2$. On the other hand, let D be a $\gamma_{dve}(T_{i+1})$ -set containing no leaves. Clearly, to dve-dominate the edges yy_1, yy_2 and x_2x_3 , we must have $u, y, x_1, x_2 \in D$. Then, $D - \{y\}$ is a dve-dominating set of T' . As $u \in W_{T_i}^4$, we deduce that $|D - \{y\}| \geq \gamma_{dve}(T_i) + 1$. Therefore, $\gamma_{dve}(T_{i+1}) \geq \gamma_{dve}(T_i) + 2$. By Observation 4 (Item 1), we obtain $\gamma_2(T_{i+1}) = \gamma_{dve}(T_{i+1})$. $\square$

Theorem 1. If $T \in \mathcal{F}$, then $\gamma_2(T) = \gamma_{dve}(T)$.

Proof. Let $T \in \mathcal{F}$. Then there exists a sequence of trees $T_1, T_2, \dots, T_k$ ($k \geq 1$), such that $T_1 \in \{P_2, P_3\}$ and $T = T_k$. Moreover, if $k \geq 2$, then T_{i+1} can be obtained from T_i by one of the aforementioned operations. We proceed by induction on the number of operations used to construct T . If $k = 1$, then $T = T_1 \in \{P_2, P_3\}$ and clearly $\gamma_2(T) = \gamma_{dve}(T)$. Let $k \geq 2$, and assume that the result holds for each tree $T \in \mathcal{F}$, which can be obtained from a sequence of operations of length $k - 1$. Let $T' = T_{k-1}$. By the induction hypothesis, $\gamma_2(T') = \gamma_{dve}(T')$. As $T = T_k$ is obtained by one of the Operations $\mathcal{T}_i$ ($i = 1, 2, 3, 4, 5, 6$) from T' , we conclude from Lemma 1 that $\gamma_2(T) = \gamma_{dve}(T)$. $\square$

Before stating the main theorem of this section, we give the following two useful observations.

Observation 6. In a connected nontrivial graph G , every 2-dominating set of G is a dve-dominating set.

Proof. Let S be a 2-dominating set of G , and let $e = xy$ be any edge of G . Clearly, if $|S \cap \{u, v\}| = 2$, then e is ve-dominated once by x and once by y . Therefore, we assume, without loss of generality, that $x \in S$. As x has at least two neighbors in S , edge e is then ve-dominated at least twice by vertices in $N(x) \cap S$. In either case, e is ve-dominated twice by vertices of S , and desired result follows. $\square$

Observation 7. Let T be a tree of order $n \geq 4$ with $\gamma_2(T) = \gamma_{dve}(T)$. Then, every support vertex of T is adjacent to at most two leaves.

Proof. Let D be a $\gamma_2(T)$ -set, and let v be a support vertex of T such that $|L_v| \geq 3$. Clearly, $L_v \subseteq D$, and for every $w \in L_v$, $D - \{w\}$ is a dve-dominating set of T . Therefore, $\gamma_{dve}(T) \leq |D| - 1 = \gamma_2(T) - 1$, contradicting the fact $\gamma_2(T) = \gamma_{dve}(T)$. $\square$

Theorem 2. Let T be a tree of order $n \geq 2$. Then $\gamma_2(T) = \gamma_{dve}(T)$ if and only if $T \in \mathcal{F}$.

Proof. By Theorem 1, we only need to prove necessity. Let T be a nontrivial tree of order n with $\gamma_2(T) = \gamma_{dve}(T)$. We proceed by induction on n . If $n \leq 3$, then $T \in \{P_2, P_3\}$ and $T \in \mathcal{F}$. Therefore, let $n \geq 4$. By Observation 5, we have $\text{diam}(T) \geq 4$, and thus $n \geq 5$. Assume that the result holds for

every nontrivial tree T' having an order less than n and satisfying $\gamma_2(T') = \gamma_{dve}(T')$. Let T be a tree of order n with $\gamma_2(T) = \gamma_{dve}(T)$. Let $v_1v_2 \dots v_k$ ($k \geq 5$) be a diametral path in T , such that $\deg_T(v_2)$ is as large as possible. Among such paths, we choose one so that $\deg_T(v_3)$ is as large as possible and root T at v_k . By Observation 7, $2 \leq \deg_T(v_2) \leq 3$. We consider the following cases.

Case 1. $\deg_T(v_2) = 2$. By the choice of diametrical path, we may assume that any child of v_3 with depth one has degree 2. We distinguish the following situations.

Subcase 1.1. $\deg_T(v_3) \geq 3$ and v_3 has a child z with depth 0.

Let $T' = T - \{z\}$. Let D be a $\gamma_2(T)$ -set. Clearly $v_1, z \in D$ and $|D \cap \{v_2, v_3\}| = 1$. Without loss of generality, we may assume that $v_3 \in D$. Therefore, $D \setminus \{z\}$ is a 2d-set of T' , which implies that $\gamma_2(T) \geq \gamma_2(T') + 1$. On the other hand, any $\gamma_{dve}(T')$ -dominating set of T' containing no leaves is a dve-dominating set of T implying that $\gamma_{dve}(T) \leq \gamma_{dve}(T')$. This leads to the contradiction $\gamma_2(T) - 1 \geq \gamma_2(T') \geq \gamma_{dve}(T') \geq \gamma_{dve}(T) = \gamma_2(T)$.

Subcase 1.2. $\deg_T(v_3) \geq 3$ and v_3 has a child with depth 1.

Let $T' = T - T_{v_2}$. As there is a $\gamma_2(T)$ -set D containing v_3 , $D \setminus \{v_1\}$ is a 2d-set of T' and so $\gamma_2(T) \geq \gamma_2(T') + 1$. On the other hand, if S is a $\gamma_{dve}(T')$ -set containing no leaves, then $v_3 \in S$ and thus $S \cup \{v_2\}$ is a dve-dominating set of T . Therefore, $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 1$. By Observation 4 (Item 2), we have $\gamma_2(T') = \gamma_{dve}(T')$. It follows from the induction hypothesis that $T' \in \mathcal{F}$. Therefore, $T \in \mathcal{F}$, as it can be obtained from T' by Operation $\mathcal{T}_1$.

Subcase 1.3. $\deg_T(v_3) = 2$ and v_4 has at least one child with depth 1 and degree 2.

Let $T' = T - T_{v_3}$. By Proposition 6 and by the assumption $\gamma_2(T) = \gamma_{dve}(T)$, we have $\gamma_2(T') = \gamma_{dve}(T')$. By the induction hypothesis, we have $T' \in \mathcal{T}$, and as T can be obtained from T' by Operation $\mathcal{T}_2$, we deduce that $T \in \mathcal{F}$.

Subcase 1.4. $\deg_T(v_3) = 2$ and v_4 has a child y with depth 1 and at least degree 3.

By Observation 7, $\deg_T(y) = 3$. Let $L_y = \{y_1, y_2\}$ and $T' = T - T_y$. Let D be a $\gamma_2(T)$ -set. Clearly, $y_1, y_2 \in D$. If $y \in D$, then $(D \setminus \{y, y_1, y_2\}) \cup \{v_4\}$ is a 2d-set of T' and if $y \notin D$, then $D \setminus \{y_1, y_2\}$ is a 2d-set of T' . In either case, we obtain $\gamma_2(T) \geq \gamma_2(T') + 2$. Now, let S be a $\gamma_{dve}(T')$ -set. If $v_4 \in S$, then $S \cup \{y\}$ is a dve-dominating set of T , which leads to a contradiction as $\gamma_2(T) \geq \gamma_2(T') + 2 \geq \gamma_{dve}(T') + 2 \geq \gamma_{dve}(T) - 1 + 2 = \gamma_2(T) + 1$. Thus $v_4 \notin S$ and so $v_4 \in W_{T'}^4$. On the other hand, as $S \cup \{v_4, y\}$ is a dve-dominating set of T , we have $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2$. By Observation 4 (Item 2), we obtain $\gamma_2(T') = \gamma_{dve}(T')$, and so by the induction hypothesis, we have $T' \in \mathcal{F}$. Therefore, $T \in \mathcal{F}$ because it can be obtained from T' by Operation $\mathcal{T}_6$.

Subcase 1.5. $\deg_T(v_3) = 2$ and v_4 has $\ell \geq 1$ children with depth 0 and $k \geq 2$ children with depth 2.

Let $T' = T - T_{v_4}$. Assume that $v_4x_1^ix_2^ix_3^i$ ($1 \leq i \leq k$) are pendant paths in T . By the choice of diametrical path, we have $\deg(x_1^i) = \deg(x_2^i) = 2$ for each $i \in \{1, \dots, k\}$. We may assume that $\{x_1^i, x_3^i \mid 1 \leq i \leq k\} \cup L_{v_4}$ is a subset of any $\gamma_2(T)$ -set. Let D be a $\gamma_2(T)$ -set. If $v_4 \in D$, then replace v_4 by v_5 in D . Therefore, we can assume that $v_4 \notin D$. It follows that $D \setminus (\{x_1^i, x_3^i \mid 1 \leq i \leq k\} \cup L_{v_4})$ is a 2d-set of T' and thus $\gamma_2(T) \geq \gamma_2(T') + 2k + \ell$. On the other hand, any $\gamma_{dve}(T')$ -set can be extended to a dve-dominating set of T by adding the vertices x_1^i, x_2^i ($1 \leq i \leq k$), which implies that $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2k$. This leads to the following contradiction,

$$\gamma_2(T) = \gamma_{dve}(T) \leq \gamma_{dve}(T') + 2k \leq \gamma_2(T') + 2k \leq \gamma_2(T) - \ell.$$

Subcase 1.6. $\deg_T(v_3) = 2$ and v_4 has $\ell \geq 2$ children with depth 0 and $k = 1$ children with depth 2.

Let $T' = T - T_{v_4}$. As in the Subcase 1.5, we can see that $\gamma_2(T) \geq \gamma_2(T') + 2 + \ell$. Also, if S is a $\gamma_{dve}(T')$ -set, then $S \cup \{v_2, v_3, v_4\}$ is a dve-dominating set of T , and thus $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 3$. Therefore,

$$\gamma_2(T) = \gamma_{dve}(T) \leq \gamma_{dve}(T') + 3 \leq \gamma_2(T') + 3 \leq \gamma_2(T) - 2 - \ell + 3,$$

a contradiction.

Subcase 1.7. $\deg_T(v_3) = 2$ and v_4 has no child with depth 0 and $k \geq 2$ children with depth 2. Let $T' = T - T_{v_4}$. Assume that $v_4x_1^ix_2^ix_3^i$ ($1 \leq i \leq k$) are pendant paths in T . By the choice of diametrical path, we have $\deg(x_1^i) = \deg(x_2^i) = 2$ for each i . Thus, T_{v_4} is isomorphic to the graph H_k . As in Subcase 1.5, we can see that $\gamma_2(T) \geq \gamma_2(T') + 2k$. On the other hand, any $\gamma_{dve}(T')$ -set can be extended to a dve-dominating set of T by adding x_1^i, x_2^i ($1 \leq i \leq k$) which implies that $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2k$. By Observation 4 (Item 2), we have $\gamma_2(T') = \gamma_{dve}(T')$. It follows from the induction hypothesis that $T' \in \mathcal{F}$. Therefore, $T \in \mathcal{F}$ because it can be obtained from T' by Operation $\mathcal{T}_3$.

Subcase 1.8. $\deg_T(v_3) = 2$ and v_4 has exactly one child of depth 0 and one child of depth 2. Let w be the leaf adjacent to v_4 and let $T' = T - T_{v_4}$. Note that T' is nontrivial, for otherwise $\gamma_2(T) \neq \gamma_{dve}(T)$. As in Subcase 1.5, we can see that $\gamma_2(T) \geq \gamma_2(T') + 3$. Now, let S be a $\gamma_{dve}(T')$ -set. If $v_5 \in S$, then $S \cup \{v_2, v_3\}$ is a dve-dominating set of T and we get the following contradiction,

$$\gamma_2(T) = \gamma_{dve}(T) \leq \gamma_{dve}(T') + 2 \leq \gamma_2(T') + 2 \leq \gamma_2(T) - 3 + 2.$$

Therefore, $v_5 \notin S$, and thus $v_5 \in W_{T'}^4$. Then, $S \cup \{v_2, v_3, v_4\}$ is a dve-dominating set of T , implying that $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 3$. By Observation 4 (item 2), we have $\gamma_2(T') = \gamma_{dve}(T')$, and thus $\gamma_2(T) = \gamma_2(T') + 3$ and $\gamma_{dve}(T) = \gamma_{dve}(T') + 3$. We deduce from the induction hypothesis that $T' \in \mathcal{T}$. Next, we need to prove that $v_5 \in W_{T'}^1$. Suppose, to the contrary, that $v_5 \notin W_{T'}^1$. Then, any $\gamma_{dve}(T'; v_5)$ -set can be extended to a dve-dominating set of T by adding v_2, v_3, v_4 , which leads to the contradiction

$$\gamma_{dve}(T) \leq \gamma_{dve}(T'; v_5) + 3 \leq \gamma_{dve}(T') - 1 + 3 = \gamma_{dve}(T') + 2.$$

Therefore, $v_5 \in W_{T'}^1$, and thus $v_5 \in W_{T'}^1 \cap W_{T'}^4$. Consequently, T can be obtained from T' by Operation $\mathcal{T}_4$, yielding $T \in \mathcal{F}$.

Subcase 1.9. $\deg_T(v_3) = 2$ and $\deg_T(v_4) = 2$. We distinguish the following situations.

- (i) $\deg_T(v_5) = 2$ or v_5 has a child with depth 0 or depth 1 or depth 2. Let $T' = T - T_{v_4}$. As in Subcase 1.5, we have $\gamma_2(T) \geq \gamma_2(T') + 2$. Now, let S be a $\gamma_{dve}(T')$ -set containing no leaf. Clearly, $S \cap N_{T'}[v_5] \neq \emptyset$, and thus $S \cup \{v_2, v_3\}$ is a dve-dominating set of T , which implies that $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2$. By Observation 4 (Item 2) and the induction hypothesis, we obtain $T' \in \mathcal{F}$. Next, we prove that $v_5 \in W_{T'}^3$. Suppose, to the contrary, that $v_5 \notin W_{T'}^3$, and let D be a $\gamma_2(T)$ -set. Without loss of generality, we may assume that $v_1, v_3, v_5 \in D$. Then, $D \setminus \{v_1, v_3\}$ is a 2d-set of T' containing v_5 , implying that $\gamma_2(T) - 2 = |D| - 2 \geq \gamma_2(T') + 1$. Therefore,

$$\gamma_{dve}(T) = \gamma_2(T) \geq \gamma_2(T') + 3 \geq \gamma_{dve}(T') + 3 \geq \gamma_{dve}(T) + 1,$$

a contradiction. Therefore, $v_5 \in W_{T'}^3$. Therefore, $T \in \mathcal{F}$ because it is obtained from T' by Operation $\mathcal{T}_5$.

- (ii) $\deg_T(v_5) \geq 3$ and all children of v_5 are of depth 3. Let $v_5y_4y_3y_2y_1$ be a path in T such that $y_4 \notin \{v_4, v_6\}$. By seeing the previous cases, we have $\deg_T(y_4) = \deg_T(y_3) = \deg_T(y_2) = 2$. Let $T' = T - T_{v_4}$. As the Subcase 1.5, we have $\gamma_2(T) \geq \gamma_2(T') + 2$. Let S be a $\gamma_{dve}(T')$ -set containing no leaf such that $|S \cap \{y_2, y_3, y_4\}|$ is as small as possible. Clearly $y_2, y_3 \in D$ and by the choice of S , we have $v_4 \notin S$. Therefore, $S \cup \{v_2, v_3\}$ is a dve-dominating set of T and thus $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2$. As in Item (i), we obtain $\gamma_2(T') = \gamma_{dve}(T')$, $T' \in \mathcal{F}$, and $v_5 \in W_{T'}^3$. Therefore, $T \in \mathcal{F}$, because it is obtained from T' by Operation $\mathcal{T}_5$.

Case 2. $\deg_T(v_2) = 3$.

Assume that $L_{v_2} = \{v_1, w\}$. We consider the following subcases.

Subcase 2.1. $\deg(v_3) \geq 3$ and v_3 has a child with depth 1, say y , with $y \neq v_2$.

Let $T' = T - T_{v_2}$. Assume that D is a $\gamma_2(T)$ -set. If $v_2 \in D$, then $D \setminus \{v_1, v_2, w\} \cup \{v_3\}$ is a 2d-set of

T' , and if $v_2 \notin D$, then $D \setminus \{v_1, w\}$ is a 2d-set of T' ; so, in either case, $\gamma_2(T) \geq \gamma_2(T') + 2$. Now, let S be a $\gamma_{dve}(T')$ -set containing no leaves. As $y, v_3 \in S$, $S \cup \{v_2\}$ is a dve-dominating set of T , and thus $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 1$, it follows that

$$\gamma_2(T) = \gamma_{dve}(T) \leq \gamma_{dve}(T') + 1 \leq \gamma_2(T') + 1 \leq \gamma_2(T) - 1,$$

a contradiction.

Subcase 2.2. $\deg(v_3) \geq 3$ and v_3 has $\ell \geq 1$ children with depth 0.

Let D be a $\gamma_2(T)$ -set. Then $\{v_1, w\} \cup L_{v_3} \subset D$, and thus $\{v_2, v_3\} \cup D - (\{v_1, w\} \cup L_{v_3})$ is a dve-dominating set of T of cardinality at most $\gamma_2(T) - 1$, a contradiction.

Subcase 2.3. $\deg(v_3) = 2$.

Let D be a $\gamma_2(T)$ -set D . Clearly, D contains v_1, w . Moreover, to 2-dominate v_3 , we must have $\{v_2, v_3\} \cap D \neq \emptyset$, say $v_3 \in D$. However, then, $\{v_2\} \cup D - \{v_1, w\}$ is a dve-dominating set of T of cardinality at most $\gamma_2(T) - 1$, a contradiction. $\square$

4. Trees T with $\gamma_{dve}(T) = \gamma_t(T)$

In this section, we provide a constructive characterization of all trees T with $\gamma_{dve}(T) = \gamma_t(T)$. For this purpose, we define the family $\mathcal{T}$ of unlabeled trees T that can be obtained from a sequence $T_1, T_2, \dots, T_k$ ($k \geq 1$) of trees, such that $T_1 \in \{P_2, P_3, P_4\}$ and $T = T_k$. If $k \geq 2$, then T_{i+1} can be obtained recursively from T_i by one of the following operations.

Operation $\mathcal{O}_1$: If $u \in V(T_i)$ is a support vertex or a non-leaf vertex adjacent to an end support vertex, then $\mathcal{O}_1$ adds a vertex x and an edge ux to obtain T_{i+1} .

Operation $\mathcal{O}_2$: If $u \in V(T_i)$ and there is a path ux_1x_2 in T_i such that $\deg_{T_i}(x_1) = 2$ and $\deg_{T_i}(x_2) = 1$, then $\mathcal{O}_2$ adds a path $P_2 = y_1y_2$ and an edge uy_1 to obtain T_{i+1} .

Operation $\mathcal{O}_3$: If $u \in W_{T_i}^2$, then $\mathcal{O}_3$ adds a path $P_4 = x_1x_2x_3x_4$ and an edge ux_1 to obtain T_{i+1} .

Operation $\mathcal{O}_4$: If $u \in V(T_i)$ and there is a path ux_1x_2 in T_i such that $\deg_{T_i}(x_1) = 2$ and $\deg_{T_i}(x_2) = 1$, then $\mathcal{O}_4$ adds a path $P_3 = y_1y_2y_3$ and an edge uy_1 to obtain T_{i+1} .

Operation $\mathcal{O}_5$: If $u \in W_{T_i}^1$ and there is a path $ux_1x_2x_3$ in T_i such that $\deg_{T_i}(x_1) = \deg_{T_i}(x_2) = 2$ and $\deg_{T_i}(x_3) = 1$, then $\mathcal{O}_5$ adds a path $y_1y_2y_3$ and an edge uy_1 to obtain T_{i+1} .

Operation $\mathcal{O}_6$: If $u \in V(T_i)$ is a leaf and there is a path $ux_1x_2x_3$ in T_i such that $\deg_{T_i}(x_3) = 1$ and no $\gamma_{dve}(T_i)$ -set contains both u and x_1 , then $\mathcal{O}_6$ adds a vertex x and an edge ux to obtain T_{i+1} .

Operation $\mathcal{O}_7$: If $u \in V(T_i)$ is a support vertex with $v \in L_u$ and there is a path uy_2y_1 such that $\deg_{T_i}(y_2) = 2$ and $\deg_{T_i}(y_1) = 1$, then $\mathcal{O}_7$ adds a new vertex w and a path $P_4 = x_1x_2x_3x_4$ and edges y_1w, vx_4 to obtain T_{i+1} .

Lemma 2. If T_i is a tree with $\gamma_{dve}(T_i) = \gamma_t(T_i)$ and T_{i+1} is a tree obtained from T_i by one of the operations $\mathcal{O}_j$ ($1 \leq j \leq 7$), then $\gamma_{dve}(T_{i+1}) = \gamma_t(T_{i+1})$.

Proof. If T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_j$, where $j \in \{1, \dots, 5\}$, then the result follows by Propositions 3, 4, 5, 6, and 7.

Let T_{i+1} be obtained from T_i by Operation $\mathcal{O}_6$. Clearly, any $\gamma_t(T_i)$ -set can be extended to a td-set of T by adding u which implies that $\gamma_t(T_{i+1}) \leq \gamma_t(T_i) + 1$. Moreover, if D is a $\gamma_{dve}(T_{i+1})$ -set containing no leaf, then clearly $u, x_1 \in D$ and D is a double ve-dominating set of T_i . As, by the assumption in Operation $\mathcal{O}_6$, no $\gamma_{dve}(T_i)$ -set contains both u and x_1 , we deduce that $|D| > \gamma_{dve}(T_i)$, that is, $\gamma_{dve}(T_{i+1}) \geq \gamma_{dve}(T_i) + 1$. Now, by Observation 3 (Item 1), we obtain $\gamma_{dve}(T_{i+1}) = \gamma_t(T_{i+1})$.

Assume now that T_{i+1} is obtained from T_i by Operation $\mathcal{O}_7$. As any $\gamma_t(T_i)$ -set can be extended to a td-set of T by adding x_2, x_3, y_1 , we have $\gamma_t(T_{i+1}) \leq \gamma_t(T_i) + 3$. Let D be a $\gamma_{dve}(T_{i+1})$ -set containing no leaf. Then, $y_1, y_2, x_2, x_3 \in D$, and to dve-dominate the edge vx_4 , we may assume that $u \in D$.

Therefore, $D - \{y_1, x_2, x_3\}$ is a dve-dominating set of T_i , implying that $\gamma_{dve}(T_{i+1}) \geq \gamma_{dve}(T_i) + 3$. By Observation 3 (Item 1), we obtain $\gamma_{dve}(T_{i+1}) = \gamma_t(T_{i+1})$. $\square$

Now we are ready to state the main theorem of this section.

Theorem 3. Let T be a tree of order $n \geq 2$. Then $\gamma_{dve}(T) = \gamma_t(T)$ if and only if $T \in \mathcal{T}$.

Proof. Let $T \in \mathcal{T}$. Then there exists a sequence of trees $T_1, T_2, \dots, T_k$ ($k \geq 1$), such that $T_1 \in \{P_2, P_3, P_4\}$ and $T = T_k$. Moreover, if $k \geq 2$, then T_{i+1} can be obtained from T_i by one of the aforementioned operations. We proceed by induction on the number of operations used to construct T . If $k = 1$, then $T \in \{P_2, P_3, P_4\}$ and clearly $\gamma_{dve}(T) = \gamma_t(T)$. Assume now that $k \geq 2$ and that the result holds for each tree, $T \in \mathcal{T}$, that can be obtained from a sequence of length $k - 1$, and let $T' = T_{k-1}$. By the induction hypothesis, $\gamma_{dve}(T') = \gamma_t(T')$. As $T = T_k$ is obtained from T' by using one of the Operations $\mathcal{O}_j$, where $j \in \{1, 2, \dots, 7\}$, we conclude from the Lemma 2 that $\gamma_{dve}(T) = \gamma_t(T)$.

Conversely, let $\gamma_{dve}(T) = \gamma_t(T)$. We proceed by induction on n . If $n \leq 3$, then $T \in \{P_2, P_3\}$ and clearly $T \in \mathcal{T}$. Let $n \geq 4$ and let the result hold for every tree T' of order less than n , satisfying $\gamma_{dve}(T') = \gamma_t(T')$. Let T be a tree of order n with $\gamma_{dve}(T) = \gamma_t(T)$. If $\text{diam}(T) = 2$, then T is a star, and it can be obtained from P_3 by frequently use of Operation $\mathcal{O}_1$, and so $T \in \mathcal{T}$. If $\text{diam}(T) = 3$, then T is a double star $DS_{p,q}$ ($q \geq p \geq 1$). If $q = 1$, then $T = P_4 \in \mathcal{T}$. If $q > 1$, then T can be obtained from P_4 by frequently use of $\mathcal{O}_1$, and thus $T \in \mathcal{T}$. Henceforth, we assume that $\text{diam}(T) \geq 4$. Let $v_1 v_2 \dots v_k$ ($k \geq 5$) be a diametral path in T such that $\deg_T(v_2)$ is as large as possible. Root T at v_k .

If $\deg_T(v_2) \geq 3$, then let $T' = T - \{v_1\}$. By Proposition 3 and by the induction hypothesis, we have $T' \in \mathcal{T}$. Now, T can be obtained from T' by operation $\mathcal{O}_1$ and so $T \in \mathcal{T}$. Therefore, we assume that $\deg_T(v_2) = 2$. By the choice of diametrical path, we may assume that any child of v_3 with depth one is of degree two. Note that using an argument similar to the previous one, we may assume that T has no strong support vertex. We consider the following cases.

Case 1. $\deg_T(v_3) \geq 3$.

If v_3 has at least one child with depth 0, say x , then let $T' = T - \{x\}$. By Proposition 3 and by the induction hypothesis, we have $T' \in \mathcal{T}$. It follows that $T \in \mathcal{T}$ because it is obtained from T' by operation $\mathcal{O}_1$. Therefore, we assume that v_3 is not a support vertex. Then, v_3 has at least one children with depth 1, say z_2 , different from v_2 . Let $v_3 z_2 z_1$ be a pendant path in T . Suppose $T' = T - \{v_1, v_2\}$. By Proposition 4 and by the induction hypothesis, we have $T' \in \mathcal{T}$. Now, as T is obtained from T' by operation $\mathcal{O}_{2,}$, we obtain $T \in \mathcal{T}$.

Case 2. $\deg_T(v_3) = 2$.

First, let $\deg_T(v_4) \geq 3$. If v_4 has a child y_2 with depth 1, then let $v_4 y_2 y_1$ be a pendant path in T , and let $T' = T - T_{v_3}$. By Proposition 6 and the inductive hypothesis on T' , we have $T' \in \mathcal{T}$. It follows that $T \in \mathcal{T}$, as it can be obtained from T' by operation $\mathcal{O}_4$.

Assume now that v_4 has a child z_3 with depth 2, different from v_3 . According to the previous case, the path $v_4 z_3 z_2 z_1$, having z_1 as a leaf, is a pendant path in T . Let $T' = T - T_{v_3}$. By Proposition 7 and Observation 3 (item 2), we have $\gamma_{dve}(T') = \gamma_t(T')$, $\gamma_t(T) = \gamma_t(T') + 2$ and $\gamma_{dve}(T) = \gamma_{dve}(T') + 2$. By the induction hypothesis on T' , we have $T' \in \mathcal{T}$. We prove now that $v_4 \in W_{T'}^1$. Suppose, to the contrary, that $v_4 \notin W_{T'}^1$. Then $\gamma_{dve}(T'; v_4) \leq \gamma_{dve}(T') - 1$ and any $\gamma_{dve}(T'; v_4)$ -set can be extended to a dve-dominating set of T by adding v_2, v_3 implying that $\gamma_{dve}(T) \leq \gamma_{dve}(T'; v_4) + 2 \leq \gamma_{dve}(T') - 1 + 2 = \gamma_{dve}(T') + 1$, a contradiction. Therefore, $v_4 \in W_{T'}^1$, and therefore $T \in \mathcal{T}$ because it can be obtained from T' by operation $\mathcal{O}_5$.

Finally, suppose v_4 is a support vertex, and let $T' = T - \{v_1\}$. Clearly, for any $\gamma_t(T)$ -set D containing no leaf, we have $v_2, v_3, v_4 \in D$, and thus $D \setminus \{v_2\}$ is a td-set of T' , implying that $\gamma_t(T) \geq \gamma_t(T') + 1$. On the other hand, any $\gamma_{dve}(T')$ -set containing no leaves, contains v_3, v_4 and thus such a set plus v_2 is a dve-dominating set of T , implying that $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 1$. By Observation 3 (Item 2), we have $\gamma_{dve}(T') = \gamma_t(T')$. By the induction hypothesis, we have $T' \in \mathcal{T}$. Note that Observation 3 (Item 2) also implies that $\gamma_{dve}(T) = \gamma_{dve}(T') + 1$. This means that no $\gamma_{dve}(T')$ -set S contains both v_2

and v_3 ; otherwise, S dve-dominates T too, which contradicts $\gamma_{dve}(T) = \gamma_{dve}(T') + 1$. Now, as T can be obtained from T' by operation $\mathcal{O}_6$, we conclude that $T \in \mathcal{T}$.

From now on, we can assume that $\deg_T(v_4) = 2$. We distinguish the following subcases.

Subcase 2.1. $\deg_T(v_5) \geq 3$.

Let $T' = T - T_{v_4}$. By Proposition 5, we have $\gamma_t(T) = \gamma_t(T') + 2$. Let S be a $\gamma_{dve}(T')$ -set containing no leaves. If v_5 is a support vertex or is adjacent to a support vertex, then clearly $|S \cap N_{T'}[v_5]| \geq 1$. If there is a path $v_5z_3z_2z_1$ such that $z_3 \notin \{v_4, v_6\}$ and $\deg(z_1) = 1$, then $z_3 \in S$, and thus $|S \cap N_{T'}[v_5]| \geq 1$. If there is a path $v_5z_4z_3z_2z_1$ such that $z_4 \notin \{v_4, v_6\}$ and $\deg(z_1) = 1$, then, as z_1 belongs to diametral path of T , each of z_4, z_3 and z_2 has degree two, and thus $z_2, z_3 \in S$. However, to ve-dominate the edge v_5z_4 we need that $|S \cap N_{T'}[v_5]| \geq 1$. In either case, we deduce that $v_5 \in W_{T'}^2$. In this case, $S \cup \{v_2, v_3\}$ is a dve-dominating set of T , implying that $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2$. By Observation 3 (Item 2), we obtain $\gamma_t(T') = \gamma_{dve}(T')$, and it follows from the induction hypothesis that $T' \in \mathcal{T}$. Therefore, $T \in \mathcal{T}$ because it is obtained from T' by Operation $\mathcal{O}_3$.

Subcase 2.1. $\deg_T(v_5) = 2$.

We note that if v_6 is a leaf, then $T = P_6$ that belongs to $\mathcal{T}$, as it can be obtained from a path P_2 (whose vertices are in $W_{P_2}^2$) by using Operation $\mathcal{O}_3$. Therefore, we assume that $\deg_T(v_6) \geq 2$. We consider some further situations.

- (a) v_6 has a child with depth 0 or 1 or $\deg_T(v_6) = 2$.

Let $T' = T - T_{v_4}$. By Proposition 5, $\gamma_t(T) = \gamma_t(T') + 2$ and $\gamma_{dve}(T) \geq \gamma_{dve}(T') + 2$. On the other hand, any $\gamma_{dve}(T')$ -set containing no leaves can be extended to a dve-dominating set of T by adding two vertices, v_2 and v_3 , which implies that $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2$. By Observation 3 (Item 2), we obtain $\gamma_t(T') = \gamma_{dve}(T')$, and it follows from the induction hypothesis that $T' \in \mathcal{T}$. Also, by assumption, it is clear that $v_5 \in W_{T'}^2$, and therefore $T \in \mathcal{T}$, as it can be obtained from T' by Operation $\mathcal{O}_3$.

- (b) v_6 has a child with depth 3.

Let y_4 be a child of v_6 with depth 3 such that $v_6y_4y_3y_2y_1$ is a path in T . As T has no strong support vertex, we have $\deg(y_2) = 2$. If $\deg(y_3) \geq 3$, then by a similar argument to that used in Case 1, we can easily see that $T \in \mathcal{T}$. Therefore, we may assume that $\deg(y_3) = 2$. If $\deg(y_4) \geq 3$, then, again, by using a similar argument as in the beginning of Case 2, we have $T \in \mathcal{T}$. Thus, we may assume that $\deg(y_4) = 2$. Now, let $T' = T - T_{y_4}$. By Proposition 5, we have $\gamma_t(T) = \gamma_t(T') + 2$. Let S be a $\gamma_{dve}(T')$ -set containing no leaves. Then, $v_2, v_3 \in S$, and to dve-dominate the edge v_4v_5 , we must have $S \cap \{v_4, v_5, v_6\} \neq \emptyset$, say $v_6 \in S$. Hence $v_6 \in W_{T'}^2$. Also, $S \cup \{y_2, y_3\}$ is a dve-dominating set of T , and thus $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2$. By Observation 3 (Item 2), we obtain $\gamma_t(T') = \gamma_{dve}(T')$, and by the induction hypothesis, we obtain $T' \in \mathcal{T}$. Therefore, $T \in \mathcal{T}$, as it is obtained from T' by Operation $\mathcal{O}_3$.

- (c) v_6 has a child with depth 4, different from v_5 .

Let y_5 be a child of v_6 with depth 4 such that $v_6y_5y_4y_3y_2y_1$ is a path in T . Note that, as y_1 belongs to a diametral path of T , and seeing the above situations, we can assume that $v_6y_5y_4y_3y_2y_1$ is a pendant path. Let $T' = T - T_{y_4}$. By Proposition 5, $\gamma_t(T) = \gamma_t(T') + 2$. Let S be a $\gamma_{dve}(T')$ -set containing no leaves. As above, $v_2, v_3 \in S$, and we may assume that $v_6 \in S$ to double ve-dominate the edge v_4v_5 . Thus $y_5 \in W_{T'}^2$, and $S \cup \{y_2, y_3\}$ is a dve-dominating set of T , implying that $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2$. By Observation 3 (Item 2), we obtain $\gamma_t(T') = \gamma_{dve}(T')$, and thus $T' \in \mathcal{T}$. Now, as T is obtained from T' by Operation $\mathcal{O}_3$, we deduce that $T \in \mathcal{T}$.

- (d) v_6 has a child with depth 2.

Considering above situations, we may assume that any child of v_6 other than v_5 has depth 2. First, let $\deg_T(v_6) = 3$. Let y_3 be the child of v_6 with depth 2 such that $v_6y_3y_2y_1$ is a path in T . As T has no strong support vertex we have $\deg_T(y_2) = 2$. Also, if $\deg_T(y_3) \geq 3$, then, as in the Case 1, we can see that $T \in \mathcal{T}$. Therefore, we may assume that $\deg_T(y_3) = 2$. Let T' be the tree obtained from T by removing the set of vertices $\{y_1, v_1, v_2, v_3, v_4\}$. If D is a $\gamma_t(T)$ -set containing no leaf,

then $y_3, y_2, v_2, v_3 \in D$ and to dominate v_5 , we may assume that $v_6 \in D$. Thus, $D - \{v_2, v_3, y_1\}$ is a td-set of T' , and thus $\gamma_t(T) \geq \gamma_t(T') + 3$. On the other hand, any $\gamma_{dve}(T')$ -set containing no leaf contains y_3, v_6 , and thus can be extended to a dve-dominating set of T by adding v_2, v_3, y_2 . Therefore, $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 3$. By Observation 3 (Item 2), we have $\gamma_t(T') = \gamma_{dve}(T')$, and thus by our inductive hypothesis, $T' \in \mathcal{T}$. As T can be obtained from T' , by using Operation $\mathcal{O}_{7,,}$, we deduce that $T \in \mathcal{T}$.

□

Author Contributions: F.M. and S.M.S. contributed to the supervision, validation, project administration and formal analysis. W.F. M.C. and R.K. contributed to the methodology and investigation and wrote the initial draft of the paper, which was investigated and approved by M.C., M.S., and the final draft was written by S.M.S. and M.S.

Funding: This work is supported by the Scientific Research Fund of SiChuan Provincial Department of Science and Technology (2019YFS0067), the Open Project Program of the State Key Lab of CAD & CG under Grant A1922 (Zhejiang University), and the Scientific Research Fund of Key Laboratory of Pattern Recognition and Intelligent Information Processing of Chengdu University (MSSB-2019-03).

Conflicts of Interest: The authors declare no conflict of interest.

References

- Haynes, T.W.; Hedetniemi, S.T.; Slater, P.J. *Fundamentals of Domination in Graphs*; Marcel Dekker, Inc.: New York, NY, USA, 1998.
- Henning, M.A.; Yeo, A. *Total Domination in Graphs (Springer Monographs in Mathematics)*; Springer: Berlin, Germany, 2013. ISBN (Print) 978-1-4614-6524-9, ISBN (Online) 978-1-4614-6525-6.
- Peters, J.W. Theoretical and Algorithmic Results on Domination and Connectivity. Ph.D. Thesis, Clemson University, Clemson, SC, USA, 1986.
- Ahangar, H.A.; Chellali, M.; Sheikholeslami, S.M.; Soroudi, M.; Volkmann, L. Total vertex-edge domination in trees. *Ars Combin* 2019, submitted.
- Boutrig, R.; Chellali, M.; Haynes, T.W.; Hedetniemi, S.T. Vertex-edge domination in graphs. *Aequat. Math.* **2016**, *90*, 355–366. [[CrossRef](#)]
- Chellali, M.; Haynes, T.W.; Hedetniemi, S.T.; Lewis, T.M. On ve-degrees and ev-degrees in graphs. *Discrete Math.* **2017**, *340*, 31–38. [[CrossRef](#)]
- Krishnakumari, B.; Venkatakrishnan, Y.B.; Krzywkowski, M. Bounds on the vertex-edge domination number of a tree. *C. R. Acad. Sci. Paris, Ser. I* **2014**, *352*, 363–366. [[CrossRef](#)]
- Lewis, J.R. Vertex-Edge and Edge-Vertex Domination in Graphs. Ph.D. Thesis, Clemson University, Clemson, SC, USA, 2007.
- Lewis, J.R.; Hedetniemi, S.T.; Haynes, T.W.; Fricke, G.H. Vertex-edge domination. *Util. Math.* **2010**, *81*, 193–213.
- Zyliński, P. Vertex-edge domination in graphs. *Aequat. Math.* **2019**, *93*, 735–742. [[CrossRef](#)]
- Krishnakumari, B.; Chellali, M.; Venkatakrishnan, Y.B.; Double vertex-edge domination. *Discrete Math. Algorithms Appl.* **2017**, *9*, 1750045. [[CrossRef](#)]

© 2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).