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Positively Continuum-Wise Expansiveness for C^1 Differentiable Maps

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Abstract: We show that if a differentiable map f of a compact smooth Riemannian manifold M is C^1 robustly positive continuum-wise expansive, then f is expanding. Moreover, C^1-generically, if a differentiable map f of a compact smooth Riemannian manifold M is positively continuum-wise expansive, then f is expanding.

Keywords: positively expansive; positively measure expansive; generic; positively continuum-wise expansive; expanding

MSC: 58C25; 37C20; 37D20

1. Introduction and Statements

Starting with Utz [1], expansive dynamical systems have been studied by researchers. Regarding this concept, many researchers suggest various expansivenesses (e.g., N-expansive [2], measure expansive [3] and continuum-wise expansive [4]). These concepts were used to show chaotic systems (see References [3,5–7]) and hyperbolic structures (see References [8–14]).

For chaoticity, Morales and Sirvent proved in Reference [3] that every Li-Yorke chaotic map in the interval or the unit circle are measure-expansive. Kato proved in Reference [7] that, if a homeomorphism f of a compactum X with $\dim X > 0$ is continuum-wise expansive and Z is a chaotic continuum of f, then either f or f^{-1} is chaotic in the sense of Li and Yorke on almost all Cantor sets $C \subset Z$. Hertz [5,6] proved that if a homeomorphism f of locally compact metric space X or Polish continua X is expansive or continuum-wise expansive then f is sensitive dependent on the initial conditions.

For hyperbolicity, Mañé proved in Reference [12] that if a diffeomorphism f of a compact smooth Riemannian manifold M is robustly expansive then it is quasi-Anosov. Arbieto proved in Reference [8] that, C^1 generically, if a diffeomorphism f of a compact smooth Riemannian manifold M is expansive then it is Axiom A and has no cycles. Sakai proved in Reference [13] that, if a diffeomorphism f of a compact smooth Riemannian manifold M is robustly expansive then it is quasi-Anosov. Lee proved in Reference [9] that, C^1 generically, if a diffeomorphism f of a compact smooth Riemannian manifold M is continuum-wise expansive then it is Axiom A and has no cycles.

Through these results, we are interested in general concepts of expansiveness. Actively researching positive expansivities (positively expansive [15], positively measure-expansive [16,17]) is a motivation of this paper. In this paper, we study positively continuum-wise expansiveness, which is the generalized notion of positive expansiveness and positive measure expansiveness.

In this paper, we assume that M is a compact smooth Riemannian manifold. A differentiable map $f : M \to M$ is *positively expansive*(write $f \in \mathcal{PE}$) if there exists a constant $\delta > 0$ such that for any $x, y \in M$, if $d(f^i(x), f^i(y)) \leq \delta \ \forall i \geq 0$ then $x = y$. From Reference [18], if a differentiable map $f \in \mathcal{PE}$ then f is open and a local homeomorphism. For any $\delta > 0$, we define a dynamical δ-ball for $x \in M$ such as $\{y \in M : d(f^i(x), f^i(y)) \leq \delta \ \forall i \geq 0\}$. Put $\Gamma^+_\delta(x) = \{y \in M : d(f^i(x), f^i(y)) \leq \delta \ \forall i \geq 0\}$.

Note that if a differentiable map $f \in \mathcal{PE}$, then $\Gamma^+_\delta(x) = \{x\}$ for any $x \in M$. Here $\delta > 0$ is called an expansive constant of f.

Let us introduce a generalization of the positively expansive called the positively measure-expansive (see Reference [3]). Let $\mathcal{M}(M)$ be the space of a Borel probability measure of M. A measure $\mu \in \mathcal{M}(M)$ is *atomic* if $\mu(\{x\}) \neq 0$, for some point $x \in M$. Let $\mathcal{A}(M)$ be the set of atomic measures of M. Note that $\mathcal{A}(M)$ is dense in $\mathcal{M}(M)$. Let $\mathcal{M}^*(M) = \mathcal{M}(M) \setminus \mathcal{A}(M)$. A differentiable map $f : M \to M$ is *positively measure-expansive* (write $f \in \mathcal{PME}$) if there exists a constant $\delta > 0$ such that $\mu(\Gamma_\delta(x)) = 0$ for any $\mu \in \mathcal{M}^*(M)$, where $\delta > 0$ is called a *measure expansive constant*. In Reference [17], the authors found that there exists a differentiable map $f : S^1 \to S^1$ that is positively μ-expansive for any $\mu \in \mathcal{M}^*_f(S^1)$ but not positively expansive where $\mathcal{M}^*_f(M)$ is the set of non-atomic invariant measures of M.

Now, we introduce another generalization of the positive expansiveness, which is called positively continuum-wise expansiveness (see Reference [4]). We say that C is a *continuum* if it is compact and connected.

Definition 1. *A differentiable map f is positively continuum-wise expansive (write $f \in \mathcal{PCWE}$) if there is a constant $e > 0$ such that if $C \subset M$ is a non-trivial continuum, then there is $n \geq 0$ such that* $\mathrm{diam} f^n(C) > e$*, where if C is a trivial, then C is a one point set.*

Note that $f \in \mathcal{PCWE}$ if and only if $f^n \in \mathcal{PCWE}$ $\forall n \geq 1$. We say that f is *countably expansive* (write $f \in \mathcal{CE}$) if there is a constant $\delta > 0$ such that for all $x \in M$, $\Gamma^+_\delta(x) = \{y \in M : d(f^i(x), f^i(y)) \leq \delta$ $\forall i \in \mathbb{Z}\}$ is countable. In Reference [19], the authors showed that if a homeomorphism $f : M \to M$ is measure expansive then f is countably expansive. Moreover, the converse is true. Then, as in the proof of Theorem 2.1 in Reference [19], it is easy to show that f is positively countable-expansive if and only if f is positively measure expansive. In this paper, we consider the relationship between the positively measure-expansive and the positively continuum-wise expansive (see Lemma 1). We can know that if f is positively measure-expansive then it is not positively continuum-wise expansive because a continuum is not countable, in general.

Definition 2. *A differentiable map $f : M \to M$ is expanding if there exist constants $C > 0$ and $\lambda > 1$ such that*

$$\|D_x f^n(v)\| \geq C\lambda^n \|v\|,$$

for any vector $v \in T_xM (x \in M)$ and any $n \geq 0$.

Note that a positively measure-expansive differentiable map is not necessarily expanding. However, under the C^1 robust or C^1 generic condition, it is true.

A differentiable map f is C^1 *robustly positive* $\mathfrak{P}$ if there exists a C^1 neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, g is positive $\mathfrak{P}$.

A point $x \in M$ is a *singular* if $D_x f : T_xM \to T_{f(x)}M$ is not injective. Denoted by S_f the set of singular points of f.

Sakai proved in Reference [15] that if a differentiable map f is C^1 robustly positive expansive then $S_f = \emptyset$ and it is an expanding map. Lee et al. [17] proved that if f is C^1 robustly positive measure-expansive, then $S_f = \emptyset$ and it is expanding. Note that if a differentiable map f is expanding then it is expansive. According to these facts, we prove the following.

Theorem A *If a differentiable map $f : M \to M$ is C^1 robustly positive continuum-wise expansive (write $f \in \mathcal{RPCWE}$) then $S_f = \emptyset$ and it is expanding.*

Let $D^1(M)$ be the set of differentiable maps $f : M \to M$. Note that $D^1(M)$ contains the set of diffeomorphisms $\mathrm{Diff}^1(M)$ on M and $\mathrm{Diff}^1(M)$ is open in $D^1(M)$. We say that a subset

$\mathcal{G} \subset D^1(M)$ is *residual* if it contains a countable intersection of open and dense subsets of $D^1(M)$. Note that the countable intersection of residual subsets is a residual subset of $D^1(M)$. A property "P" holds *generically* if there exists a residual subset $\mathcal{G} \subset D^1(M)$ such that for any $f \in \mathcal{G}$, f has the "P". Some times we write for C^1 generic $f \in D^1(M)$ which means that there exists a residual set $\mathcal{G} \subset D^1(M)$ such that for any $f \in \mathcal{G}$. Arbieto [8] and Sakai [15] proved that, C^1 generically, a positively expansive map is expanding. Ahn et al. [16] proved that for a C^1 generic $f \in D^1(M)$, if $S_f = \emptyset$ and f is positively measure expansive, then it is expanding. Recently, Lee et al. [17] showed that, C^1 generically, if $f \in D^1(M)$ is positively measure-expansive then $S_f = \emptyset$ and f is expanding. According to these results, we consider C^1 generic positively continuum-wise expansive for $f \in D^1(M)$ and prove the following.

Theorem B *For C^1 generic $f \in D^1(M)$, if f is positively continuum-wise expansive then $S_f = \emptyset$ and it is expanding.*

2. The Proof of Theorem A

The following proof is similar to Lemma 2.2 in Reference [19].

Lemma 1. *Let $C \subset M$ be compact and connected. A differentiable map $f \in \mathcal{PCWE}$ if and only if there is a constant $\delta > 0$ such that for all $x \in M$, if a continuum $C \subset \Gamma^+_\delta(x)$ then C is a trivial continuum set.*

Proof. Let $\delta > 0$ be a continuum-wise expansive constant and C be compact and connected (that is, a continuum). Take $c = \delta/2$.. We assume that for any $x \in M$, if $C \subset \Gamma^+_c(x)$ then $\mathrm{diam} f^n(C) \leq 2c$ for all $n \geq 0$. Since f is positively continuum-wise expansive, C should be a trivial continuum set. Thus, if $f \in \mathcal{PCWE}$, then for all $x \in M$, if a continuum $C \subset \Gamma^+_c(x)$, then C is a trivial continuum set.

For the converse part, suppose that $f \in \mathcal{PCWE}$. Then, there is a constant $c > 0$ such that $\mathrm{diam} f^n(C) \leq c \; \forall n \geq 0$, where C is a continuum. Let $x \in C$ be given. Since $\mathrm{diam} f^n(C) \leq c$, for all $y \in C$ we have

$$d(f^n(x), f^n(y)) \leq c \forall n \geq 0.$$

Thus, we know $y \in \Gamma_c(x)$. Since $y \in C$ and y is arbitrary, we have $C \subset \Gamma_c(x)$. Since a continuum $C \subset \Gamma_c(x)$, we have that C is a trivial continuum set. □

A periodic point $p \in P(f)$ is *hyperbolic* if $D_p f^{\pi(p)} : T_pM \to T_pM$ has no eigenvalue with a modulus equal to 0 or 1, where $\pi(p)$ is the period of p. Then, $T_pM = E^s_p \oplus E^u_p$ of subspaces such that

(a) $D_p f^{\pi(p)}(E^\sigma_p) = E^\sigma_p (\sigma = s, u)$, and
(b) there exist constants $C > 0$, and $\lambda \in (0,1)$ satisfies for all positive integer $n \in \mathbb{N}$,

- $\| D_p f^n(v) \| \leq C\lambda^n \| v \|$ for any $v \in E^s_p$, and
- $\| D_p f^{-n}(v) \| \leq C\lambda^n \| v \|$ for any $v \in E^u_p$

A hyperbolic point $p \in P(f)$ is a *sink* if $E^u_p = \{0\}$, a *source* if $E^s_p = \{0\}$, and a *saddle* if $E^s_p \neq \{0\}$ and $E^u_p \neq \{0\}$. Let $P_h(f)$ be the set of hyperbolic periodic points of f. The dimension of the stable manifold $W^s(p) = \{x \in M : d(f^i(x), f^i(p)) \to 0 \text{ as } i \to \infty\}$ is written by the *index* of p, and denoted by $\mathrm{ind}(p)$. Then, we know $0 \leq \mathrm{ind}(p) \leq \dim M$. Let $P_i(f)$ be the set of all $p \in P_h(f)$ with $\mathrm{ind}(p) = i$.

Lemma 2. *If a differentiable map $f \in \mathcal{PCWE}$ then $P_i(f) = \emptyset$ for $1 \leq i \leq \dim M$.*

Proof. By contradiction, we assume that there is $i \in [1, \dim M]$ such that $P_i(f) \neq \emptyset$. Take $p \in P_i(f)$ and $\delta > 0$. Then, we can find a local stable manifold $W^s_\delta(p)$ of p such that $W^s_\delta(p) \neq \emptyset$. We can construct a continuum $\mathcal{J}_p \subset W^s_\delta(p)$ centered at p such that $\mathrm{diam}\mathcal{J}_p = \delta/4$. Let $\Gamma^+_{\delta/2}(p) = \{y \in M :$

$d(f^i(p), f^i(y)) \leq \delta/2\ \forall i \geq 0\}$. Then, we know $\mathcal{J}_p \subset \Gamma^+_{\delta/2}(p)$. By Lemma 1, $\mathcal{J}_p$ should be a trivial continuum set. This is a contradiction since $\mathcal{J}_p$ is not a trivial continuum set. □

In Reference [17], the authors showed that there is a positively expansive differentiable map $f : S^1 \to S^1$ such that $S_f \neq \emptyset$. Thus, if f is positively measure-expansive then $S_f \neq \emptyset$. But if f is C^1 robustly positive measure-expansive then $S_f = \emptyset$. For that, we consider that f is C^1 robustly positive continuum-wise expansive.

The following is a version of differentiable maps of Franks' lemma (see Lemma 2.1 in Reference [8]).

Lemma 3 ([20])**.** *Let $f : M \to M$ be a differentiable map and let $\mathcal{U}(f)$ be a C^1 neighborhood of f. Then, there exists $\delta > 0$ such that for a finite set $A = \{x_1, x_2, \ldots, x_n\} \subset M$, a neighborhood U of A and a linear map $L_i : T_{x_i}M \to T_{f(x_i)}M$ satisfying $\|L_i - D_{x_i}f\| < \delta$ for $1 \leq i \leq n$, there exist $\varepsilon_0 > 0$ and $g \in \mathcal{U}(f)$ having the following properties;*

(a) *$g(x) = f(x)$ if $x \in A$, and*
(b) *$g(x) = \exp_{f(x_i)} \circ L_i \circ \exp^{-1}_{x_i}(x)$ if $x \in B_{\varepsilon_0}(x_i)$ and $\forall i \in \{1, \ldots, n\}$.*

It is clear that assertion (b) implies that

$$g(x) = f(x) \quad \text{if} \quad x \in A$$

and that $D_{x_i}g = L_i, \forall i \in \{1, \ldots, n\}$.

Theorem 1. *If a differentiable map $f \in \mathcal{RPCWE}$ then $S_f = \emptyset$.*

Proof. Suppose that there is $x \in S_f$. Then, by Lemma 3, we can take g C^1 close to f such that g has a closed connected small arc $B_\epsilon(x)$ centered at x with radius $\epsilon > 0$, such that $\dim B_\epsilon(x) = 1$ and $g(B_\epsilon(x))$ is one point. Take $\delta = 2\epsilon$. Let $\Gamma^+_\delta(x) = \{y \in M : d(g^i(x), g^i(y)) \leq \delta\ \forall i \geq 0\}$. It is clear $B_\epsilon(x) \subset \Gamma^+_\delta(x)$. Since $g(B_\epsilon(x))$ is one point, for any $y \in B_\epsilon(x)$, we know that $\operatorname{diam} g^i(B_\epsilon(x)) \leq \delta$ for all $i \geq 0$. However, $B_\epsilon(x)$ is not a trivial continuum set, by Lemma 1 this is a contradiction. □

Recall that a differentiable map $f : M \to M$ is *star* if every periodic point of $g(C^1$ nearby $f)$ is hyperbolic.

Lemma 4. *If a differentiable map $f \in \mathcal{RPCWE}$ then f is star.*

Proof. Suppose that f is not star. Then, we can take g C^1 close to f such that g has a non-hyperbolic $p \in P(g)$. As Lemma 3, we can find g_1 C^1 close to g (g_1 C^1 close to f) such that $D_p g_1^{\pi(p)}$ has an eigenvalue λ with $|\lambda| = 1$. For simplicity, we assume that $g_1^{\pi(p)}(p) = g_1(p) = p$. Let E^c_p be associated with λ. If $\lambda \in \mathbb{R}$ then $\dim E^c_p = 1$, and if $\lambda \in \mathbb{C}$ then $\dim E^c_p = 2$.

First, we consider $\dim E^c_p = 1$. Then, we assume that $\lambda = 1$ (the other case can be proved similarly). By Lemma 3, there are $\epsilon > 0$ and h C^1 close to g_1 (also, C^1 close to f), having the following properties;

- $h(p) = g_1(p) = p$,
- $h(x) = \exp_p \circ D_p g_1 \circ \exp^{-1}_p(x)$ if $x \in B_\epsilon(p)$, and
- $h(x) = g_1(x)$ if $x \notin B_{4\epsilon}(p)$.

Since $\lambda = 1$, we can construct a closed connected small arc $\mathcal{I}_p \subset B_\epsilon(p) \cap \exp_p(E^c_p(\epsilon))$ with its center at p such that

- $\operatorname{diam}\mathcal{I}_p = \epsilon/4$,
- $h(\mathcal{I}_p) = \mathcal{I}_p$, and
- the map $h|_{\mathcal{I}_p} : \mathcal{I}_p \to \mathcal{I}_p$ which is the identity.

Take $\delta = \epsilon/2$. Let $\Gamma^+_\delta(p) = \{x \in M : d(h^i(x), h^i(p)) \le \delta \ \forall i \ge 0\}$. Then, it is clear $\mathcal{I}_p \subset \Gamma_\delta(p)$, and diam$h^i(\mathcal{I}_p) =$ diam$\mathcal{I}_p$ for all $i \ge 0$. Since $f \in \mathcal{RPCWE}$, according to Lemma 1, $\mathcal{I}_p$ has to be just a trivial continuum set. This is a contradiction since $\mathcal{I}_p$ is not a trivial continuum set.

Finally, we consider dim$E^c_p = 2$. For convenience, we assume that $g^{\pi(p)}(p) = g(p) = p$. As Lemma 3, we can find $\epsilon > 0$ and $g_1 \in \mathcal{U}(f)$, which has the following properties;

- $g_1(p) = g(p) = p$,
- $g_1(x) = \exp_p \circ D_p g \circ \exp_p^{-1}(x)$ if $x \in B_\epsilon(p)$, and
- $g_1(x) = g(x)$ if $x \notin B_{4\epsilon}(p)$.

For any $v \in E^c_p(\epsilon)$, there is $l > 0$ such that $D_p g^l(v) = v$. Take $u \in E^c_p(\epsilon)$ such that $\|u\| = \epsilon/2$. As in the previous arguments, we can construct a closed connected small arc $\mathcal{J}_p \subset B_\epsilon(p) \cap \exp_p(E^c_p(\epsilon))$ such that

- diam$\mathcal{J}_p = \epsilon/4$,
- $g^l_1(\mathcal{J}_p) = \mathcal{J}_p$, and
- $g^l_1|_{\mathcal{J}_p} : \mathcal{J}_p \to \mathcal{J}_p$ is the identity map.

As in the proof of the first case, take $\delta = \epsilon/2$. Let $\Gamma^+_\delta(p) = \{x \in M : d(g^{li}_1(x), g^{li}_1(p) \le \delta \ \forall i \ge 0\}$. It is clear that $\mathcal{J}_p \subset \Gamma^+_\delta(p)$. Then, by Lemma 1, $\mathcal{J}_p$ must be a trivial continuum set but it is not possible since $\mathcal{J}_p$ is a closed connected small arc. Thus, if $f \in \mathcal{RPCWE}$ then f is star. □

The differentiable maps $f, g : M \to M$ are *conjugate* if there is a homeomorphism $h : M \to M$ such that $f \circ h = h \circ g$. We say that a differentiable map f is *structurally stable* if there is a C^1 neighborhood $\mathcal{U}(f)$ of $f \in D^1(M)$ such that for any $g \in \mathcal{U}(f)$, g is conjugate to f. A differentiable map f is Ω *stable* if there is a C^1 neighborhood $\mathcal{U}(f)$ of $f \in D^1(M)$ such that for any $g \in \mathcal{U}(f)$, $g|_{\Omega(g)}$ is conjugate to $f|_{\Omega(f)}$, where $\Omega(f)$ denotes the nonwandering points of f. Przytycki proved in Reference [21] that if f is an Anosov differentiable map then it is not an Anosov diffeomorphism or expandings which are not structurally stable. Moreover, assume that f is Axiom A (i.e., $\overline{P(f)} = \Omega(f)$ is hyperbolic) and has no singular points in the nonwandering set $\Omega(f)$. Then f is Ω stable if and only if f is strong Axiom A and has no cycles (see Reference [22]). Here, f is *strong Axiom A* means that f is Axiom A and $\Omega(f)$ is the disjoint union $\Lambda_1 \cup \Lambda_2$ of two closed f invariant sets.

According to the above results of a diffeomorphism $f \in \mathrm{Diff}^1(M)$, one can consider the case of a differentiable $f \in D^1(M)$ which is an extension of a diffeomorphism. For instance, a diffeomorphism $f \in \mathrm{Diff}(M)$ is said to be *star* if we can choose a C^1 neighborhood $\mathcal{U}(f)$ of f such that every periodic point of g is hyperbolic, for all $g \in \mathcal{U}(f)$.

If a diffeomorphism f is star then f is Axiom A and has no cycles (see References [23,24]). Aoki et al. Theorem A in Reference [25] proved that if a differentiable map f is star and the nonwandering set $\Omega(f) \cap S_f \subset \{p \in P(f) : p \text{ is a sink }\}$ then f is Axiom A and has no cycles.

Theorem 2. *Let $f \in D^1(M)$. If $f \in \mathcal{RPCWE}$ then f is Axiom A and has no cycles.*

Proof. Suppose that $f \in \mathcal{RPCWE}$. As Lemma 4, f is star. By Theorem 1, we know $S_f = \emptyset$, and so, $\Omega(f) \cap S_f = \emptyset$. By Lemma 2, there do not exist sinks in $P(f)$, that is, $\{p \in P(f) : p \text{ is a sink }\} = \emptyset$. Thus, by Theorem A in Reference [25], f is Axiom A and has no cycles. □

Proof of Theorem A. Suppose that $f \in \mathcal{RPCWE}$. Then, by Lemma 2, Theorem 2 and Proposition 2.7 in [17], $\Omega(f) = \overline{P_0(f)}$ is hyperbolic and $\overline{P_0(f)}$ is expanding. Then, by Lemma 2.8 in Reference [17], $M = \overline{P_0(f)}$. Thus, f is expanding. □

3. The Proof of Theorem B

Denote by $\mathcal{KS}$ the set of Kupka–Smale C^1 maps of M. By Shub [26], $\mathcal{KS}$ is a residual set of $D^1(M)$. If $f \in \mathcal{KS}$ then every $p \in P(f)$ is hyperbolic. Then, we can see the following.

Lemma 5. *Let $f \in \mathcal{KS}$. If $f \in \mathcal{PCWE}$ then $P(f) = P_0(f)$.*

Proof. Let $f \in \mathcal{PCWE}$. Suppose, by contradiction, that $P_i(f) \neq \emptyset$ for some $1 \leq i \leq \dim M$. Take $p \in P_i(f)$ and $\delta > 0$. Then, we can define a local stable manifold $W^s_\delta(p)$ of p such that $W^s_\delta(p) \neq \emptyset$. We can construct a closed connected small arc $\mathcal{J}_p \subset W^s_\delta(p)$ with its center at p such that $\mathrm{diam}\mathcal{J}_p = \delta/4$. Let $\Gamma^+_\delta(p) = \{x \in M : d(f^i(x), f^i(p)) \leq \delta \text{ for all } i \geq 0\}$. Then, it is clear $\mathcal{J}_p \subset \Gamma^+_\delta(p)$. Since $f \in \mathcal{PCWE}$, by Lemma 1, $\mathcal{J}_p$ must be a trivial continuum set. This is a contradiction since $\mathcal{J}_p$ is not a trivial continuum set. Thus, every $p \in P(f)$ is a source so that $P(f) = P_0(f)$. □

Lemma 6. Lemma 8 in [15]. *There exists a residual set $\mathcal{G}_1 \subset D^1(M)$ such that for given $f \in \mathcal{G}_1$, if for any C^1 neighborhood $\mathcal{U}(f)$ of f there exist $g \in \mathcal{U}(f)$ and $p \in P_h(g)$ with $\mathrm{ind}(p) = i (0 \leq i \leq \dim M)$, then there is $p' \in P_h(f)$ with $\mathrm{ind}(p') = i$.*

Lemma 7. *There exists a residual subset $\mathcal{G}_2 \subset D^1(M)$ such that for a given $f \in \mathcal{G}_2$, if $f \in \mathcal{PCWE}$ then $S_f \cap \overline{P_0(f)} = \emptyset$.*

Proof. Let $f \in \mathcal{G}_2 = \mathcal{KS} \cap \mathcal{G}_1$ and $f \in \mathcal{PCWE}$. Suppose, by contradiction, that $S_f \cap \overline{P_0(f)} \neq \emptyset$. Since $S_f \cap \overline{P_0(f)} \neq \emptyset$, we can choose a point $x \in S_f \cap \overline{P_0(f)}$. Then, we can find a sequence of periodic points $\{p_n\} \subset P_0(f)$ with period $\pi(p_n)$ such that $p_n \to x$ as $n \to \infty$. As Lemma 3, there exists g C^1 close to f such that $g^{\pi(p_n)}(p_n) = p_n$ and $p_n \in S_g$. Again using Lemma 3, there exists g_1 C^1 closed to g such that g_1 C^1 is close to f, $g_1^{\pi(p_n)}(p_n) = p_n$, and $\mathrm{ind}(p_n) = i(1 \leq i \leq \dim M)$. Since $f \in \mathcal{G}_1$, by Lemma 6, f has a hyperbolic saddle periodic point q with $\mathrm{index}(q) = i(1 \leq i \leq \dim M)$. This is a contradiction by Lemma 2. □

For a $\delta > 0$, a point $p \in P(f)(f^{\pi(p)}(p) = p)$ said to be a *δ-hyperbolic* (see Reference [27]) if for an eigenvalue of $Df^{\pi(p)}(p)$, we can take an eigenvalue λ of $Df^{\pi(p)}(p)$ such that

$$(1-\delta)^{\pi(p)} < |\lambda| < (1+\delta)^{\pi(p)}.$$

Lemma 8. *There exists a residual subset $\mathcal{G}_3 \subset D^1(M)$ such that for a given $f \in \mathcal{G}_3$, if $f \in \mathcal{PCWE}$, then we can take $\delta > 0$ such that f has no δ-hyperbolic.*

Proof. Let $f \in \mathcal{G}_3 = \mathcal{KS} \cap \mathcal{G}_1 \cap \mathcal{G}_2$, and let $f \in \mathcal{PCWE}$. Since $f \in \mathcal{KS} \cap \mathcal{G}_1 \cap \mathcal{G}_2$, by Lemma 2 and Lemma 7, we know $S_f \cap \overline{P_0(f)} = \emptyset$. Assume that for any $\delta > 0$, there is a $p \in P_h(f)$ with a δ-hyperbolic. By Lemma 3, we can take g C^1 close to f such that p has an eigenvalue with modulus one. Again using Lemma 3, there exists g_1 C^1 close to g (g_1 C^1 close to f) such that g_1 has a saddle $q \in P_h(g_1)$ with $\mathrm{ind}(q) = i(1 \leq i \leq \dim M)$, where $P_h(g_1)$ is the set of all hyperbolic periodic points of g_1. Since $f \in \mathcal{G}_1$, f has a saddle $q' \in P_h(f)$ with $\mathrm{ind}(q') = i(1 \leq i \leq \dim M)$. This is a contradiction by Lemma 2. □

Lemma 9. Lemma 7 in Reference [15]. *There exists a residual subset $\mathcal{G}_4 \subset D^1(M)$ such that for a given $f \in \mathcal{G}_4$ and $\delta > 0$, if any C^1 neighborhood $\mathcal{U}(f)$ of f there exist $g \in \mathcal{U}(f)$ and $p \in P_h(g)$ with a δ-hyperbolic, then we can find $p' \in P_h(f)$ with a 2δ-hyperbolic.*

Lemma 10. *There exists a residual subset $\mathcal{G}_5 \subset D^1(M)$ such that for a given $f \in \mathcal{G}_5$, if $f \in \mathcal{PCWE}$ then f is star.*

Proof. Let $f \in \mathcal{G}_5 = \mathcal{G}_3 \cap \mathcal{G}_4$ and $f \in \mathcal{PCWE}$. Suppose that f is not star. Then, as Lemma 3, we can take g C^1 close to f such that g has a $q \in P_h(g)$ with a $\delta/2$-hyperbolic for some $\delta > 0$. Since $f \in \mathcal{G}_4$, f has a hyperbolic periodic point p' with a δ-hyperbolic. This is a contradiction by Lemma 8. □

The following is a differentiable version of closing Lemma under the generic sense (see Theorem 1 in Reference [28]). Then we set $\mathcal{CL}$ is the residual subset in $D^1(M)$ such that for any $f \in \mathcal{CL}$,

$\Omega(f) = \overline{P}(f)$.

Proof of Theorem B. Let $f \in \mathcal{G} = \mathcal{G}_5 \cap \mathcal{CL}$ and $f \in \mathcal{PCWE}$. It is enough to show that $M = \overline{P_0(f)}$. By Lemmas 5 and 7, $P(f) = P_0(f)$ and $S_f \cap \overline{P_0(f)} = \emptyset$. Since $f \in \mathcal{CL}$, $\Omega(f) = \overline{P(f)}$. According to Lemma 10, f is star, and so $\{\Omega(f) \setminus \overline{P(f)}\} \cap S_f = \emptyset$. Thus we have $\Omega(f) = \overline{P(f)} = \overline{P_0(f)}$ is hyperbolic. As Proposition 2.7 in Reference [17], we have that $\overline{P_0(f)}$ is expanding. Then, as in the proof of Lemma 3.8 in Reference [17], we have $M = \overline{P_0(f)}$. □

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