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Hybrid Contractions on Branciari Type Distance Spaces

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Abstract: In this manuscript, we consider some hybrid contractions that merge linear and nonlinear contractions in the abstract spaces induced by the Branciari distance and the Branciari b-distance. More precisely, we introduce the notion of a (p,c)-weight type ψ-contraction in the setting of Branciari distance spaces and the concept of a (p,c)-weight type contraction in Branciari b-distance spaces. We investigate the existence of a fixed point of such operators in Branciari type distance spaces and illustrate some examples to show that the presented results are genuine in the literature.

Keywords: Branciari type metric space; hybrid contraction; (p,c)-weight type ψ-contraction; (p,c)-weight type contraction; fixed point

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1. Introduction

The notion of metric spaces has many generalizations, in which each puts in the limelight the importance of the conditions that define them. Most of the generalizations of metric are obtained by relaxing one of its three axioms: self-distance, symmetry and the triangle inequality. In the literature, there are several extensions of metric spaces, such as symmetric, quasi-metric, fuzzy metric, cone-metric, G-metric, b-metric and so on. In this manuscript, we prefer to investigate hybrid contractions in the abstract spaces induced by Branciari distance. Indeed, Branciari distance [1] (respectively, Branciari b-distance [2]) is obtained by replacing the triangle inequality axiom with the quadrilateral inequality (quadrilateral inequality multiplied by a constant s) axiom in the definition of a standard metric. Despite the apparent similarity between the definitions of the standard metric and Branciari distance (respectively, Branciari b-distance), the corresponding topologies are quite different. Therefore, we name this abstract space as Branciari distance space instead of Branciari metric space. In addition, in the literature, this space has been called a rectangular metric space or a generalized metric space. We assert that the abstract space is described perfectly by Branciari distance spaces. Furthermore, despite the appearance purpose, Branciari distance is neither a generalization nor an extension of the standard metric space. On the other hand, interesting fixed point features have been appointed in these frameworks, see e.g., [3–31].

In this manuscript, we aim to give two hybrid contractions, namely the (p,c)-weight type ψ-contraction and the (p,c)-weight type contraction in the setting of two abstract constructions:

Branciari distance spaces and Branciari b-distance spaces. We obtain the existence of a fixed point for these hybrid contractions and we consider examples to support our obtained results.

Definition 1 ([1])**.** *Let $\mathcal{X} \neq \emptyset$ and $d \colon \mathcal{X} \times \mathcal{X} \to [0, \infty)$ a function which fulfills the next assumptions for each $s, t \in \mathcal{X}$ and all distinct $u, v \in \mathcal{X}$ each of which is different from s and t*

$$\begin{array}{ll} (b1) & d(s,t) = 0 \text{ if and only if } s = t; \\ (b2) & d(s,t) = d(t,s); \\ (b3) & d(s,t) \leq d(s,u) + d(u,v) + d(v,t) \text{(the quadrilateral inequality)}. \end{array}$$

Then d is a Branciari distance (a generalized metric). The pair $(\mathcal{X}, d)$ is called a Branciari distance space (BDS).

Throughout the paper, the couple letters $(\mathcal{X}, d)$ refers to a Branciari distance space.

Herein after, the symbol $\mathbb{R}_0^+$ represents the set of non-negative real numbers. Further, the symbol $\mathbb{N}_0$ denotes the non-negative integers.

In what follows, we recollect the important tools of topology in the framework of Branciari distance spaces.

Definition 2.

1. *$\{\varkappa_n\}$ in $(\mathcal{X}, d)$ is convergent to $\varkappa$ if and only if $d(\varkappa_n, \varkappa) \to 0$ as $n \to \infty$.*
2. *$\{\varkappa_n\}$ in $(\mathcal{X}, d)$ is Cauchy if and only if for each $\varepsilon > 0$ we may find $N(\varepsilon)$ such that $d(\varkappa_n, \varkappa_m) < \varepsilon$ for all $n > m > N(\varepsilon)$.*
3. *A Branciari distance space $(\mathcal{X}, d)$ is complete if each Cauchy (fundamental) sequence in $(\mathcal{X}, d)$ is convergent.*
4. *A mapping $T \colon (\mathcal{X}, d) \to (\mathcal{X}, d)$ is continuous if for any sequence $\{\varkappa_n\}$ in $\mathcal{X}$ such that $d(\varkappa_n, \varkappa) \to 0$ as $n \to \infty$, we have $d(T\varkappa_n, T\varkappa) \to 0$ as $n \to \infty$.*

Branciari introduced the open ball, the closed ball (and hence the corresponding topology) which are different than that of metric spaces. In addition, the structures of these two abstract notions are quite different from each other. Indeed, the following interesting properties of the Branciari distance space are the main motivation why we consider our new hybrid contractions in these abstract spaces

1. The limit of a sequence in a Branciari distance space is not necessarily unique.
2. A convergent sequence in a Branciari distance space may not be a Cauchy sequence.
3. A Branciari distance space may not be continuous.
4. The topologies of a Branciari distance space and a metric space are incompatible.

For more details, see e.g., [13,26,28–31].

Next, we provide an example of a genuine BDS.

Example 1 ([28])**.** *Let $\mathcal{X} = \{(0,0)\} \cup ((0,1] \times [0,1])$. Define a function $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_0^+$ by*

$$\begin{array}{rll} d((\varkappa,\omega),(\varkappa,\omega)) & = 0, & \\ d((0,0),(\varkappa,0)) = d((\varkappa,0),(0,0)) & = \varkappa, & \varkappa \in (0,1], \\ d((\varkappa,0),(\omega,\eta)) = d((\omega,\eta),(\varkappa,0)) & = |\varkappa - \omega| + \eta, & \varkappa,\omega,\eta \in (0,1] \\ d((\varkappa,\theta),(\omega,\eta)) & = 3, & \text{otherwise.} \end{array}$$

It is evident that $(\mathcal{X}, d)$ forms a Branciari distance space.

Proposition 1 ([19])**.** *Suppose that $\{\varkappa_n\}$ is a Cauchy sequence in a BDS $(\mathcal{X}, d)$ with $\lim_{n\to\infty} d(\varkappa_n, u) = 0$, where $u \in \mathcal{X}$. Then $\lim_{n\to\infty} d(\varkappa_n, z) = d(u, z)$ for all $z \in \mathcal{X}$. In particular, the sequence $\{\varkappa_n\}$ does not converge to z if $z \neq u$.*

Lemma 1 (See e.g., [15]). *Let $(\mathcal{X}, d)$ be a BDS and let $\{\varkappa_n\}$ be a Cauchy sequence in $\mathcal{X}$ such that $\varkappa_m \neq \varkappa_n$ whenever $m \neq n$. Then the sequence $\{\varkappa_n\}$ converges to at most one point.*

2. Results on Branciari Distance Spaces

We start this section by giving a definition of the set Ψ of auxiliary functions, known as (c)-comparison function, (see e.g., [8,24]) that shall be used in the main result.

$$c := \{\psi \colon [0, \infty) \to [0, \infty) : \psi \text{ satisfies } (\Psi_1) - (\Psi_2)\},$$

where

(Ψ_1) ψ is nondecreasing;
(Ψ_2) there are $i_0 \in \mathbb{N}$ and $\delta \in (0, 1)$ and a convergent series $\sum_{i=1}^{\infty} v_i$ such that $v_i \geq 0$ and

$$\psi^{i+1}(t) \leq \delta \psi^i(t) + v_i,$$

for $i \geq i_0$ and $t \geq 0$.

Lemma 2 ([24]). *If $\psi \in \Psi$, then*

(i) $\psi(t) < t$, *for any* $t \in \mathbb{R}^+$;
(ii) ψ *is continuous at* 0;
(iii) $(\psi^n(t))_{n \in \mathbb{N}}$ *converges to* 0 *as* $n \to \infty$ *for* $t \geq 0$;
(iv) *the series* $\sum_{k=1}^{\infty} \psi^k(t)$ *is convergent for* $t \geq 0$.

First, by utilizing a (c)-comparison function, we introduce a new type contraction that combines both the linear and nonlinear type contractions in the context of Branciari distance spaces.

Definition 3. *A self-mapping T on $(\mathcal{X}, d)$ is said to be a (p, c)-weight type ψ-contraction, if there exists $\psi \in \Psi$ so that the following inequality holds for any $s, t \in \mathcal{X}$ which are not fixed points of T*

$$d(Ts, Tt) \leq \psi(\mathcal{W}_T^{p,c}(s, t)), \tag{1}$$

where $p \geq 0$, $c = (c_1, c_2, c_3)$, and c_1, c_2, and c_3 are positive numbers such that $c_1 + c_2 + c_3 = 1$, and

$$\mathcal{W}_T^{p,c}(s, t) = \begin{cases} (c_1 d^p(s, t) + c_2 d^p(s, Ts) + c_3 d^p(t, Tt))^{\frac{1}{p}}, & \text{if } p > 0 \\ d^{c_1}(s, t) d^{c_2}(s, Ts) d^{c_3}(t, Tt), & \text{if } p = 0. \end{cases}$$

Note that such contractions, defined in Definition 3, were initiated in the recent paper [21] in the setting of b-metric spaces.

Theorem 1. *Let $(\mathcal{X}, d)$ be a complete BDS and $T \colon \mathcal{X} \to \mathcal{X}$ be a (p, c)-weight type ψ-contraction mapping. Then the mapping T possesses a fixed point $\varkappa^*$.*

Proof. Starting with $\varkappa \in \mathcal{X}$, put $\varkappa_0 = \varkappa$ and define $\varkappa_{n+1} = T\varkappa_n$. Without loss of generality, we may assume that for any $n \in \mathbb{N}_0$, $\varkappa_n \neq \varkappa_{n+1}$. Indeed, in case of $\varkappa_{n_0} = \varkappa_{n_0+1} = T\varkappa_{n_0}$ for some $n_0 \in \mathbb{N}$, then $\varkappa_{n_0}$ is a fixed point of T that finalize the proof.

Let us take into consideration the situation in which $p > 0$. The proof of this situation consists of three steps.

First step: We shall indicate that

$$\lim_{n \to \infty} d(\varkappa_n, \varkappa_{n+1}) = 0 \text{ and } \lim_{n \to \infty} d(\varkappa_n, \varkappa_{n+2}) = 0.$$

Using the contraction condition, we get, for $n \geq 1$,

$$\begin{aligned} d(\varkappa_n, \varkappa_{n+1}) &\leq \psi(\mathcal{W}_T^{p,c}(\varkappa_n, \varkappa_{n+1})) \\ &= \psi\left([c_1 d^p(\varkappa_{n-1}, \varkappa_n) + c_2 d^p(\varkappa_n, \varkappa_{n+1}) + c_3 d^p(\varkappa_{n-1}, \varkappa_n)]^{\frac{1}{p}} \right). \end{aligned}$$

If $d(\varkappa_{n_0-1}, \varkappa_{n_0}) \leq d(\varkappa_{n_0}, \varkappa_{n_0+1})$ for some $n_0 \in \mathbb{N}$, then we get

$$\begin{aligned} d(\varkappa_{n_0}, \varkappa_{n_0+1}) &\leq \psi\left(\left[c_1 d^p(\varkappa_{n_0}, \varkappa_{n_0+1}) + c_2 d^p(\varkappa_{n_0}, \varkappa_{n_0+1}) + c_3 d^p(\varkappa_{n_0-1}, \varkappa_{n_0})\right]^{\frac{1}{p}} \right) \\ &= \psi\left(\left[(c_1 + c_2 + c_3) d^p(\varkappa_{n_0+1}, \varkappa_{n_0})\right]^{\frac{1}{p}} \right) \\ &= \psi\left(d(\varkappa_{n_0+1}, \varkappa_{n_0})\right) \\ &< d(\varkappa_{n_0+1}, \varkappa_{n_0}), \end{aligned}$$

a contradiction. Consequently, we find that $d(\varkappa_n, \varkappa_{n+1}) \leq d(\varkappa_{n-1}, \varkappa_n)$, $n \in \mathbb{N}$, and further,

$$d(\varkappa_n, \varkappa_{n+1}) \leq \psi(d(\varkappa_{n-1}, \varkappa_n)) \leq d(\varkappa_{n-1}, \varkappa_n). \tag{2}$$

In addition, we find that

$$d(\varkappa_n, \varkappa_{n+1}) \leq \psi^n(d(\varkappa_0, \varkappa_1)).$$

On account of Lemma 2,

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \tag{3}$$

We prove that

$$\lim_{n \to \infty} d(\varkappa_n, \varkappa_{n+2}) = 0. \tag{4}$$

Regarding (1), we find that

$$d(\varkappa_n, \varkappa_{n+2}) = d(Tx_{n-1}, Tx_{n+1}) \leq \psi(\mathcal{W}_T^{p,c}(\varkappa_{n-1}, \varkappa_{n+1})), \tag{5}$$

for all $n \geq 1$, where

$$\mathcal{W}_T^{p,c}(\varkappa_{n-1}, \varkappa_{n+1}) = (c_1 d^p(\varkappa_{n-1}, \varkappa_{n+1}) + c_2 d^p(\varkappa_{n-1}, T\varkappa_{n-1}) + c_3 d^p(\varkappa_{n+1}, T\varkappa_{n+1}))^{\frac{1}{p}}. \tag{6}$$

Now, we shall consider the possible cases. If

$$\max\{d(\varkappa_{n-1}, \varkappa_{n+1}), d(\varkappa_{n-1}, \varkappa_n), d(\varkappa_n, \varkappa_{n+2})\} = d(\varkappa_n, \varkappa_{n+2}).$$

as in the above case, we get $d(\varkappa_n, \varkappa_{n+2}) \leq \psi(d(\varkappa_n, \varkappa_{n+2})) < d(\varkappa_n, \varkappa_{n+2})$, a contradiction. Thus, we have the following estimation

$$\begin{aligned} \mathcal{W}_T^{p,c}(\varkappa_{n-1}, \varkappa_{n+1}) &\leq [(c_1 + c_2 + c_3) \max\{d^p(\varkappa_{n-1}, \varkappa_{n+1}), d^p(\varkappa_{n-1}, \varkappa_n)\}]^{\frac{1}{p}}, \\ &= [\max\{d^p(\varkappa_{n-1}, \varkappa_{n+1}), d^p(\varkappa_{n-1}, \varkappa_n)\}]^{\frac{1}{p}}. \end{aligned}$$

Take $a_n = d(\varkappa_n, \varkappa_{n+2})$ and $b_n = d(\varkappa_n, \varkappa_{n+1})$. Thus, from (6) and (5)

$$a_n = d(\varkappa_n, \varkappa_{n+2}) \leq \psi(M(\varkappa_{n-1}, \varkappa_{n+1})) = \psi([\max\{a_{n-1}^p, b_{n-1}^p\}]^{\frac{1}{p}}) \quad \text{for all } n \in \mathbb{N}. \tag{7}$$

Again, by (2)

$$b_n \leq b_{n-1} \leq [\max\{a_{n-1}^p, b_{n-1}^p\}]^{\frac{1}{p}}.$$

Therefore,

$$\max\{a_n, b_n\} \leq [\max\{a_{n-1}^p, b_{n-1}^p\}]^{\frac{1}{p}} = \max\{a_{n-1}, b_{n-1}\} \quad \text{for all } \; n \in \mathbb{N}.$$

The sequence $\{\max\{a_n, b_n\}\}$ is monotone nonincreasing, so it converges to some $t \geq 0$. Taking the limit as $n \to \infty$, we get

$$t = \limsup_{n\to\infty} \max\{a_n, b_n\} \leq \limsup_{n\to\infty} \psi(\max\{a_{n-1}, b_{n-1}\}) \leq \psi(\lim_{n\to\infty} \max\{a_{n-1}, b_{n-1}\}) = \psi(t) < t,$$

which is a contradiction; that is, (4) is proved.

Second step: We aim to indicate that the sequence $\{x_n\}$ is not periodic; that is,

$$\varkappa_n \neq \varkappa_m \quad \text{for all } n \neq m.$$

We shall use the method of *Reductio ad Absurdum*. We presume that $\varkappa_n = \varkappa_m$ for some $m, n \in \mathbb{N}$ with $m \neq n$. Regarding that $d(\varkappa_p, \varkappa_{p+1}) > 0$ for each $p \in \mathbb{N}$, without loss of generality, we may assume that $m > n + 1$.

By employing the contraction inequality, we find

$$\begin{aligned} d(\varkappa_n, \varkappa_{n+1}) = \quad & d(\varkappa_n, T\varkappa_n) = d(\varkappa_m, T\varkappa_m) \\ & = d(T\varkappa_{m-1}, T\varkappa_m) \leq \psi(\mathcal{W}_T^{p,c}(\varkappa_{m-1}, \varkappa_m)), \end{aligned} \tag{8}$$

where

$$\begin{aligned} \mathcal{W}_T^{p,c}(\varkappa_{m-1}, \varkappa_m) = \quad & [c_1 d^p(\varkappa_{m-1}, \varkappa_m) + c_2 d^p(\varkappa_{m-1}, Tx_{m-1}) + c_3 d^p(\varkappa_m, Tx_m)]^{\frac{1}{p}} \\ = \quad & [c_1 d^p(\varkappa_{m-1}, \varkappa_m) + c_2 d^p(\varkappa_{m-1}, x_m) + c_3 d^p(\varkappa_m, \varkappa_{m+1})]^{\frac{1}{p}} \end{aligned} \tag{9}$$

Since $d(\varkappa_m, \varkappa_{m+1}) \leq d(\varkappa_{m-1}, \varkappa_m)$, then from (8) we get

$$\begin{aligned} d(\varkappa_n, \varkappa_{n+1}) \quad & = d(\varkappa_n, Tx_n) = d(\varkappa_m, Tx_m) \\ & = d(\varkappa_m, \varkappa_{m+1}) \\ & \leq \psi(\mathcal{W}_T^{p,c}(\varkappa_{m-1}, \varkappa_m)) \\ & \leq \psi(d(\varkappa_{m-1}, \varkappa_m)) \\ & \leq \psi^{m-n}(d(\varkappa_n, \varkappa_{n+1})). \end{aligned} \tag{10}$$

Since ψ is monotone, inequality (10) yields

$$d(\varkappa_n, \varkappa_{n+1}) \leq \psi^{m-n}(d(\varkappa_n, \varkappa_{n+1})) < d(\varkappa_n, \varkappa_{n+1}), \tag{11}$$

a contradiction.

Third and last step: We assert that the recursive sequence $\{\varkappa_n\}$ is a Cauchy sequence, i.e.

$$\lim_{n\to\infty} d(\varkappa_n, \varkappa_{n+k}) = 0 \quad \text{for all } \; k \in \mathbb{N}. \tag{12}$$

Note that the above inequality holds for $k = 1$ and $k = 2$ due to (12) and (4). So, we investigate relation (12) for $k \geq 3$. Owing to the nature of Branciari distances, we need to examine the following two possibilities.

Case (I): Assume that $k = 2m + 1$ where $m \geq 1$. Then, by utilizing the second step together with relation (3) and the quadrilateral inequality, we observe

$$\begin{aligned} d(\varkappa_n, \varkappa_{n+k}) = d(\varkappa_n, \varkappa_{n+2m+1}) &\leq d(\varkappa_n, \varkappa_{n+1}) + d(\varkappa_{n+1}, \varkappa_{n+2}) + \ldots + d(\varkappa_{n+2m}, \varkappa_{n+2m+1}) \\ &\leq \sum_{p=n}^{n+2m} \psi^p(d(\varkappa_0, \varkappa_1)) \\ &\leq \sum_{p=n}^{+\infty} \psi^p(d(\varkappa_0, \varkappa_1)) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{13}$$

Case (II): Assume that $k = 2m$ where $m \geq 2$. Again by (3) and employing the quadrilateral inequality and keeping second step in mind, we derive

$$\begin{aligned} d(\varkappa_n, \varkappa_{n+k}) = d(\varkappa_n, \varkappa_{n+2m}) &\leq d(\varkappa_n, \varkappa_{n+2}) + d(\varkappa_{n+2}, \varkappa_{n+3}) + \ldots + d(\varkappa_{n+2m-1}, \varkappa_{n+2m}) \\ &\leq d(\varkappa_n, \varkappa_{n+2}) + \sum_{p=n+2}^{n+2m-1} \psi^p(d(\varkappa_0, \varkappa_1)) \\ &\leq d(\varkappa_n, \varkappa_{n+2}) + \sum_{p=n}^{+\infty} \psi^p(d(\varkappa_0, \varkappa_1)) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{14}$$

By combining relations (13) and (14), we have

$$\lim_{n\to\infty} d(\varkappa_n, \varkappa_{n+k}) = 0 \quad \text{for all } k \geq 3.$$

We conclude that $\{\varkappa_n\}$ is a Cauchy sequence in $(\mathcal{X}, d)$. From the completeness of $\mathcal{X}$, the iterative sequence $\{\varkappa_n\}$ is convergent to $\varkappa$.

Observe that $c_3^{\frac{1}{p}} < 1$. Suppose $T\varkappa \neq \varkappa$.

Going back now to the contractive condition, it follows

$$\begin{aligned} d(T\varkappa_n, T\varkappa) &\leq \psi\left(\mathcal{W}_T^{p,c}(\varkappa_n, \varkappa)\right) < \mathcal{W}_T^{p,c}(\varkappa_n, \varkappa) \\ &= (c_1 d^p(\varkappa_n, \varkappa) + c_2 d^p(\varkappa_n, \varkappa_{n+1}) + c_3 d^p(\varkappa, T\varkappa))^{\frac{1}{p}}. \end{aligned}$$

Consider $n \to \infty$; we get that $d(\varkappa, T\varkappa) \leq c_3^{\frac{1}{p}} d(\varkappa, T\varkappa)$, a contradiction. Hence $T\varkappa = \varkappa$.

Having now in view the case $p = 0$, we have

$$d(\varkappa_n, \varkappa_{n+1}) \quad \leq \psi(d^{c_1}(\varkappa_{n-1}, \varkappa_n) d^{c_2}(\varkappa_{n-1}, \varkappa_n) d^{c_3}(\varkappa_n, \varkappa_{n+1})).$$

As it is mentioned above, in case of $d(\varkappa_{n-1}, \varkappa_n) \leq d(\varkappa_n, \varkappa_{n+1})$ we get

$$d(\varkappa_n, \varkappa_{n+1}) \leq \psi(d(\varkappa_n, \varkappa_{n+1})) < d(\varkappa_n, \varkappa_{n+1}),$$

a contradiction. Accordingly, we conclude that

$$d(\varkappa_n, \varkappa_{n+1}) \leq \psi(d(\varkappa_{n-1}, \varkappa_n)) < d(\varkappa_{n-1}, \varkappa_n).$$

In addition, we have

$$d(\varkappa_n, \varkappa_{n+1}) \leq \psi^{n-1}(d(\varkappa_0, \varkappa_1)).$$

By following the necessary steps as above, we obtain that $\varkappa_n \to \varkappa$. It follows that

$$d(T\varkappa_n, T\varkappa) \leq d^{c_1}(\varkappa_n, \varkappa) d^{c_2}(\varkappa_{n-1}, \varkappa_n) d^{c_3}(\varkappa, T\varkappa).$$

Letting $n \to \infty$, we find that $d(\varkappa, T\varkappa) = 0$ and hence this case is also proved. □

In what follows, we define the second hybrid contraction in BDS as follows:

Definition 4. *A self-mapping T on $(\mathcal{X}, d)$ is said to be a (p, c)-weight type contraction, if there is a constant q in $(0, 1)$ so that the following inequality holds for any $s, t \in \mathcal{X}$ which are not fixed points of T*

$$d(Ts, Tt) \leq q\mathcal{W}^p(T, s, t, c),$$

where $p \geq 0$, $c = (c_1, c_2, c_3)$, and c_1, c_2, and c_3 are positive numbers such that $c_1 + c_2 + c_3 = 1$, and

$$\mathcal{W}^p(T, s, t, c) = \begin{cases} (c_1 d^p(s, t) + c_2 d^p(s, Ts) + c_3 d^p(t, Tt))^{\frac{1}{p}}, & \text{if } p > 0 \\ d^{c_1}(s, t) d^{c_2}(s, Ts) d^{c_3}(t, Tt), & \text{if } p = 0. \end{cases}$$

Note that such contractions, as in Definition 4, were initiated in the recent paper [21] in the setting of b-metric space.

Example 2. *Consider the set $\mathcal{X} = \{0, 1, 2, 3\}$, and the BDS $d \colon \mathcal{X} \times \mathcal{X} \to [0, \infty)$, defined by*

$$d(\varkappa, \omega) = \begin{cases} 0, & \text{if } \varkappa = \omega; \\ 3, & \text{if } (\varkappa, \omega) \in \{(1, 2), (2, 1)\}; \\ 1, & \textit{otherwise}. \end{cases}$$

$(\mathcal{X}, d)$ is a BDS, but not a usual metric space, since $d(1, 2) > d(1, 0) + d(0, 2)$. Furthermore, consider $T \colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, $T0 = 1$, $T1 = 2$, $T2 = 2$, $T3 = 3$. T is a (p, c)-weight type contraction with $q \geq \frac{\sqrt{11}}{11}$, and $c_1 = c_2 = c_3 = \frac{1}{3}$, fact which can be easily checked.

Corollary 1. *Let $(\mathcal{X}, d)$ be a complete BDS and $T \colon \mathcal{X} \to \mathcal{X}$ be a (p, c)-weight type ψ-contraction mapping. Then the mapping T possesses a fixed point $\varkappa^*$.*

Proof. It is sufficient to take $\psi(t) = qt$ where $q \in (0, 1)$. □

Remark 1. *It is clear that by a proper choice of c_1, c_2, c_3, q and $p > 0$, several existing results are found in the literature. Among them we can list the original Branciari contraction and Kannan-type, Chatterjea type, Ćirić-Reih-Rus type linear contractions as well as nonlinear (interpolative) contractions for $p = 0$ with a suitable choice of c_1, c_2, c_3.*

The uniqueness is not a feature of such kind of a generalized contraction; for a counterexample see [6].

3. Results on Branciari b-Distance Spaces

We start by the recollecting definition of Branciari b-metric spaces.

Definition 5 (See e.g., [18])**.** *Let $\mathcal{S}$ be a nonempty set, $s \geq 1$, and $\delta \colon \mathcal{S} \times \mathcal{S} \to [0, \infty)$ a function which fulfills the following conditions for all $w, t \in \mathcal{S}$ and all distinct $u, v \in \mathcal{S}$ each of which is different from s and t*

- *(b1)* $\delta(w, t) = 0$ *if and only if* $w = t$;
- *(b2)* $\delta(w, t) = \delta(t, w)$;
- *(b3)* $\delta(w, t) \leq s[\delta(w, u) + \delta(u, v) + \delta(v, t)]$, *(the extended quadrilateral inequality).*

Then δ is a Branciari b-distance. The pair $(\mathcal{S}, \delta)$ is called a Branciari b-distance space (in short, BbDS).

Throughout the paper, $(\mathcal{S}, \delta)$ refers to a Branciari b-distance space.

For an example of such a space, we cite [23].

Example 3. *Let $(\mathcal{S}, D)$ be a BDS. Consider the mapping $\delta\colon \mathcal{S}\times\mathcal{S}\to[0,\infty)$, $\delta(\varkappa,\rho)=D^p(\varkappa,\rho)$ for any $p\in(0,\infty)\setminus\{1\}$. Then δ is a BbDS.*

Convergence, Cauchy property and completeness are defined as in the case of BDS. More precisely,

Definition 6.

1. *$\{\varkappa_n\}$ in a $(\mathcal{S},\delta)$ is convergent to $\varkappa$ if and only if $\delta(\varkappa_n,\varkappa)\to 0$ as $n\to\infty$.*
2. *$\{\varkappa_n\}$ in a $(\mathcal{S},\delta)$ is Cauchy if and only if for each $\varepsilon>0$ we may find $N(\varepsilon)$ such that $\delta(\varkappa_n,\varkappa_m)<\varepsilon$ for all $n>m>N(\varepsilon)$.*
3. *A Branciari b-distance space $(\mathcal{S},\delta)$ is complete if each Cauchy (fundamental) sequence in $(\mathcal{S},\delta)$ is convergent.*
4. *A mapping $T\colon(\mathcal{S},\delta)\to(\mathcal{S},\delta)$ is continuous if for any sequence $\{\varkappa_n\}$ in $\mathcal{X}$ such that $\delta(\varkappa_n,\varkappa)\to 0$ as $n\to\infty$, we have $\delta(T\varkappa_n,T\varkappa)\to 0$ as $n\to\infty$.*

Definition 7. *Consider $(\mathcal{S},\delta)$ a BbDS. $T\colon\mathcal{S}\to\mathcal{S}$ is a (p,c)-weight type contraction if there is a constant q in $(0,1)$ so that the following inequality holds for any s, $t\in\mathcal{S}$ which are not fixed points of the mapping T*

$$\delta(Ts,Tt)\le q\mathcal{W}^p(T,s,t,c),$$

where $p\ge 0$, $c=(c_1,c_2,c_3)$, and c_1, c_2, and c_3 are positive numbers such that $c_1+c_2+c_3=1$, and

$$\mathcal{W}^p(T,s,t,c)=\begin{cases}\left(c_1\delta^p(s,t)+c_2\delta^p(s,Ts)+c_3\delta^p(t,Tt)\right)^{\frac{1}{p}}, & \text{if } p>0\\ \\ \delta^{c_1}(s,t)\delta^{c_2}(s,Tt)\delta^{c_3}(t,Tt), & \text{if } p=0.\end{cases}$$

Example 4. *Consider the set $\mathcal{X}=\{0,1,2,3\}$, and the BbDS $d\colon\mathcal{X}\times\mathcal{X}\to[0,\infty)$, defined by*

$$d(\varkappa,\omega)=\begin{cases}0, & \text{if } \varkappa=\omega;\\ 4, & \text{if } (\varkappa,\omega)\in\{(1,3),(3,1)\};\\ 1, & \textit{otherwise}.\end{cases}$$

d is a BbDS with $s=2$, but not an BbDS metric space, since $d(1,3)>d(1,0)+d(0,1)+d(1,3)$. Furthermore, consider $T\colon\mathcal{S}\to\mathcal{S}$, defined by $T0=0$, $T1=3$, $T2=0$, $T3=3$. Then T is a (p,c)-weight type contraction, which can be easily checked.

Theorem 2. *Suppose that a self-mapping T on a complete BbDS $(\mathcal{S},\delta)$ forms a (p,c)-weight type contraction mapping for which $s^pq^{2p}c_1^2<1$, $q^pc_3s<1$, $sC^2<1$, where $C=q\left(\frac{c_1+c_3}{1-q^pc_2}\right)^{\frac{1}{p}}<1$. If T is a continuous mapping, then the Picard iteration sequence $\{T^n\varkappa_0\}$, $\varkappa_0\in\mathcal{S}$, is convergent to a fixed point $\varkappa$.*

Proof. Consider $\varkappa_{n+1}=T\varkappa_n$, for any $\varkappa\in\mathcal{S}$. We may assume that for any $n\in\mathbb{N}_0$, $\varkappa_n\ne\varkappa_{n+1}$. Let us take into consideration the situation in which $p>0$. By the use of the contraction condition, we get, for $n\ge 1$,

$$\delta(\varkappa_n,\varkappa_{n+1})\le q\left(c_1\delta^p(\varkappa_{n-1},\varkappa_n)+c_2\delta^p(\varkappa_n,\varkappa_{n+1})+c_3\delta^p(\varkappa_{n-1},\varkappa_n)\right)^{\frac{1}{p}},$$

that is, for $n\ge 1$,

$$\delta(\varkappa_n,\varkappa_{n+1})\le\left(\frac{q^p(c_1+c_3)}{1-q^pc_2}\right)^{\frac{1}{p}}\delta(\varkappa_{n-1},\varkappa_n).$$

By the same means as in the previous theorem, we get, for $n\in\mathbb{N}_0$,

$$\delta(\varkappa_n,\varkappa_{n+2})\le C^{np}q^p(1-c_1)\frac{C^p}{C^p-q^pc_1}\delta^p(x_0,x_1)+q^{np}c_1^n\delta^p(x_0,x_2).$$

Let us now consider $n \in \mathbb{N}_0$, $m \geq 2$. Taking advantage of the quadrilateral inequality, we obtain

$$\begin{aligned}
\delta(\varkappa_n, \varkappa_{n+2m+1}) &\leq \sum_{k=0}^{m-1} s^{k+1}\left(\delta(\varkappa_{n+2k}, \varkappa_{n+2k+1}) + \delta(\varkappa_{n+2k+1}, \varkappa_{n+2k+2})\right) + s^m \delta(\varkappa_{n+2m}, \varkappa_{n+2m+1}) \\
&\leq 2\sum_{k=0}^{m} C^{n+2k} s^{k+1} \delta(\varkappa_0, \varkappa_1) \\
&\leq \frac{2s}{1 - sC^2} C^n s \delta(\varkappa_0, \varkappa_1).
\end{aligned} \tag{15}$$

On the other hand, it can be observed that

$$\begin{aligned}
\delta(\varkappa_n, \varkappa_{n+2m}) &\leq s(\delta(\varkappa_n, \varkappa_{n+1}) + \delta(\varkappa_{n+1}, \varkappa_{n+2}) + \delta(\varkappa_{n+2}, \varkappa_{n+2m})) \\
&\leq \sum_{k=0}^{m-2} s^{k+1}\left(\delta(\varkappa_{n+2k}, \varkappa_{n+2k+1}) + \delta(\varkappa_{n+2k+1}, \varkappa_{n+2k+2})\right) \\
&\qquad + s^{m-1}\delta(\varkappa_{n+2m-2}, \varkappa_{n+2m}) \\
&\leq 2\sum_{k=0}^{m-2} s^{k+1} C^{n+k} \delta(\varkappa_0, \varkappa_1) + s^{m-1}\Big(C^{(n+2m-2)p} q^p (1 - c_1) \frac{C^p}{C^p - q^p c_1} \delta^p(\varkappa_0, \varkappa_1) \\
&\qquad + q^{(n+2m-2)p} c_1^{n+2m-2} \delta^p(x_0, x_2)\Big)^{\frac{1}{p}} \\
&\leq \frac{1}{1 - sC} s C^n \delta(\varkappa_0, \varkappa_1) + \left(C^{n-1} q^p (1 - c_1) \frac{C^p}{C^p - q^p c_1} \delta^p(\varkappa_0, \varkappa_1)\right. \\
&\qquad \left. + q^{np} c_1^n \delta^p(\varkappa_0, \varkappa_2)\right)^{\frac{1}{p}}.
\end{aligned} \tag{16}$$

Having in mind inequalities (15) and (16), it follows that $\{\varkappa_n\}$ is a Cauchy sequence. The completeness of $\mathcal{X}$ implies that the sequence $\{\varkappa_n\}$ is convergent to $\varkappa \in \mathcal{X}$.

Going back now to the contractive condition, it follows

$$\begin{aligned}
\delta(T\varkappa_n, T\varkappa) &\leq q\mathcal{M}^p(T, \varkappa_n, \varkappa, c) \\
&= q\left(c_1\delta^p(\varkappa_n, \varkappa) + c_2\delta^p(\varkappa_n, \varkappa_{n+1}) + c_3\delta^p(\varkappa, T\varkappa)\right)^{\frac{1}{p}}.
\end{aligned}$$

Having in mind also that

$$\delta(\varkappa, T\varkappa) \leq s\left(\delta(\varkappa, \varkappa_n) + \delta(\varkappa_n, \varkappa_{n+1}) + \delta(\varkappa_{n+1}, T\varkappa)\right),$$

consider $n \to \infty$; we get that $\delta(\varkappa, Tt^*) \leq qc_3^{\frac{1}{p}} s\delta(t^*, T\varkappa)$, hence $T\varkappa = \varkappa$.

In the case $p = 0$ might be treated in a similar way as in the proof of the previous theorem. □

Here, we underline the importance of Remark 1 and can easily derive the analog of it.

Remark 2. *On account of a proper choice of c_1, c_2, c_3, q and $p > 0$, several results are extracted in the framework of the linear contractions as well as nonlinear (interpolative) contractions for $p = 0$ with a suitable choice of c_1, c_2, c_3.*

4. Conclusions

Regarding the basic three axioms (self-distance, symmetry and the triangle inequality) of the standard metric space, we notice that almost all generalization and extension presume the first of them. The distance function is called symmetric if it satisfies the axioms of self-distance and symmetry. This crucial notion is very weak to construct a topology on which we can consider nonlinear analysis problems. Investigation of Branciari distance space has a crucial role in order to comprehend the possibility and the impossibility of the fundamental notion: semimetric spaces. The presented results are considered a stone in construction of this road. On the other hand, this result may lead to new research topics. For example considering the following publications, [3,4,10,12,16,25,27] one can consider the characterization of these results in the Branciari type distance spaces.

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