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On the Generalization for Some Power-Exponential-Trigonometric Inequalities

Aníbal Coronel [1,*], Peter Kórus [2], Esperanza Lozada [1] and Elias Irazoqui [1]

1 Departamento de Ciencias Básicas, Facultad de Ciencias, Universidad del Bío-Bío, Campus Fernando May, 3780000 Chillán, Chile; elozada@udec.cl (E.L.); eliasirazoqui@gmail.com (E.I.)

2 Department of Mathematics, Juhász Gyula Faculty of Education, University of Szeged, Hattyas utca 10, H-6725 Szeged, Hungary, korpet@jgypk.szte.hu

* Correspondence: acoronel@ubiobio.cl; Tel.: +56-42-2463259

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Abstract: In this paper, we introduce and prove several generalized algebraic-trigonometric inequalities by considering negative exponents in the inequalities.

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1. Introduction

In recent years, an increasing amount of attention has been paid to the study of power-exponential inequalities [1–10]. A review of some problems and historical landmarks are given in [2,11]. In particular, in order to contextualize, we recall that the basic problem of comparing a^b and b^a for all positive real numbers a and b was presented in [12–14]. Increasing in algebraic difficulty, the comparison of $a^a + b^b$ and $a^b + b^a$ was studied independently by Laub–Ilani and Zeikii–Cirtoaje–Berndt, see [15–18], respectively. The result is the fact that the inequality

$$a^a + b^b \geq a^b + b^a, \quad a, b \in [0, \infty[\tag{1}$$

holds. An extension of (1) was proposed, analyzed and proved by Matejíčka, Cîrtoaje and Coronel-Huancas in [2,17,19] obtaining the inequality

$$a^{ra} + b^{rb} \geq a^{rb} + b^{ra}, \quad a, b \in [0, \infty[, \quad r \in [0, e[. \tag{2}$$

More recently, other extensions and generalizations of (1) were introduced, proved and conjectured by Özban in [11], where, in particular, the author proved the following inequalities:

$$\begin{aligned}
&(\sin x)^{\sin x} + (\sin y)^{\sin y} > (\sin x)^{\sin y} + (\sin y)^{\sin x}, \quad 0 < x < y < \pi/2,\\
&(\cos x)^{\cos x} + (\cos y)^{\cos y} > (\cos x)^{\cos y} + (\cos y)^{\cos x}, \quad 0 < x < y < \pi/2,\\
&(\cos x)^{\sin x} + (\cos y)^{\sin y} < (\cos x)^{\sin y} + (\cos y)^{\sin x}, \quad 0 < x < y \leq 1,\\
&(\cos x)^{x} + (\cos y)^{y} < (\cos x)^{y} + (\cos y)^{x}, \quad 0 < x < y \leq \pi/2,\\
&(\sin x)^{x} + (\sin y)^{y} > (\sin x)^{y} + (\sin y)^{x}, \quad 0 < x < y \leq \pi/2,\\
&x^{\cos x} + y^{\cos y} < x^{\cos y} + y^{\cos x}, \quad 0 < x < y, \ 1 \leq y \leq \pi/2,\\
&x^{\sin x} + y^{\sin y} > x^{\sin y} + y^{\sin x}, \quad 0 < x < y \leq \pi/2.
\end{aligned} \tag{3}$$

In order to extend or generalize (2) and (3), it seems natural to ask some questions: What happens with the inequality (2) when $r \in \mathbb{R} - [0, e[$? and what happens with the inequalities in (3) if we include a negative power r?. We note that the powers in question exist, since the basis of powers in (2) and (3) are positive. Indeed, in this article, we study (2) for $r \in]-\infty, 0[$ and establish reverse inequalities for some cases. Moreover, we study the generalization of the inequalities in (3) with negative power r.

The main results of the paper are the following theorems:

Theorem 1. *Let the function $\varphi_\alpha : \mathbb{R} \to \mathbb{R}$ be defined by $\varphi_\alpha(m) = m\alpha^m$ for each $\alpha > 1$ and consider the following sets:*

$$\begin{aligned} A_{old} &= \Big\{(a,b,r) \in \mathbb{R}^3 \quad : \quad a \geq 0, \quad b \geq 0, \quad r \in [0, e[\Big\}, \\ A^d_{new} &= \Big\{(a,b,r) \in \mathbb{R}^3 \quad : \quad a > 1, \quad b > 1, \quad r < 0, \quad \varphi_b(rb) > \varphi_b(ra)\Big\} \\ &\quad \bigcup \Big\{(a,b,r) \in \mathbb{R}^3 \quad : \quad a > 1, \quad b > 1, \quad r < 0, \quad \varphi_b(rb) < \varphi_b(ra), \quad a^{rb} < \overline{\gamma}\Big\}, \\ A^r_{new} &= \Big\{(a,b,r) \in \mathbb{R}^3 \quad : \quad 0 \leq a \leq 1, \quad 0 \leq b \leq 1, \quad r < 0\Big\} \\ &\quad \bigcup \Big\{(a,b,r) \in \mathbb{R}^3 \quad : \quad a > 1, \quad b > 1, \quad r < 0, \quad \varphi_b(rb) < \varphi_b(ra), \quad a^{rb} > \overline{\gamma}\Big\}, \end{aligned} \tag{4}$$

where $\overline{\gamma} \in]0,1[$ is such that $\overline{\gamma} \neq b^{rb}$ and $(\overline{\gamma})^{a/b} - \overline{\gamma} - b^{ra} + b^{rb} = 0$. Then, the following inequalities

$$a^{ra} + b^{rb} \geq a^{rb} + b^{ra}, \quad (a,b,r) \in A_{old} \cup A^d_{new}, \tag{5}$$

$$a^{ra} + b^{rb} \leq a^{rb} + b^{ra}, \quad (a,b,r) \in A^r_{new} \tag{6}$$

are satisfied.

Remark 1. *The inclusion of the notation $\overline{\gamma}$ is related with the fact that the argumentation of the proof is based on the properties of function $f(t) = (t)^s - t - \gamma^s + \gamma$ with $t = a^{rb}$ $s = a/b$ and $\gamma = b^{rb}$. In particular, we observe that, if $0 < t < \gamma < 1$, there are two solutions of $f(t) = 0$ on the interval $]0,1[$; one solution is clearly γ and the other solution is difficult to get explicitly and is denoted by $\overline{\gamma}$.*

Theorem 2. *If $x, y \in (0, \pi/2)$ and $r < 0$, then*

$$(\sin x)^{r \sin x} + (\sin y)^{r \sin y} \leq (\sin x)^{r \sin y} + (\sin y)^{r \sin x}, \tag{7}$$

$$(\cos x)^{r \cos x} + (\cos y)^{r \cos y} \leq (\cos x)^{r \cos y} + (\cos y)^{r \cos x}, \tag{8}$$

$$(\cos x)^{r \sin x} + (\cos y)^{r \sin y} \geq (\cos x)^{r \sin y} + (\cos y)^{r \sin x}. \tag{9}$$

Theorem 3. *If $x, y \in (0, \pi/2)$ and $r < 0$, then*

$$(\cos x)^{rx} + (\cos y)^{ry} \geq (\cos x)^{ry} + (\cos y)^{rx}, \tag{10}$$

$$(\sin x)^{rx} + (\sin y)^{ry} \leq (\sin x)^{ry} + (\sin y)^{rx}. \tag{11}$$

Theorem 4. *If $x, y \in (0, \pi/2)$, $\min\{x, y\} \in (0, 1]$ and $r < 0$, then*

$$x^{r \cos x} + y^{r \cos y} \geq x^{r \cos y} + y^{r \cos x}, \tag{12}$$

$$x^{r \sin x} + y^{r \sin y} \leq x^{r \sin y} + y^{r \sin x}. \tag{13}$$

The rest of the paper is dedicated to the proof of Theorems 1–4.

2. Proofs of Main Results

2.1. Proof of Theorem 1

For completeness and self-contained structure of the proof, we recall the notation and a result given in [1]. Indeed, let us consider $s \in \mathbb{R}^+$ and we define the functions f and g from $\mathbb{R}^+$ to $\mathbb{R}$ by the relations

$$\begin{aligned} f(t) &= t^s - t - \gamma^s + \gamma, \\ g(t) &= \begin{cases} e^{-\ln(t)/(t-1)}, & \text{for } t \notin \{0,1\}, \\ e^{-1}, & \text{for } t = 1, \\ 0, & \text{for } t = 0. \end{cases} \end{aligned}$$

Then, the following properties are satisfied: $f(\gamma) = 0$ and $f(0) = f(1) = -\gamma^s + \gamma$; if $s > 1$ (resp. $s < 1$), f is strictly increasing (resp. decreasing) on $]g(s), \infty[$ and strictly decreasing (resp. increasing) on $]0, g(s)[$; and g is continuous on $\mathbb{R}^+ \cup \{0\}$, strictly increasing on $\mathbb{R}^+$, $y = 1$ is a horizontal asymptote of $y = g(t)$, and the range of g is $[0,1]$. Moreover, if we consider the function $\xi : \mathbb{R}^+ \to \mathbb{R}$ $\xi(m) = -m^s + m$ and φ_α defined in the enunciate of the theorem, we observe that the following following assertions are satisfied: $\xi(0) = \xi(1) = 0$; if $s > 1$ (resp. $s < 1$) w has a maximum at $g(s)$ (resp. minimum at $g(s)$); $\varphi_\alpha(0) = 0$; φ_α has a minimum at $m^* = -1/\ln(\alpha)$; φ_α has a inflection point at $m^{**} = -2/\ln(\alpha)$; $y = 0$ is a left horizontal asymptote of φ_α and the range of g is $[\varphi_\alpha(m^*), \infty[$ with $\varphi_\alpha(m^*) < 0$.

Let us consider $t = a^{rb}$, $\gamma = b^{rb}$, and $s = a/b$ and we observe that

$$f(t) = (a^{rb})^{a/b} - a^{rb} - (b^{rb})^{a/b} + b^{rb} = a^{ra} - a^{rb} - b^{ra} + b^{rb}. \tag{14}$$

Then, the proofs of (5) and (6) are reduced to analyze the sign of $f(t)$ for $t \in [0, \gamma]$. Indeed, without loss of generality and by the symmetric form of the inequalities in (5) and (6), we assume that $0 \le b < a$ (i.e., $s = a/b > 1$) and consider three cases:

(i) Let a, b such that $1 > a > b \ge 0$. Then, for $r < 0$, we note that $1 < a^r < b^r$ or equivalently we have that $1 < t < \gamma$. Moreover, observing that $s > 1$ and $g(s) < 1$, by the strictly increasing behavior of f on $[g(s), \infty)$, we deduce that $f(g(s)) < f(1) < f(t) < f(\gamma) = 0$. Thus, from (14) and $f(t) < 0$, we follow that the inequality $a^{ra} + b^{rb} < a^{rb} + b^{ra}$ is satisfied.

(ii) Let a, b such that $a > 1 > b \ge 0$. In this case, we have that $a^r < 1 < b^r$ or equivalently $t < 1 < \gamma$. We note that $s > 1$ implies the strictly decreasing behavior of f on $[0, g(s)]$ and the strictly increasing behavior of f on $[g(s), \infty[$. Moreover, observing that $g(s) \in [0,1]$, we deduce that $f(t) < f(1) = -\gamma^s + \gamma := \xi(\gamma)$ for any $t < 1 < \gamma$. Now, by the fact that ξ is decreasing on $[g(s), \infty[$, we have that $\xi(\gamma) < \xi(1) = 0$ for any $\gamma > 1$. Thus, $f(t) < \xi(\gamma) < 0$ for $t < 1 < \gamma$ and, from (14), the inequality $a^{ra} + b^{rb} < a^{rb} + b^{ra}$ is satisfied.

(iii) Let a, b such that $a > b > 1$. Similarly to cases (i) and (ii), we have that $s > 1$ and $0 < a^r < 1 < b^r < 1$ or equivalently $0 < t < \gamma < 1$. Here, we distinguish two subcases: $\gamma \le g(s)$ and $g(s) < \gamma < 1$. First, if $\gamma \le g(s)$, we have that f is strictly decreasing on $[0, \gamma]$ and consequently $f(t) \ge f(\gamma) = 0$ for $t \in [0, \gamma]$. Second, if $g(s) < \gamma < 1$, by the fact that $f(0) = \xi(\gamma) > 0 = f(\gamma) > f(g(s))$, we have that there exists $\overline{\gamma} \in [0, g(s)[$ such that $f(\overline{\gamma}) = 0$. Then, $f(t) \ge f(\overline{\gamma}) = 0$ for $t \in [0, \overline{\gamma}]$ and $f(t) \le f(\gamma) = f(\overline{\gamma}) = 0$ for $t \in [\overline{\gamma}, \gamma]$. Thus, from both subcases, we conclude that the inequality $a^{ra} + b^{rb} < a^{rb} + b^{ra}$ is satisfied for $t \in [\overline{\gamma}, \gamma]$ with $\gamma \in]g(s), 1[$ and the inequality $a^{ra} + b^{rb} > a^{rb} + b^{ra}$ is satisfied for $t \in [0, \overline{\gamma}]$ with $\gamma \in]g(s), 1[$ or for $t \in [0, \gamma]$ with $\gamma \in]0, g(s)]$.

On the other hand, by the definition of γ, s, g and φ_b, we observe that $\gamma < g(s)$ (resp. $\gamma > g(s)$) is equivalent to $\varphi_b(rb) > \varphi_b(ra)$ (resp. $\varphi_b(rb) < \varphi_b(ra)$). Moreover, the relation $t > \overline{\gamma}$ (resp. $t < \overline{\gamma}$) is equivalent to $a^{rb} > \overline{\gamma}$ (resp. $a^{rb} < \overline{\gamma}$). Thus, the subcases can be characterized in terms of the function φ_b and $a^{rb} > \overline{\gamma}$ or $a^{rb} < \overline{\gamma}$.

Hence, translating (i), (ii) and (iii) to the corresponding notation in (4) and observing that the set A_{old} is the set for the inequality in (2), we conclude the proof the theorem.

2.2. Proof of Theorem 2

Since $\sin t, \cos t > 0$ for $t \in (0, \pi/2)$, Theorem 1 immediately implies inequalities (7) and (8). To prove (9), we define

$$f(t) = (\cos t)^{r \sin t} + (\cos y)^{r \sin y} - (\cos t)^{r \sin y} - (\cos y)^{r \sin t}$$

for y is fixed and arbitrarily selected such that $y \in (0, \pi/2)$ and $0 < t \le y$. We note that $f(y) = 0$, then the result follows if f is decreasing. Indeed, to see this, we write

$$f'(t) = r\left[g(t)\cos t + \frac{\sin t}{\cos t} h(t)\right],$$

where

$$\begin{aligned} g(t) &= (\cos t)^{r \sin t} \ln(\cos t) - (\cos y)^{r \sin t} \ln(\cos y), \\ h(t) &= (\cos t)^{r \sin y} \sin y - (\cos t)^{r \sin t} \sin t. \end{aligned}$$

Now, since $r < 0$, it is enough to show that $g(t), h(t) > 0$. For g, we have that

$$\begin{aligned} g(t) &= -\int_t^y \frac{d}{ds}(\cos s)^{r \sin t} \ln(\cos s) \\ &= \int_t^y ((\cos s)^{r \sin t - 1} \sin s)(1 + r \sin t \ln(\cos s))\, ds > 0 \end{aligned}$$

and, similarly for h, we deduce that

$$\begin{aligned} h(t) &= \int_t^y \frac{d}{ds}(\cos t)^{r \sin s} \sin s \\ &= \int_t^y ((\cos t)^{r \sin s} \cos s)(1 + r \sin s \ln(\cos t))\, ds > 0. \end{aligned}$$

2.3. Proof of Theorem 3

Set $0 < t \le y < \pi/2$ and $r < 0$ arbitrarily. Along the proofs, we will use that $\sin s, \cos s > 0$ for $s \in (0, \pi/2)$.

In order to prove (10), let us consider $f_1(t) = (\cos t)^{rt} + (\cos y)^{ry} - (\cos t)^{ry} - (\cos y)^{rt}$. Observing that $f_1(y) = 0$, it is enough to show that f_1 is decreasing. Indeed, the decreasing behavior of f_1 follows immediately since

$$f_1'(t) = r\left[g_1(t) + \frac{\sin t}{\cos t} h_1(t)\right],$$

where

$$\begin{aligned} g_1(t) &= (\cos t)^{rt} \ln(\cos t) - (\cos y)^{rt} \ln(\cos y) = -\int_t^y \frac{d}{ds}(\cos s)^{rt} \ln(\cos s) \\ &= \int_t^y ((\cos s)^{rt-1} \sin s)(1 + rt \ln(\cos s))\, ds > 0 \end{aligned}$$

and

$$
\begin{aligned}
h_1(t) = y(\cos t)^{ry} - t(\cos t)^{rt} &= \int_t^y \frac{d}{ds} s(\cos t)^{rs} \\
&= \int_t^y (\cos t)^{rs}(1 + rs\ln(\cos t))\, ds > 0.
\end{aligned}
$$

We prove (11) by analogous arguments to the proof of (10). Indeed, let us introduce the notation $f_2(t) = (\sin t)^{ry} + (\sin y)^{rt} - (\sin t)^{rt} - (\sin y)^{ry}$. We observe that

$$
f_2'(t) = r\left[g_2(t) + \frac{\cos t}{\sin t} h_2(t)\right] < 0,
$$

since

$$
\begin{aligned}
g_2(t) = (\sin y)^{rt}\ln(\sin y) - (\sin t)^{rt}\ln(\sin t) &= \int_t^y \frac{d}{ds}(\sin s)^{rt}\ln(\sin s) \\
&= \int_t^y ((\sin s)^{rt-1}\cos s)(1 + rt\ln(\sin s))\, ds > 0
\end{aligned}
$$

and

$$
\begin{aligned}
h_2(t) = y(\sin t)^{ry} - t(\sin t)^{rt} &= \int_t^y \frac{d}{ds} s(\sin t)^{rs} \\
&= \int_t^y (\sin t)^{rs}(1 + rs\ln(\sin t))\, ds > 0.
\end{aligned}
$$

Thus, (11) is a consequence of the decreasing behavior of f_2 and the fact that $f_2(y) = 0$.

2.4. *Proof of Theorem 4*

We set $0 < x \le y < \pi/2$ with $x \le 1$ and $r < 0$ arbitrarily selected. Then, by the fact that $\cos x \ge \cos y > 0$, we deduce the following estimate:

$$
\begin{aligned}
x^{r\cos x} - x^{r\cos y} &= x^{r\cos y}(x^{r(\cos x - \cos y)} - 1) \\
&\ge y^{r\cos y}(y^{r(\cos x - \cos y)} - 1) = y^{r\cos x} - y^{r\cos y},
\end{aligned}
$$

which implies (12). Similarly, using the fact that $\sin y \ge \sin x > 0$ implies that

$$
\begin{aligned}
x^{r\sin y} - x^{r\sin x} &= x^{r\sin x}(x^{r(\sin y - \sin x)} - 1) \\
&\ge y^{r\sin x}(y^{r(\sin y - \sin x)} - 1) = y^{r\sin y} - y^{r\sin x},
\end{aligned}
$$

and we get the proof of (13).

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References

1. Coronel, A.; Huancas, F. On the inequality $a^{2a} + b^{2b} + c^{2c} \ge a^{2b} + b^{2c} + c^{2a}$. *Aust. J. Math. Anal. Appl.* **2012**, *9*, 3.

2. Coronel, A.; Huancas, F. Proof of three power-exponential inequalitie. *J. Inequal. Appl.* **2014**, *2014*, 509, doi:10.1186/1029-242X-2014-509.
3. Cîrtoaje, V. Proofs of three open inequalities with power-exponential functions. *J. Nonlinear Sci. Appl.* **2011**, *4*, 130–137.
4. Matejíčka, L. Some remarks on Cîrtoaje's conjecture. *J. Inequal. Appl.* **2016**, *159*, 269, doi:10.1186/s13660-016-1092-2.
5. Matejíčka, L. On the Cîrtoaje's conjecture. *J. Inequal. Appl.* **2016**, *159*, 152, doi:10.1186/s13660-016-1211-0.
6. Matejíčka, L. Next, generalization of Cîrtoaje's inequality. *J. Inequal. Appl.* **2017**, *159*, 159, doi:10.1186/s13660-017-1436-6.
7. Mitrinović, D.S. *Analytic Inequalities*; In Cooperation with P. M. Vasić. Die Grundlehren der Mathematischen Wissenschaften, Band 165; Springer: New York, NY, USA; Berlin, Germany, 1970.
8. Miyagi, M.; Nishizawa, Y. A short proof of an open inequality with power-exponential functions. *Aust. J. Math. Anal. Appl.* **2014**, *11*, 6.
9. Miyagi, M.; Nishizawa, Y. Extension of an inequality with power exponential functions. *Tamkang J. Math.* **2015**, *46*, 427–433.
10. Miyagi, M.; Nishizawa, Y. A stronger inequality of Cîrtoaje's one with power exponential functions. *J. Nonlinear Sci. Appl.* **2015**, *8*, 224–230
11. Özban, A.Y. New algebraic-trigonometric inequalities of Laub-Ilani type. *Bull. Aust. Math. Soc.* **2017**, *96*, 87–97, doi:10.1017/S0004972717000156
12. Bullen, P.S. *A Dictionary of Inequalities*; Volume 97 of Pitman Monographs and Surveys in Pure and Applied Mathematics; Longman: Harlow, UK, 1998.
13. Luo, J.; Wen, J.J. A power-mean discriminance of comparing a^b and b^a. In *Research Inequalities*; Yand, X.-Z., Ed.; People's Press of Tibet: Lhasa, China, 2000; pp. 83–88.
14. Qi, F.; Debnath, L. Inequalities for power-exponential functions. *J. Inequal. Pure Appl. Math.* **2000**, *1*, 15.
15. Laub, M. Problem E3116. *Am. Math. Mon.* **1985**, *92*, 666, doi: 10.2307/2323718.
16. Laub, M.; Ilani, I. A subtle inequality. *Am. Math. Mon.* **1990**, *97*, 65–67, doi: 10.2307/2324012
17. Cîrtoaje, V. On some inequalities with power-exponential functions. *J. Inequal. Pure Appl. Math.* **2009**, *10*, 21.
18. Zeikii, A.; Cirtoaje, V.; Berndt, W. Mathlinks. Forum. Available online: http://www.mathlinks.ro/Forum/viewtopic.php?t=118722 (accessed on 11 November 2006).
19. Matejicka, L. Solution of one conjecture on inequalities with power-exponential functions. *J. Inequal. Pure Appl. Math.* **2009**, *10*, 72.