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Positive Solutions for a System of Fractional Integral Boundary Value Problems of Riemann–Liouville Type Involving Semipositone Nonlinearities

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Abstract: In this work by the index of fixed point and matrix theory, we discuss the positive solutions for the system of Riemann–Liouville type fractional boundary value problems

$$D_{0+}^{\alpha}u(t)+f_1(t,u(t),v(t),w(t))=0, t\in(0,1),$$

$$D_{0+}^{\alpha}v(t)+f_2(t,u(t),v(t),w(t))=0, t\in(0,1),$$

$$D_{0+}^{\alpha}w(t)+f_3(t,u(t),v(t),w(t))=0, t\in(0,1),$$

$$u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, D_{0+}^{p}u(t)|_{t=1}=\int_0^1 h(t)D_{0+}^{q}u(t)dt,$$

$$v(0)=v'(0)=\cdots=v^{(n-2)}(0)=0, D_{0+}^{p}v(t)|_{t=1}=\int_0^1 h(t)D_{0+}^{q}v(t)dt,$$

$$w(0)=w'(0)=\cdots=w^{(n-2)}(0)=0, D_{0+}^{p}w(t)|_{t=1}=\int_0^1 h(t)D_{0+}^{q}w(t)dt,$$

where $\alpha\in(n-1,n]$ with $n\in\mathbb{N}$, $n\geq 3$, $p,q\in\mathbb{R}$ with $p\in[1,n-2]$, $q\in[0,p]$, D_{0+}^{α} is the α order Riemann–Liouville type fractional derivative, and $f_i(i=1,2,3)\in C([0,1]\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R})$ are semipositone nonlinearities.

Keywords: Riemann–Liouville type fractional problem; positive solutions; the index of fixed point; matrix theory

1. Introduction

In this work the positive solutions for the system of fractional boundary value problems involving Riemann–Liouville type are considered:

$$\begin{cases} D_{0+}^{\alpha}u(t)+f_1(t,u(t),v(t),w(t))=0, t\in(0,1),\\ D_{0+}^{\alpha}v(t)+f_2(t,u(t),v(t),w(t))=0, t\in(0,1),\\ D_{0+}^{\alpha}w(t)+f_3(t,u(t),v(t),w(t))=0, t\in(0,1),\\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, D_{0+}^{p}u(t)|_{t=1}=\int_0^1 h(t)D_{0+}^{q}u(t)dt,\\ v(0)=v'(0)=\cdots=v^{(n-2)}(0)=0, D_{0+}^{p}v(t)|_{t=1}=\int_0^1 h(t)D_{0+}^{q}v(t)dt,\\ w(0)=w'(0)=\cdots=w^{(n-2)}(0)=0, D_{0+}^{p}w(t)|_{t=1}=\int_0^1 h(t)D_{0+}^{q}w(t)dt, \end{cases} \tag{1}$$

where D^{α}_{0+} is the α order Riemann–Liouville type fractional derivative, the constants α, p, q, n, and the functions h, $f_i(i = 1,2,3)$ satisfy the assumptions

(C0) $n \in \mathbb{N}, n \geq 3, \alpha \in (n-1, n], p \in [1, n-2], q \in [0, p]$,

(C1) there exists h with $h(t) \geq 0(\not\equiv 0)$ on $[0,1]$ such that $A := \frac{\Gamma(\alpha)}{\Gamma(\alpha-p)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \int_0^1 h(t)t^{\alpha-q-1}dt > 0$,

(C2) $f_i(i = 1,2,3) \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$, and there is a $M > 0$ such that

$$f_i(t, x_1, x_2, x_3) \geq -M, \text{ for } (t, x_1, x_2, x_3) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, i = 1,2,3.$$

Fractional calculus theory shows undoubted advantages in aerodynamics, electrodynamics in complex medium, the theory of control, signal and image processing, rheology, and many other issues, see the books [1–3]. The study of such kind of problems has received considerable attention in the previous studies, see for instance [4–79] and the references therein.

In [4] by the fixed point theorem of Guo–Krasnosel'skii, the authors discussed the positive solutions for the multi-point Riemann–Liouville fractional boundary value problems

$$\begin{cases} D^{\alpha}_{0+}u(t) + \lambda f(t, u(t)) = 0, t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ D^{p}_{0+}u(t)|_{t=1} = \sum_{i=1}^{m} a_i D^{q}_{0+}u(t)|_{t=\xi_i} \end{cases} \tag{2}$$

where f is a sign-changing nonlinearity. In [5], the authors studied the multiple positive solutions for the problem (2) ($\lambda = 1$), where f is a sign-changing nonlinearity, and permits singularities on t and u. In [6], by means of the index of fixed point, the authors researched the positive solutions for the boundary value problems of Hadamard fractional equations

$$\begin{cases} -{}^{H}D^{\alpha}u(t) = f(t, u(t)), \quad t \in [1, e], \\ u(1) = \delta u(1) = \delta u(e) = 0, \end{cases} \tag{3}$$

where f is a sign-changing nonlinearity, and may grow superlinearly and sublinearly at ∞.

The fractional-order equations in systems have also been widely investigated in the literature, see for example [52–79]. In [52], the authors studied the system of Hadamard fractional integral boundary value problems

$$\begin{cases} {}^{H}D^{\beta}u(t) + f_1(t, u(t), v(t)) = 0, \quad 1 < t < e, \\ {}^{H}D^{\beta}v(t) + f_2(t, u(t), v(t)) = 0, \quad 1 < t < e, \\ u(1) = v(1) = u'(1) = v'(1) = 0, \\ u(e) = \int_1^e h(s)v(s)\frac{ds}{s}, \\ v(e) = \int_1^e g(s)u(s)\frac{ds}{s}, \end{cases} \tag{4}$$

where the nonlinearities $f_i(i = 1,2) \in C([1,e] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$.

In [53], by means of the alternative of Leray–Schauder, the authors obtained the uniqueness and existence of solutions for the system of fractional integral boundary value problems

$$\begin{cases} D^{\alpha}x(t) = f(t, x(t), y(t), D^{\gamma}y(t)), t \in [0, T], \\ D^{\beta}y(t) = g(t, x(t), D^{\delta}x(t), y(t)), t \in [0, T], \end{cases} \tag{5}$$

with the integral boundary conditions

$$\begin{cases} x(0) = h(y), \int_0^T y(s)ds = \mu_1 x(\eta), \\ y(0) = \phi(x), \int_0^T x(s)ds = \mu_2 y(\xi), \end{cases}$$

where $D^{\alpha}, D^{\beta}, D^{\delta}, D^{\gamma}$ are the fractional derivatives of Caputo type.

In [54], the authors studied the positive solutions of the abstract fractional semipositone differential system with integral boundary conditions, which arises from HIV infection models

$$\begin{cases} D_{0+}^{\alpha}u(t)+\lambda f\left(t,u(t),D_{0+}^{\beta}u(t),v(t)\right)=0, \\ D_{0+}^{\gamma}v(t)+\lambda g(t,u(t))=0,\ 0<t<1, \\ D_{0+}^{\beta}u(0)=D_{0+}^{\beta+1}u(0)=0,\ D_{0+}^{\beta}u(1)=\int_0^1 D_{0+}^{\beta}u(s)dA(s), \\ v(0)=v'(0)=0,\ v(1)=\int_0^1 v(s)dB(s), \end{cases} \tag{6}$$

where f,g are the semipositone nonlinearities (so-called semipositone problems), which originally modeled nonlinear phenomena of chemical reactions by Dutch chemist Aris [80]. For some relevant work, we refer the reader to [4–7,71–75].

Motivated by the works aforementioned, in this work we use the index of fixed point and nonnegative matrix theory to study the positive solutions for the system of Riemann–Liouville type fractional boundary value problems (1). We first transform our problem into the equivalent system of Hammerstein type integral equations, and establish some nonnegative operator equations. Then, using some superlinear and sublinear conditions for our nonlinearities, we obtain two existence theorems. Finally, we offer two examples to explain our main theorems.

2. Preliminaries

Now, we offer the definition of the $\alpha(>0)$ order Riemann–Liouville type fractional derivative, which is given by

$$D_{0+}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_0^t(t-s)^{n-\alpha-1}f(s)ds,$$

where $f:(0,+\infty)\to(-\infty,+\infty)$ is a continuous function, and $n=[\alpha]+1$. For more materials, we refer to the books [1–3].

Lemma 1. *lSuppose that (C0)–(C1) hold. Let $f\in C[0,1]$, then the problem*

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t)=0, t\in(0,1), \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, D_{0+}^{p}u(t)|_{t=1}=\int_0^1 h(t)D_{0+}^{q}u(t)dt, \end{cases} \tag{7}$$

has a solution, which can take the form

$$u(t)=\int_0^1 G(t,s)f(s)ds,$$

where

$$G(t,s)=g_1(t,s)+\frac{t^{\alpha-1}}{A}\int_0^1 h(t)g_2(t,s)dt,$$

and

$$g_1(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-p-1}-(t-s)^{\alpha-1}, & 0\le s\le t\le 1, \\ t^{\alpha-1}(1-s)^{\alpha-p-1}, & 0\le t\le s\le 1, \end{cases} \tag{8}$$

$$g_2(t,s)=\frac{1}{\Gamma(\alpha-q)}\begin{cases} t^{\alpha-q-1}(1-s)^{\alpha-p-1}-(t-s)^{\alpha-q-1}, & 0\le s\le t\le 1, \\ t^{\alpha-q-1}(1-s)^{\alpha-p-1}, & 0\le t\le s\le 1. \end{cases} \tag{9}$$

Proof. Using similar arguments in ([4], [Lemma 1 and 2]), we have

$$u(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n}-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds,$$

where $c_i \in \mathbb{R}$, $i = 1, 2, ..., n$. Note that $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, and thus $c_2 = \cdots = c_n = 0$. Consequently, we get

$$u(t) = c_1 t^{\alpha-1} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds.$$

Therefore, we find

$$D_{0+}^{p} u(t) = c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-p)} t^{\alpha-p-1} - I_{0+}^{\alpha-p} f(t),\ D_{0+}^{q} u(t) = c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} t^{\alpha-q-1} - I_{0+}^{\alpha-q} f(t).$$

Using the condition $D_{0+}^{p} u(t)|_{t=1} = \int_0^1 h(t) D_{0+}^{q} u(t)dt$, we have

$$c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-p)} - \frac{1}{\Gamma(\alpha-p)} \int_0^1 (1-s)^{\alpha-p-1} f(s)ds = c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \int_0^1 h(t) t^{\alpha-q-1} dt - \frac{1}{\Gamma(\alpha-q)} \int_0^1 h(t) \int_0^t (t-s)^{\alpha-q-1} f(s)dsdt.$$

Solving this equation, we obtain

$$c_1 = \frac{1}{A\Gamma(\alpha-p)} \int_0^1 (1-s)^{\alpha-p-1} f(s)ds - \frac{1}{A\Gamma(\alpha-q)} \int_0^1 h(t) \int_0^t (t-s)^{\alpha-q-1} f(s)dsdt.$$

As a result, we get

$$\begin{aligned}
u(t) &= \frac{1}{A\Gamma(\alpha-p)} \int_0^1 t^{\alpha-1}(1-s)^{\alpha-p-1} f(s)ds - \frac{t^{\alpha-1}}{A\Gamma(\alpha-q)} \int_0^1 h(t) \int_0^t (t-s)^{\alpha-q-1} f(s)dsdt - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1}(1-s)^{\alpha-p-1} f(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds + \left[\frac{1}{A\Gamma(\alpha-p)} - \frac{1}{\Gamma(\alpha)}\right] \int_0^1 t^{\alpha-1}(1-s)^{\alpha-p-1} f(s)ds \\
&\quad - \frac{t^{\alpha-1}}{A\Gamma(\alpha-q)} \int_0^1 h(t) \int_0^t (t-s)^{\alpha-q-1} f(s)dsdt \\
&= \int_0^1 g_1(t,s) f(s)ds + \frac{t^{\alpha-1}}{A\Gamma(\alpha-q)} \left[\int_0^1 \int_0^1 h(t) t^{\alpha-q-1}(1-s)^{\alpha-p-1} f(s)dsdt - \int_0^1 h(t) \int_0^t (t-s)^{\alpha-q-1} f(s)dsdt\right] \\
&= \int_0^1 g_1(t,s) f(s)ds + \frac{t^{\alpha-1}}{A} \int_0^1 \int_0^1 h(t) g_2(t,s)dt f(s)ds \\
&= \int_0^1 G(t,s) f(s)ds.
\end{aligned}$$

□

Lemma 2. *(see ([4], [Lemma 3])). Suppose that (C0) holds. The functions $g_i (i = 1, 2)$ have the properties*
(i) $g_i \in C([0,1] \times [0,1], \mathbb{R}^+)$, and $g_i(t,s) > 0$ for $t, s \in (0,1)$, $i = 1, 2$,
(ii) $t^{\alpha-1} \widetilde{\varphi}(s) \le g_1(t,s) \le \widetilde{\varphi}(s)$ for all $t, s \in [0,1]$, where

$$\widetilde{\varphi}(s) = \frac{(1-s)^{\alpha-p-1} \left(1-(1-s)^p\right)}{\Gamma(\alpha)}, s \in [0,1],$$

(iii) $g_1(t,s) \le \frac{t^{\alpha-1}(1-s)^{\alpha-p-1}}{\Gamma(\alpha)}$, $t, s \in [0,1]$.

Lemma 3. *Suppose that (C0)–(C1) hold. The Green's function G has the properties*
(i) $G \in C([0,1] \times [0,1], \mathbb{R}^+)$, and $G(t,s) > 0$ for $t, s \in (0,1)$,
(ii) $t^{\alpha-1} \varphi(s) \le G(t,s) \le \varphi(s), \forall t, s \in [0,1]$, where

$$\varphi(s) = \widetilde{\varphi}(s) + \frac{1}{A} \int_0^1 h(t) g_2(t,s)dt, s \in [0,1],$$

(iii) $G(t,s) \le t^{\alpha-1} \left[\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha)} + \frac{1}{A} \int_0^1 h(t) g_2(t,s)dt\right], \forall t, s \in [0,1]$.
This is a direct result of Lemma 2, so we omit its proof.

Lemma 4. *Let $\kappa_1 = \int_0^1 t^{\alpha-1}\varphi(t)dt$, $\kappa_2 = \int_0^1 \varphi(t)dt$. Then we have the following inequalities*

$$\kappa_1\varphi(s) \le \int_0^1 G(t,s)\varphi(t)dt \le \kappa_2\varphi(s),\ \forall s \in [0,1]. \tag{10}$$

From Lemma 3(ii), we easily obtain (10).

Next we will consider the problem

$$\begin{cases} D_{0+}^{\alpha}u(t) + \widetilde{f}(t,u(t)) = 0, t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, D_{0+}^{p}u(t)|_{t=1} = \int_0^1 h(t)D_{0+}^{q}u(t)dt, \end{cases} \tag{11}$$

where $\widetilde{f}$ satisfies the condition

(C2)$'$ $\widetilde{f} \in C([0,1] \times \mathbb{R}^+, \mathbb{R})$, and there is a $M > 0$ such that

$$\widetilde{f}(t,x_1) \ge -M, \text{ for } (t,x_1) \in [0,1] \times \mathbb{R}^+.$$

Lemma 5. *Suppose that (C0)–(C1) and (C2)$'$. Then the problem (11) is equivalent to*

$$u(t) = \int_0^1 G(t,s)\widetilde{f}(s,u(s))ds, \tag{12}$$

where G is defined in Lemma 1.

Now, we take care of the following auxiliary problem associated with (11):

$$\begin{cases} D_{0+}^{\alpha}u(t) + \widetilde{F}(t,u(t)-z(t)) = 0, t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, D_{0+}^{p}u(t)|_{t=1} = \int_0^1 h(t)D_{0+}^{q}u(t)dt, \end{cases} \tag{13}$$

where $\widetilde{F}(t,x_1) = \begin{cases} \widetilde{f}(t,x_1) + M, & t \in [0,1], x_1 \ge 0, \\ \widetilde{f}(t,0) + M, & t \in [0,1], x_1 < 0, \end{cases}$ *and* $z(t) = M\int_0^1 G(t,s)ds$, *for* $t \in [0,1]$. *Then* $\widetilde{F}$ *is nonnegative continuous on* $[0,1] \times \mathbb{R}^+$, *and from Lemma 5 we have (13) is equivalent to*

$$u(t) = \int_0^1 G(t,s)\widetilde{F}(s,u(s)-z(s))ds, \tag{14}$$

where G is as in Lemma 1.

Lemma 6. *(i) If (11) has a positive solution u^*, then (13) has a solution $u^* + z$.*

(ii) If u^* is a solution for (13), and $u^*(t) \ge z(t)$ for $t \in [0,1]$, then $u^* - z$ is a positive solution for (11).

Proof. Note that z satisfies the fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha}z(t) + M = 0, t \in (0,1), \\ z(0) = z'(0) = \cdots = z^{(n-2)}(0) = 0, D_{0+}^{p}z(t)|_{t=1} = \int_0^1 h(t)D_{0+}^{q}z(t)dt. \end{cases} \tag{15}$$

Substituting $u^* + z$ into (13), we have

$$D_{0+}^{\alpha}(u^*+z)(t) + \widetilde{F}(t,u^*(t)+z(t)-z(t)) = 0 \Longrightarrow D_{0+}^{\alpha}u^*(t) + D_{0+}^{\alpha}(z)(t) + \widetilde{f}(t,u^*(t)) + M = 0.$$

Using $D_{0+}^{\alpha}(z)(t) = -M$, we have $D_{0+}^{\alpha}u^*(t) + \widetilde{f}(t,u^*(t)) = 0$, and note that u^*, z satisfy the boundary conditions in (11), (15), we obtain Lemma 6(i) holds.

Next, substituting $u^* - z$ into (11), and using $D_{0+}^{\alpha}(z)(t) = -M$ we have

$$D_{0+}^{\alpha}(u^*-z)(t) + \widetilde{f}(t,u^*(t)-z(t)) = 0 \Longrightarrow D_{0+}^{\alpha}u^*(t) - D_{0+}^{\alpha}z(t) + \widetilde{f}(t,u^*(t)-z(t)) = 0,$$

and

$$D_{0+}^{\alpha}u^*(t)+\widetilde{F}(t,u^*(t)-z(t))=0.$$

Note that u^*,z satisfy the boundary conditions in (13), (15), we obtain Lemma 6(ii) holds.

Lemma 6 implies that we only need to seek the solution u^* for (13), which is greater than z, we can obtain the positive solution u^*-z for (11).

Let $E:=C[0,1]$, $\|u\|:=\max\limits_{t\in[0,1]}|u(t)|$, $P:=\{u\in E:u(t)\geq 0,\forall t\in[0,1]\}$, $P_0=\{u\in P:u(t)\geq t^{\alpha-1}\|u\|,\forall t\in[0,1]\}$. Then $(E,\|\cdot\|)$ is a real Banach space, and P,P_0 are cones on E. Note that the relations between (13) and (14), we let an operator $T:P\to P$ as follows:

$$(Tu)(t)=\int_0^1 G(t,s)\widetilde{F}(s,u(s)-z(s))ds,\ \text{for } u\in P,t\in[0,1].$$

From the continuity of $G,\widetilde{F}$ we obtain $T:P\to P$ is a completely continuous operator, and if there exists $\overline{u}\in P\backslash\{0\}$ such that $T\overline{u}=\overline{u}$, then this $\overline{u}$ is a positive solution for (13). □

Lemma 7. $T(P)\subset P_0$.

By Lemma 3(ii) we can easily obtain this conclusion, so we omit its proof.

Note that if $\overline{u}$ is a positive fixed point of T, from Lemma 7 we have $\overline{u}\in P_0$. Moreover, when

$$\|\overline{u}\|\geq\widetilde{M}=M\int_0^1\left[\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha)}+\frac{1}{A}\int_0^1 h(t)g_2(t,s)dt\right]ds>0,$$

we have

$$\begin{aligned}\overline{u}(t)-z(t)&\geq t^{\alpha-1}\|\overline{u}\|-M\int_0^1 G(t,s)ds\\&\geq t^{\alpha-1}\|\overline{u}\|-M\int_0^1 t^{\alpha-1}\left[\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha)}+\frac{1}{A}\int_0^1 h(t)g_2(t,s)dt\right]ds\\&\geq 0.\end{aligned}$$

Then from Lemma 6 we have $\overline{u}-z$ is a positive solution for (11)*. Therefore, we only need to study the positive fixed point u^* for T, which the norm is greater than $\widetilde{M}$, then u^*-z is a positive solution for* (11)*.*

In the following two lemmas, we let X be a real Banach space and P a cone on X.

Lemma 8. *(see [81]). Let $\Omega\subset X$ be a bounded open set, and $T:\overline{\Omega}\cap P\to P$ a continuous compact operator. If there exists $\mu_0\in P\backslash\{0\}$ such that*

$$u-Tu\neq\lambda\mu_0,\forall\lambda\geq 0,u\in\partial\Omega\cap P,$$

then $i(T,\Omega\cap P,P)=0$, where i is the index of fixed point on P.

Lemma 9. *(see [81]). Let $\Omega\subset X$ be a bounded open set with $0\in\Omega$, and $T:\overline{\Omega}\cap P\to P$ a continuous compact operator. If*

$$u-\lambda Tu\neq 0,\forall\lambda\in[0,1],u\in\partial\Omega\cap P,$$

then $i(T,\Omega\cap P,P)=1$.

In what follows, in order to build our main theorems, we need to introduce some basic knowledge for nonnegative matrices, for more details see [82,83].

Definition 1. *Let $\mathcal{M}$ be a real matrix. If all elements of $\mathcal{M}$ are nonnegative, then $\mathcal{M}$ is called to be nonnegative.*

Definition 2. *A real square matrix $\mathcal{M} = (m_{ij})_{n\times n}$ is called $\mathbb{R}^n_+$-monotone, if for every column vector $x \in \mathbb{R}^n, \mathcal{M}x \in \mathbb{R}^n_+ \Longrightarrow x \in \mathbb{R}^n_+$.*

Lemma 10. *A real square matrix $\mathcal{M}$ is $\mathbb{R}^n_+$-monotone $\Longleftrightarrow \det \mathcal{M} \neq 0$, and $\mathcal{M}^{-1}$ is nonnegative.*

Remark 1. *Note that our boundary condition at $t = 1$ is integral and generalizes multi-point fractional boundary conditions. However, our problem (7) can be considered as a perturbation of the two-point boundary value problem*

$$\begin{cases} D^\alpha_{0+}u(t) + f(t) = 0, t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = D^p_{0+}u(t)|_{t=1} = 0, \end{cases} \tag{16}$$

which is equivalent to

$$u(t) = \int_0^1 g_1(t,s)f(s)ds,$$

where g_1 is defined by (8). Therefore, our method, by making good use of the original Green's function for the problem (16), will dispense with constructing a new Green's function, in contrast to some papers dealing with multi-point boundary value problems. For example, in [50] the author studied the problem

$$\begin{cases} D^\alpha_{0+}u(t) + f(t,u(t)) = 0, 0 < t < 1, \\ u(0) = 0, \beta u(\eta) = u(1), \end{cases} \tag{17}$$

where $\alpha \in (1,2], \beta\eta^{\alpha-1}, \eta \in (0,1)$. The author obtained the Green's function associated with (17) is

$$G_{Bai}(t,s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1} - \beta t^{\alpha-1}(\eta-s)^{\alpha-1} - (t-s)^{\alpha-1}\left(1-\beta\eta^{\alpha-1}\right)}{\left(1-\beta\eta^{\alpha-1}\right)\Gamma(\alpha)}, & 0 \le s \le t \le 1, s \le \eta, \\ \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}\left(1-\beta\eta^{\alpha-1}\right)}{\left(1-\beta\eta^{\alpha-1}\right)\Gamma(\alpha)}, & 0 < \eta \le s \le t \le 1, \\ \frac{[t(1-s)]^{\alpha-1} - \beta t^{\alpha-1}(\eta-s)^{\alpha-1}}{\left(1-\beta\eta^{\alpha-1}\right)\Gamma(\alpha)}, & 0 \le t \le s \le \eta < 1, \\ \frac{[t(1-s)]^{\alpha-1}}{\left(1-\beta\eta^{\alpha-1}\right)\Gamma(\alpha)}, & 0 \le t \le s \le 1, \eta \le s. \end{cases} \tag{18}$$

This function is very complicated. However, we note that this function can be expressed by

$$G_{Bai}(t,s) = g_{Bai}(t,s) + \frac{\beta t^{\alpha-1}}{1-\beta\eta^{\alpha-1}} g_{Bai}(\eta,s),$$
$$g_{Bai}(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ [t(1-s)]^{\alpha-1} & 0 \le t \le s \le 1, \end{cases}$$

where g_{Bai} is the Green's function for the problem

$$\begin{cases} D^\alpha_{0+}u(t) + f(t,u(t)) = 0, 0 < t < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{19}$$

Compared with G_{Bai}, g_{Bai} is much simpler.

3. Main Results

From the discission of Section 2, we can define the operators $T_i (i = 1,2,3) : P \times P \times P \to P$ and $T : P \times P \times P \to P \times P \times P$ as follows:

$$T_i(u,v,w)(t) = \int_0^1 G(t,s)F_i(s,u(s)-z(s),v(s)-z(s),w(s)-z(s))ds,$$
$$T(u,v,w)(t) = (T_1,T_2,T_3)(u,v,w)(t), \text{ for } t \in [0,1],$$

where $F_i(t,x_1,x_2,x_3)=\begin{cases} f_i(t,x_1,x_2,x_3)+M, & t\in[0,1],\ \text{for } x_i\geq 0, i=1,2,3,\\ f_i(t,0,0,0)+M, & t\in[0,1],\ \text{for else cases.}\end{cases}$ Consequently, if there exists $(\overline{u},\overline{v},\overline{w})$ is a positive fixed pint of T with $\|\overline{u}\|,\|\overline{v}\|,\|\overline{w}\|\geq\widetilde{M}$, then we obtain $(\overline{u}-z,\overline{v}-z,\overline{w}-z)$ is a positive solution for (1).

Now, we list our assumptions for $F_i(i=1,2,3)$:

(C3) There exist $a_{ji},b_{ji}\geq 0$ and $l_j>0(i,j=1,2,3)$ such that

$$\begin{pmatrix} F_1(t,x_1,x_2,x_3)\\ F_2(t,x_1,x_2,x_3)\\ F_3(t,x_1,x_2,x_3)\end{pmatrix}\geq\begin{pmatrix} a_{11}x_1+a_{12}x_2+a_{13}x_3-l_1\\ a_{21}x_1+a_{22}x_2+a_{23}x_3-l_2\\ a_{31}x_1+a_{32}x_2+a_{33}x_3-l_3\end{pmatrix},\forall(t,x_1,x_2,x_3)\in[0,1]\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+,$$

and the matrix $\mathcal{M}_1$ is a $\mathbb{R}^3_+$-monotone matrix, where

$$\mathcal{M}_1=\begin{pmatrix}\kappa_1a_{11}-1 & \kappa_1a_{12} & \kappa_1a_{13}\\ \kappa_1a_{21} & \kappa_1a_{22}-1 & \kappa_1a_{23}\\ \kappa_1a_{31} & \kappa_1a_{32} & \kappa_1a_{33}-1\end{pmatrix}.$$

(C4) There exists $Q_i(t)$ in $[0,1]$ such that

$$\int_0^1\varphi(t)Q_i(t)dt<\widetilde{M},\ \text{and } F_i(t,x_1,x_2,x_3)\leq Q_i(t),\forall(t,x_1,x_2,x_3)\in[0,1]\times\left[0,\widetilde{M}\right]^3,i=1,2,3.$$

(C5) There exist $\widetilde{a}_{ji},\widetilde{b}_{ji}\geq 0$ and $\widetilde{l}_j>0(i,j=1,2,3)$ such that

$$\begin{pmatrix} F_1(t,x_1,x_2,x_3)\\ F_2(t,x_1,x_2,x_3)\\ F_3(t,x_1,x_2,x_3)\end{pmatrix}\leq\begin{pmatrix} \widetilde{a}_{11}x_1+\widetilde{a}_{12}x_2+\widetilde{a}_{13}x_3+\widetilde{l}_1\\ \widetilde{a}_{21}x_1+\widetilde{a}_{22}x_2+\widetilde{a}_{23}x_3+\widetilde{l}_2\\ \widetilde{a}_{31}x_1+\widetilde{a}_{32}x_2+\widetilde{a}_{33}x_3+\widetilde{l}_3\end{pmatrix},\forall(t,x_1,x_2,x_3)\in[0,1]\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+,$$

and the matrix $\mathcal{M}_2$ is a $\mathbb{R}^3_+$-monotone matrix, where

$$\mathcal{M}_2=\begin{pmatrix}1-\kappa_2\widetilde{a}_{11} & -\kappa_2\widetilde{a}_{12} & -\kappa_2\widetilde{a}_{13}\\ -\kappa_2\widetilde{a}_{21} & 1-\kappa_2\widetilde{a}_{22} & -\kappa_2\widetilde{a}_{23}\\ -\kappa_2\widetilde{a}_{31} & -\kappa_2\widetilde{a}_{32} & 1-\kappa_2\widetilde{a}_{33}\end{pmatrix}.$$

(C6) There exists $\widetilde{Q}_i(t)$ in $[0,1]$, and $t_0\in(0,1)$ such that

$$\int_0^1\varphi(t)\widetilde{Q}_i(t)dt>\widetilde{M}t_0^{1-\alpha},\ \text{and } F_i(t,x_1,x_2,x_3)\geq\widetilde{Q}_i(t),\forall(t,x_1,x_2,x_3)\in[1,e]\times\left[0,\widetilde{M}\right]^3,i=1,2,3.$$

Let $B_\rho=\{u\in P:\|u\|<\rho\}$ for $\rho>0$ in the sequel. Then we easily have $\partial B_\rho=\{u\in P:\|u\|=\rho\},\overline{B}_\rho=\{u\in P:\|u\|\leq\rho\}$.

Theorem 1. *Suppose that (C0)–(C4) hold. Then* (1) *has a positive solution.*

Proof. We first show that:

$$(u,v,w)\neq T(u,v,w)+\lambda(\phi_1,\phi_2,\phi_3),\ \text{for } u,v,w\in\partial B_{R_1}\cap P,\ \lambda\geq 0, \tag{20}$$

where $\phi_i(i=1,2,3)$ are given elements in cone P_0, and $R_1>\widetilde{M}$. Argument by contrary, there exists $u,v,w\in\partial B_{R_1}\cap P$ and $\lambda_0\geq 0$ such that

$$(u,v,w)=T(u,v,w)+\lambda_0(\phi_1,\phi_2,\phi_3),\ \text{for } u,v,w\in\partial B_{R_1}\cap P,\ \lambda\geq 0. \tag{21}$$

This implies that

$$\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} T_1(u,v,w)(t) + \lambda_0\phi_1(t) \\ T_2(u,v,w)(t) + \lambda_0\phi_2(t) \\ T_3(u,v,w)(t) + \lambda_0\phi_3(t) \end{pmatrix} \geq \begin{pmatrix} \int_0^1 G(t,s)F_1(s,u(s)-z(s),v(s)-z(s),w(s)-z(s))ds \\ \int_0^1 G(t,s)F_2(s,u(s)-z(s),v(s)-z(s),w(s)-z(s))ds \\ \int_0^1 G(t,s)F_3(s,u(s)-z(s),v(s)-z(s),w(s)-z(s))ds \end{pmatrix}.$$

Note that Lemma 7 we have

$$u,v,w \in P_0. \tag{22}$$

From (C3) we have

$$\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} \geq \begin{pmatrix} \int_0^1 G(t,s)(a_{11}(u(s)-z(s)) + a_{12}(v(s)-z(s)) + a_{13}(w(s)-z(s)) - l_1)ds \\ \int_0^1 G(t,s)(a_{21}(u(s)-z(s)) + a_{22}(v(s)-z(s)) + a_{23}(w(s)-z(s)) - l_2)ds \\ \int_0^1 G(t,s)(a_{31}(u(s)-z(s)) + a_{32}(v(s)-z(s)) + a_{33}(w(s)-z(s)) - l_3)ds \end{pmatrix}.$$

Multiplying by $\varphi(t)$ for the above both sides, and integrating on $[0,1]$, by Lemma 4 we get

$$\begin{pmatrix} \int_0^1 u(t)\varphi(t)dt \\ \int_0^1 v(t)\varphi(t)dt \\ \int_0^1 w(t)\varphi(t)dt \end{pmatrix} \geq \begin{pmatrix} \int_0^1 \kappa_1\varphi(t)(a_{11}(u(t)-z(t)) + a_{12}(v(t)-z(t)) + a_{13}(w(t)-z(t)))dt - l_1\kappa_2^2 \\ \int_0^1 \kappa_1\varphi(t)(a_{21}(u(t)-z(t)) + a_{22}(v(t)-z(t)) + a_{23}(w(t)-z(t)))dt - l_2\kappa_2^2 \\ \int_0^1 \kappa_1\varphi(t)(a_{31}(u(t)-z(t)) + a_{32}(v(t)-z(t)) + a_{33}(w(t)-z(t)))dt - l_3\kappa_2^2 \end{pmatrix}.$$

Consequently, we find

$$\begin{aligned} \begin{pmatrix} \kappa_1 a_{11} - 1 & \kappa_1 a_{12} & \kappa_1 a_{13} \\ \kappa_1 a_{21} & \kappa_1 a_{22} - 1 & \kappa_1 a_{23} \\ \kappa_1 a_{31} & \kappa_1 a_{32} & \kappa_1 a_{33} - 1 \end{pmatrix} \begin{pmatrix} \int_0^1 u(t)\varphi(t)dt \\ \int_0^1 v(t)\varphi(t)dt \\ \int_0^1 w(t)\varphi(t)dt \end{pmatrix} &\leq \begin{pmatrix} \kappa_1(a_{11}+a_{12}+a_{13})\int_0^1 \varphi(t)z(t)dt + l_1\kappa_2^2 \\ \kappa_1(a_{21}+a_{22}+a_{23})\int_0^1 \varphi(t)z(t)dt + l_2\kappa_2^2 \\ \kappa_1(a_{31}+a_{32}+a_{33})\int_0^1 \varphi(t)z(t)dt + l_3\kappa_2^2 \end{pmatrix} \\ &\leq \begin{pmatrix} \kappa_1(a_{11}+a_{12}+a_{13})M\kappa_2^2 + l_1\kappa_2^2 \\ \kappa_1(a_{21}+a_{22}+a_{23})M\kappa_2^2 + l_2\kappa_2^2 \\ \kappa_1(a_{31}+a_{32}+a_{33})M\kappa_2^2 + l_3\kappa_2^2 \end{pmatrix}. \end{aligned}$$

Therefore, we obtain

$$\begin{pmatrix} \int_0^1 u(t)\varphi(t)dt \\ \int_0^1 v(t)\varphi(t)dt \\ \int_0^1 w(t)\varphi(t)dt \end{pmatrix} \leq \begin{pmatrix} \kappa_1 a_{11}-1 & \kappa_1 a_{12} & \kappa_1 a_{13} \\ \kappa_1 a_{21} & \kappa_1 a_{22}-1 & \kappa_1 a_{23} \\ \kappa_1 a_{31} & \kappa_1 a_{32} & \kappa_1 a_{33}-1 \end{pmatrix}^{-1} \begin{pmatrix} \kappa_1(a_{11}+a_{12}+a_{13})M\kappa_2^2 + l_1\kappa_2^2 \\ \kappa_1(a_{21}+a_{22}+a_{23})M\kappa_2^2 + l_2\kappa_2^2 \\ \kappa_1(a_{31}+a_{32}+a_{33})M\kappa_2^2 + l_3\kappa_2^2 \end{pmatrix}$$

$$= \frac{1}{\Delta_1} \begin{pmatrix} (\kappa_1 a_{22}-1)(\kappa_1 a_{33}-1)-\kappa_1^2 a_{23}a_{32} & \kappa_1^2 a_{13}a_{32}-\kappa_1 a_{12}(\kappa_1 a_{33}-1) & \kappa_1^2 a_{12}a_{23}-\kappa_1 a_{13}(\kappa_1 a_{22}-1) \\ \kappa_1^2 a_{23}a_{31}-\kappa_1 a_{21}(\kappa_1 a_{33}-1) & (\kappa_1 a_{11}-1)(\kappa_1 a_{33}-1)-\kappa_1^2 a_{13}a_{31} & \kappa_1^2 a_{13}a_{21}-\kappa_1 a_{23}(\kappa_1 a_{11}-1) \\ \kappa_1^2 a_{21}a_{32}-\kappa_1 a_{31}(\kappa_1 a_{22}-1) & \kappa_1^2 a_{12}a_{31}-\kappa_1 a_{32}(\kappa_1 a_{11}-1) & (\kappa_1 a_{11}-1)(\kappa_1 a_{22}-1)-\kappa_1^2 a_{12}a_{21} \end{pmatrix}$$

$$\cdot \begin{pmatrix} \kappa_1(a_{11}+a_{12}+a_{13})M\kappa_2^2 + l_1\kappa_2^2 \\ \kappa_1(a_{21}+a_{22}+a_{23})M\kappa_2^2 + l_2\kappa_2^2 \\ \kappa_1(a_{31}+a_{32}+a_{33})M\kappa_2^2 + l_3\kappa_2^2 \end{pmatrix},$$

where

$$\Delta_1 = \det \begin{pmatrix} \kappa_1 a_{11}-1 & \kappa_1 a_{12} & \kappa_1 a_{13} \\ \kappa_1 a_{21} & \kappa_1 a_{22}-1 & \kappa_1 a_{23} \\ \kappa_1 a_{31} & \kappa_1 a_{32} & \kappa_1 a_{33}-1 \end{pmatrix}.$$

As a result of this, there exist $\mathcal{N}_i > 0 (i = 1, 2, 3)$ such that

$$\begin{pmatrix} \int_0^1 u(t)\varphi(t)dt \\ \int_0^1 v(t)\varphi(t)dt \\ \int_0^1 w(t)\varphi(t)dt \end{pmatrix} \leq \begin{pmatrix} \mathcal{N}_1 \\ \mathcal{N}_2 \\ \mathcal{N}_3 \end{pmatrix},$$

where $\mathcal{N}_1 = \frac{1}{\Delta_1}[((\kappa_1 a_{22}-1)(\kappa_1 a_{33}-1)-\kappa_1^2 a_{23}a_{32})(\kappa_1(a_{11}+a_{12}+a_{13})M\kappa_2^2+l_1\kappa_2^2)+(\kappa_1^2 a_{13}a_{32}-\kappa_1 a_{12}(\kappa_1 a_{33}-1))(\kappa_1(a_{21}+a_{22}+a_{23})M\kappa_2^2+l_2\kappa_2^2)+(\kappa_1^2 a_{12}a_{23}-\kappa_1 a_{13}(\kappa_1 a_{22}-1))(\kappa_1(a_{31}+a_{32}+a_{33})M\kappa_2^2+l_3\kappa_2^2)]$, $\mathcal{N}_2 = \frac{1}{\Delta_1}[(\kappa_1^2 a_{23}a_{31}-\kappa_1 a_{21}(\kappa_1 a_{33}-1))(\kappa_1(a_{11}+a_{12}+a_{13})M\kappa_2^2+l_1\kappa_2^2)+((\kappa_1 a_{11}-1)(\kappa_1 a_{33}-1)-\kappa_1^2 a_{13}a_{31})(\kappa_1(a_{21}+a_{22}+a_{23})M\kappa_2^2+l_2\kappa_2^2)+(\kappa_1^2 a_{13}a_{21}-\kappa_1 a_{23}(\kappa_1 a_{11}-1))(\kappa_1(a_{31}+a_{32}+a_{33})M\kappa_2^2+l_3\kappa_2^2)]$, $\mathcal{N}_3 = \frac{1}{\Delta_1}[(\kappa_1^2 a_{21}a_{32}-\kappa_1 a_{31}(\kappa_1 a_{22}-1))(\kappa_1(a_{11}+a_{12}+a_{13})M\kappa_2^2+l_1\kappa_2^2)+(\kappa_1^2 a_{12}a_{31}-\kappa_1 a_{32}(\kappa_1 a_{11}-1))(\kappa_1(a_{21}+a_{22}+a_{23})M\kappa_2^2+l_2\kappa_2^2)+((\kappa_1 a_{11}-1)(\kappa_1 a_{22}-1)-\kappa_1^2 a_{21}a_{21})(\kappa_1(a_{31}+a_{32}+a_{33})M\kappa_2^2+l_3\kappa_2^2)]$.

Note that (22), we have

$$\begin{pmatrix} \|u\| \\ \|v\| \\ \|w\| \end{pmatrix} \leq \begin{pmatrix} \mathcal{N}_1\kappa_1^{-1} \\ \mathcal{N}_2\kappa_1^{-1} \\ \mathcal{N}_3\kappa_1^{-1} \end{pmatrix}.$$

Therefore, we can choose $R_1 > \max\{\widetilde{M}, \mathcal{N}_1\kappa_1^{-1}, \mathcal{N}_2\kappa_1^{-1}, \mathcal{N}_3\kappa_1^{-1}\}$ such that when $u, v, w \in \partial B_{R_1} \cap P$, (21) is not satisfied. This also indicates that (20) holds for $u, v, w \in \partial B_{R_1} \cap P$, and Lemma 8 indicates that

$$i(T, B_{R_1} \cap (P \times P \times P), P \times P \times P) = 0. \tag{23}$$

On the other hand, we prove that

$$(u, v, w) \neq \lambda T(u, v, w), \text{ for } u, v, w \in \partial B_{\widetilde{M}} \cap P, \lambda \in [0, 1]. \tag{24}$$

If this claim is not true, there exist $u, v, w \in \partial B_{\widetilde{M}} \cap P, \lambda_1 \in [0,1]$ such that

$$(u, v, w) = \lambda_1 T(u, v, w).$$

This implies that

$$\|u\| \leq \|T_1(u, v, w)\|, \ \|v\| \leq \|T_2(u, v, w)\|, \text{ and } \|w\| \leq \|T_3(u, v, w)\|.$$

However, from (C4) we have

$$\begin{aligned} T_1(u, v, w)(t) &= \int_0^1 G(t, s) F_1(s, u(s) - z(s), v(s) - z(s), w(s) - z(s)) ds \\ &\leq \int_0^1 \varphi(s) Q_1(s) \frac{ds}{s} \\ &< \widetilde{M}. \end{aligned}$$

Note that by (C4), $\|u\| = \widetilde{M}$. Hence, we obtain $\|T_1(u, v, w)\| < \|u\|$. Similarly, $\|T_2(u, v, w)\| < \|v\|$ and $\|T_3(u, v, w)\| < \|w\|$. This has a contradiction. Hence (24) holds. By Lemma 9 we get

$$i(T, B_{\widetilde{M}} \cap (P \times P \times P), P \times P \times P) = 1. \tag{25}$$

By use of (23) and (25) we can calculate

$$\begin{aligned} & i(T, (B_{R_1} \backslash \overline{B}_{\widetilde{M}}) \cap (P \times P \times P), P \times P \times P) \\ & = i(T, B_{R_1} \cap (P \times P \times P), P \times P \times P) - i(T, B_{\widetilde{M}} \cap (P \times P \times P), P \times P \times P) \\ & = -1. \end{aligned}$$

Therefore, T has a fixed point (u^*, v^*, w^*) on $(B_{R_1} \backslash \overline{B}_{\widetilde{M}}) \cap (P \times P \times P)$. Consequently, $(u^* - z, v^* - z, w^* - z)$ is a positive solution for (1), i.e., (1) has a positive solution. □

Theorem 2. *Suppose that (C0)–(C2), (C5)–(C6) hold. Then* (1) *has a positive solution.*

Proof. We first claim that:

$$(u, v, w) \neq \lambda T(u, v, w), \text{ for } u, v, w \in \partial B_{R_2} \cap P, \lambda \in [0,1], \tag{26}$$

where $R_2 > \widetilde{M}$. If this claim does not hold, there exist $u, v, w \in \partial B_{R_2} \cap P, \lambda_2 \in [0,1]$ such that

$$(u, v, w) = \lambda_2 T(u, v, w). \tag{27}$$

This indicates that

$$\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} \lambda_2 T_1(u, v, w)(t) \\ \lambda_2 T_2(u, v, w)(t) \\ \lambda_2 T_3(u, v, w)(t) \end{pmatrix} \leq \begin{pmatrix} \int_0^1 G(t, s) F_1(s, u(s) - z(s), v(s) - z(s), w(s) - z(s)) ds \\ \int_0^1 G(t, s) F_2(s, u(s) - z(s), v(s) - z(s), w(s) - z(s)) ds \\ \int_0^1 G(t, s) F_3(s, u(s) - z(s), v(s) - z(s), w(s) - z(s)) ds \end{pmatrix}.$$

Using Lemma 7, we know $u, v, w \in P_0$. By virtue of (C5), we obtain

$$\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} \leq \begin{pmatrix} \int_0^1 G(t,s)(\widetilde{a}_{11}(u(s)-z(s))+\widetilde{a}_{12}(v(s)-z(s))+\widetilde{a}_{13}(w(s)-z(s))+\widetilde{l}_1)ds \\ \int_0^1 G(t,s)(\widetilde{a}_{21}(u(s)-z(s))+\widetilde{a}_{22}(v(s)-z(s))+\widetilde{a}_{23}(w(s)-z(s))+\widetilde{l}_2)ds \\ \int_0^1 G(t,s)(\widetilde{a}_{31}(u(s)-z(s))+\widetilde{a}_{32}(v(s)-z(s))+\widetilde{a}_{33}(w(s)-z(s))+\widetilde{l}_3)ds \end{pmatrix}.$$

Multiplying by $\varphi(t)$, and integrating over $[0,1]$, Lemma 4 enables us to get

$$\begin{aligned} \begin{pmatrix} \int_0^1 u(t)\varphi(t)dt \\ \int_0^1 v(t)\varphi(t)dt \\ \int_0^1 w(t)\varphi(t)dt \end{pmatrix} &\leq \begin{pmatrix} \int_0^1 \kappa_2\varphi(t)(\widetilde{a}_{11}(u(t)-z(t))+\widetilde{a}_{12}(v(t)-z(t))+\widetilde{a}_{13}(w(t)-z(t))+\widetilde{l}_1)dt \\ \int_0^1 \kappa_2\varphi(t)(\widetilde{a}_{21}(u(t)-z(t))+\widetilde{a}_{22}(v(t)-z(t))+\widetilde{a}_{23}(w(t)-z(t))+\widetilde{l}_2)dt \\ \int_0^1 \kappa_2\varphi(t)(\widetilde{a}_{31}(u(t)-z(t))+\widetilde{a}_{32}(v(t)-z(t))+\widetilde{a}_{33}(w(t)-z(t))+\widetilde{l}_3)dt \end{pmatrix} \\ &\leq \begin{pmatrix} \int_0^1 \kappa_2\varphi(t)(\widetilde{a}_{11}u(t)+\widetilde{a}_{12}v(t)+\widetilde{a}_{13}w(t))dt+\widetilde{l}_1\kappa_2^2 \\ \int_0^1 \kappa_2\varphi(t)(\widetilde{a}_{21}u(t)+\widetilde{a}_{22}v(t)+\widetilde{a}_{23}w(t))dt+\widetilde{l}_2\kappa_2^2 \\ \int_0^1 \kappa_2\varphi(t)(\widetilde{a}_{31}u(t)+\widetilde{a}_{32}v(t)+\widetilde{a}_{33}w(t))dt+\widetilde{l}_3\kappa_2^2 \end{pmatrix}. \end{aligned}$$

Therefore, we find

$$\begin{pmatrix} 1-\kappa_2\widetilde{a}_{11} & -\kappa_2\widetilde{a}_{12} & -\kappa_2\widetilde{a}_{13} \\ -\kappa_2\widetilde{a}_{21} & 1-\kappa_2\widetilde{a}_{22} & -\kappa_2\widetilde{a}_{23} \\ -\kappa_2\widetilde{a}_{31} & -\kappa_2\widetilde{a}_{32} & 1-\kappa_2\widetilde{a}_{33} \end{pmatrix} \begin{pmatrix} \int_0^1 u(t)\varphi(t)dt \\ \int_0^1 v(t)\varphi(t)dt \\ \int_0^1 w(t)\varphi(t)dt \end{pmatrix} \leq \begin{pmatrix} \widetilde{l}_1\kappa_2^2 \\ \widetilde{l}_2\kappa_2^2 \\ \widetilde{l}_3\kappa_2^2 \end{pmatrix},$$

and

$$\begin{aligned} \begin{pmatrix} \int_0^1 u(t)\varphi(t)dt \\ \int_0^1 v(t)\varphi(t)dt \\ \int_0^1 w(t)\varphi(t)dt \end{pmatrix} &\leq \begin{pmatrix} 1-\kappa_2\widetilde{a}_{11} & -\kappa_2\widetilde{a}_{12} & -\kappa_2\widetilde{a}_{13} \\ -\kappa_2\widetilde{a}_{21} & 1-\kappa_2\widetilde{a}_{22} & -\kappa_2\widetilde{a}_{23} \\ -\kappa_2\widetilde{a}_{31} & -\kappa_2\widetilde{a}_{32} & 1-\kappa_2\widetilde{a}_{33} \end{pmatrix}^{-1} \begin{pmatrix} \widetilde{l}_1\kappa_2^2 \\ \widetilde{l}_2\kappa_2^2 \\ \widetilde{l}_3\kappa_2^2 \end{pmatrix} \\ = \frac{1}{\Delta_2} &\begin{pmatrix} (1-\kappa_2\widetilde{a}_{22})(1-\kappa_2\widetilde{a}_{33})-\kappa_2^2\widetilde{a}_{23}\widetilde{a}_{32} & \kappa_2^2\widetilde{a}_{13}\widetilde{a}_{32}+\kappa_2\widetilde{a}_{12}(1-\kappa_2\widetilde{a}_{33}) & \kappa_2^2\widetilde{a}_{12}\widetilde{a}_{23}+\kappa_2\widetilde{a}_{13}(1-\kappa_2\widetilde{a}_{22}) \\ \kappa_2^2\widetilde{a}_{23}\widetilde{a}_{31}+\kappa_2\widetilde{a}_{21}(1-\kappa_2\widetilde{a}_{33}) & (1-\kappa_2\widetilde{a}_{11})(1-\kappa_2\widetilde{a}_{33})-\kappa_2^2\widetilde{a}_{13}\widetilde{a}_{31} & \kappa_2^2\widetilde{a}_{13}\widetilde{a}_{21}+\kappa_2\widetilde{a}_{23}(1-\kappa_2\widetilde{a}_{11}) \\ \kappa_2^2\widetilde{a}_{21}\widetilde{a}_{32}+\kappa_2\widetilde{a}_{31}(1-\kappa_2\widetilde{a}_{22}) & \kappa_2^2\widetilde{a}_{12}\widetilde{a}_{31}+\kappa_2\widetilde{a}_{32}(1-\kappa_2\widetilde{a}_{11}) & (1-\kappa_2\widetilde{a}_{11})(1-\kappa_2\widetilde{a}_{22})-\kappa_2^2\widetilde{a}_{12}\widetilde{a}_{21} \end{pmatrix} \\ &\cdot \begin{pmatrix} \widetilde{l}_1\kappa_2^2 \\ \widetilde{l}_2\kappa_2^2 \\ \widetilde{l}_3\kappa_2^2 \end{pmatrix}, \end{aligned}$$

where

$$\Delta_2 = \det \begin{pmatrix} 1-\kappa_2\widetilde{a}_{11} & -\kappa_2\widetilde{a}_{12} & -\kappa_2\widetilde{a}_{13} \\ -\kappa_2\widetilde{a}_{21} & 1-\kappa_2\widetilde{a}_{22} & -\kappa_2\widetilde{a}_{23} \\ -\kappa_2\widetilde{a}_{31} & -\kappa_2\widetilde{a}_{32} & 1-\kappa_2\widetilde{a}_{33} \end{pmatrix}.$$

Hence, there exist $\mathcal{N}_i > 0 (i = 4, 5, 6)$ such that

$$\begin{pmatrix} \int_0^1 u(t)\varphi(t)dt \\ \int_0^1 v(t)\varphi(t)dt \\ \int_0^1 w(t)\varphi(t)dt \end{pmatrix} \leq \begin{pmatrix} \mathcal{N}_4 \\ \mathcal{N}_5 \\ \mathcal{N}_6 \end{pmatrix},$$

where $\mathcal{N}_4 = \frac{\kappa_2^2}{\Delta_2}[\widetilde{l}_1((1-\kappa_2\widetilde{a}_{22})(1-\kappa_2\widetilde{a}_{33}) - \kappa_2^2\widetilde{a}_{23}\widetilde{a}_{32}) + \widetilde{l}_2(\kappa_2^2\widetilde{a}_{13}\widetilde{a}_{32} + \kappa_2\widetilde{a}_{12}(1-\kappa_2\widetilde{a}_{33})) + \widetilde{l}_3(\kappa_2^2\widetilde{a}_{12}\widetilde{a}_{23} + \kappa_2\widetilde{a}_{13}(1-\kappa_2\widetilde{a}_{22}))]$, $\mathcal{N}_5 = \frac{\kappa_2^2}{\Delta_2}[\widetilde{l}_1(\kappa_2^2\widetilde{a}_{23}\widetilde{a}_{31} + \kappa_2\widetilde{a}_{21}(1-\kappa_2\widetilde{a}_{33})) + \widetilde{l}_2((1-\kappa_2\widetilde{a}_{11})(1-\kappa_2\widetilde{a}_{33}) - \kappa_2^2\widetilde{a}_{13}\widetilde{a}_{31}) + \widetilde{l}_3(\kappa_2^2\widetilde{a}_{13}\widetilde{a}_{21} + \kappa_2\widetilde{a}_{23}(1-\kappa_2\widetilde{a}_{11}))]$, $\mathcal{N}_6 = \frac{\kappa_2^2}{\Delta_2}[\widetilde{l}_1(\kappa_2^2\widetilde{a}_{21}\widetilde{a}_{32} + \kappa_2\widetilde{a}_{31}(1-\kappa_2\widetilde{a}_{22})) + \widetilde{l}_2(\kappa_2^2\widetilde{a}_{12}\widetilde{a}_{31} + \kappa_2\widetilde{a}_{32}(1-\kappa_2\widetilde{a}_{11})) + \widetilde{l}_3((1-\kappa_2\widetilde{a}_{11})(1-\kappa_2\widetilde{a}_{22}) - \kappa_2^2\widetilde{a}_{12}\widetilde{a}_{21})]$. Note that $u, v, w \in P_0$, we have

$$\begin{pmatrix} \|u\| \\ \|v\| \\ \|w\| \end{pmatrix} \leq \begin{pmatrix} \mathcal{N}_4\kappa_1^{-1} \\ \mathcal{N}_5\kappa_1^{-1} \\ \mathcal{N}_6\kappa_1^{-1} \end{pmatrix}.$$

Therefore, we can choose $R_2 > \max\{\widetilde{M}, \mathcal{N}_4\kappa_1^{-1}, \mathcal{N}_5\kappa_1^{-1}, \mathcal{N}_6\kappa_1^{-1}\}$ such that when $u, v, w \in \partial B_{R_2} \cap P$, (27) is not satisfied. This also indicates that (26) holds for $u, v, w \in \partial B_{R_2} \cap P$, and by Lemma 9 we get

$$i(T, B_{R_2} \cap (P \times P \times P), P \times P \times P) = 1. \tag{28}$$

On the other hand, we prove that

$$(u, v, w) \neq T(u, v, w) + \lambda(\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3), \text{ for } u, v, w \in \partial B_{\widetilde{M}} \cap P, \ \forall \lambda \geq 0, \tag{29}$$

where $\widetilde{\phi}_i \in P (i = 1, 2, 3)$ are fixed elements. Otherwise, there exist $u, v, w \in \partial B_{\widetilde{M}} \cap P, \lambda_3 \geq 0$ such that

$$(u, v, w) = T(u, v, w) + \lambda_3(\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3).$$

This implies that

$$\begin{pmatrix} \|u\| \\ \|v\| \\ \|w\| \end{pmatrix} \geq \begin{pmatrix} \|T_1(u, v, w)\| \\ \|T_2(u, v, w)\| \\ \|T_3(u, v, w)\| \end{pmatrix}. \tag{30}$$

However, from (C6) we have

$$\begin{aligned} T_i(u, v, w)(t_0) &= \int_0^1 G(t_0, s)F_i(s, u(s) - z(s), v(s) - z(s), w(s) - z(s))ds \\ &\geq t_0^{\alpha-1} \int_0^1 \varphi(s)\widetilde{Q}_i(s)ds \\ &> \widetilde{M}, i = 1, 2, 3. \end{aligned}$$

Note that from (C6), we have $\|u\| = \widetilde{M}$. Hence, we obtain

$$\begin{pmatrix} \|T_1(u, v, w)\| \\ \|T_2(u, v, w)\| \\ \|T_3(u, v, w)\| \end{pmatrix} \geq \begin{pmatrix} T_1(u, v, w)(t_0) \\ T_2(u, v, w)(t_0) \\ T_3(u, v, w)(t_0) \end{pmatrix} > \begin{pmatrix} \|u\| \\ \|v\| \\ \|w\| \end{pmatrix}.$$

This has a contradiction with (30), and thus (29) holds. By Lemma 8 we find

$$i(T, B_{\widetilde{M}} \cap (P \times P \times P), P \times P \times P) = 0. \tag{31}$$

From (28) and (31) we can calculate

$$\begin{aligned} & i(T,(B_{R_2} \backslash \overline{B}_{\widetilde{M}}) \cap (P \times P \times P), P \times P \times P) \\ & \quad = i(T, B_{R_2} \cap (P \times P \times P), P \times P \times P) - i(T, B_{\widetilde{M}} \cap (P \times P \times P), P \times P \times P) \\ & \quad = 1. \end{aligned}$$

Therefore T has a fixed point (u^*, v^*, w^*) on $(B_{R_2} \backslash \overline{B}_{\widetilde{M}}) \cap (P \times P \times P)$. Therefore, $(u^* - z, v^* - z, w^* - z)$ is a positive solution for (1), i.e., (1) has a positive solution.

Let $n = 4, \alpha = 3.5, p = 1.5, q = 0.5$, and $h(t) = t, t \in [0,1]$. Then we have $A = 2.91$, and $\int_0^1 h(t) g_2(t,s) dt = \frac{5}{24}s - \frac{1}{4}s^2 + \frac{1}{24}s^4, s \in [0,1]$. This implies that (C0)–(C1) hold. Moreover, we can calculate

$$\kappa_1 = 0.017, \kappa_2 = 0.075, \widetilde{M} = 0.16M.$$

□

Example 1. *Let* $\kappa_1 a_{11} - 1 = \kappa_1 a_{22} - 1 = \kappa_1 a_{33} - 1 = \kappa_1$*, and we have* $a_{11} = a_{22} = a_{33} = \frac{\kappa_1 + 1}{\kappa_1} = 59.82$*. Moreover, we take the matrix*

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 59.82 & 0 & 0 \\ 0 & 59.82 & 0 \\ 0 & 0 & 59.82 \end{pmatrix},$$

and

$$\begin{pmatrix} F_1(t, x_1, x_2, x_3) \\ F_2(t, x_1, x_2, x_3) \\ F_3(t, x_1, x_2, x_3) \end{pmatrix} = \begin{pmatrix} 2M(9.57M)^{-\gamma_1}(a_{11}x_1 + a_{12}x_2 + a_{13}x_3)^{\gamma_1} \\ 1.8M(9.57M)^{-\gamma_2}(a_{21}x_1 + a_{22}x_2 + a_{23}x_3)^{\gamma_2} \\ 1.5M(9.57M)^{-\gamma_3}(a_{31}x_1 + a_{32}x_2 + a_{33}x_3)^{\gamma_3} \end{pmatrix}, \forall (t, x_1, x_2, x_3) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+,$$

where $\gamma_i > 1 (i = 1,2,3)$*. Note that*

$$\mathcal{M}_1 = \begin{pmatrix} \kappa_1 a_{11} - 1 & \kappa_1 a_{12} & \kappa_1 a_{13} \\ \kappa_1 a_{21} & \kappa_1 a_{22} - 1 & \kappa_1 a_{23} \\ \kappa_1 a_{31} & \kappa_1 a_{32} & \kappa_1 a_{33} - 1 \end{pmatrix} = \begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_1 & 0 \\ 0 & 0 & \kappa_1 \end{pmatrix}.$$

Hence, $\mathcal{M}_1$ *is a* $\mathbb{R}^3_+$*-monotone matrix. Furthermore, for all* $t \in [0,1]$ *we have*

$$\begin{aligned} \liminf_{a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \to +\infty} \frac{F_1(x_1, x_2, x_3)}{a_{11}x_1 + a_{12}x_2 + a_{13}x_3} &= \liminf_{a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \to +\infty} \frac{2M(9.57M)^{-\gamma_1}(a_{11}x_1 + a_{12}x_2 + a_{13}x_3)^{\gamma_1}}{a_{11}x_1 + a_{12}x_2 + a_{13}x_3} = +\infty, \\ \liminf_{a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \to +\infty} \frac{F_2(x_1, x_2, x_3)}{a_{21}x_1 + a_{22}x_2 + a_{23}x_3} &= \liminf_{a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \to +\infty} \frac{1.8M(9.57M)^{-\gamma_2}(a_{21}x_1 + a_{22}x_2 + a_{23}x_3)^{\gamma_2}}{a_{21}x_1 + a_{22}x_2 + a_{23}x_3} = +\infty, \\ \liminf_{a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \to +\infty} \frac{F_3(x_1, x_2, x_3)}{a_{31}x_1 + a_{32}x_2 + a_{33}x_3} &= \liminf_{a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \to +\infty} \frac{1.5M(9.57M)^{-\gamma_3}(a_{31}x_1 + a_{32}x_2 + a_{33}x_3)^{\gamma_3}}{a_{31}x_1 + a_{32}x_2 + a_{33}x_3} = +\infty. \end{aligned}$$

On the other hand, if $(t, x_1, x_2, x_3) \in [0,1] \times [0, 0.16M]^3$*, we have*

$$F_1 \leq 2M, \; F_2 \leq 1.8M, F_3 \leq 1.5M.$$

If we choose $Q_1(t) \equiv 2M, Q_2(t) \equiv 1.8M, Q_3(t) \equiv 1.5M$ *for* $t \in [0,1]$, *and we have*

$$\int_0^1 \varphi(t)Q_i(t)dt \le \int_0^1 \varphi(t)Q_1(t)dt = 2\kappa_2 M = 0.15M < \widetilde{M}, i = 1,2,3.$$

Therefore, (C3)–(C4) hold.

Example 2. *Let* $t_0 = 0.5$, $\widetilde{Q}_1(t) = 13M, \widetilde{Q}_2(t) = 14M, \widetilde{Q}_3(t) = 15M$ *for* $t \in [0,1]$, *and*

$$\begin{pmatrix} \widetilde{a}_{11} & \widetilde{a}_{12} & \widetilde{a}_{13} \\ \widetilde{a}_{21} & \widetilde{a}_{22} & \widetilde{a}_{23} \\ \widetilde{a}_{31} & \widetilde{a}_{32} & \widetilde{a}_{33} \end{pmatrix} = \begin{pmatrix} 2 & 5 & 3 \\ 8 & 3 & 4 \\ 6 & 3 & 4 \end{pmatrix},$$

and

$$\begin{pmatrix} F_1(t,x_1,x_2,x_3) \\ F_2(t,x_1,x_2,x_3) \\ F_3(t,x_1,x_2,x_3) \end{pmatrix} = \begin{pmatrix} 13Me^{1.6M}e^{-2x_1-5x_2-3x_3} \\ 14Me^{2.4M}e^{-8x_1-3x_2-4x_3} \\ 15Me^{2.08M}e^{-6x_1-3x_2-4x_3} \end{pmatrix}, \forall (t,x_1,x_2,x_3) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+.$$

Then if $(t,x_1,x_2,x_3) \in [0,1] \times [0, 0.16M]^3$, *we have* $F_1 \ge 13M, F_2 \ge 14M, F_3 \ge 15M$, *and* $\int_0^1 \varphi(t)\widetilde{Q}_i(t)dt \ge \int_0^1 \varphi(t)\widetilde{Q}_1(t)dt = 13\kappa_2 M > 0.16 \times 5.6569M.$

On the other hand, we can calculate

$$\det \mathcal{M}_2 = \det \begin{pmatrix} 1 - 0.075 \times 2 & -0.075 \times 5 & -0.075 \times 3 \\ -0.075 \times 8 & 1 - 0.075 \times 3 & -0.075 \times 4 \\ -0.075 \times 6 & -0.075 \times 3 & 1 - 0.075 \times 4 \end{pmatrix} = 0.0868,$$

and

$$\mathcal{M}_2^{-1} = \frac{1}{0.0868} \begin{pmatrix} 0.475 & 0.313 & 0.287 \\ 0.555 & 0.494 & 0.39 \\ 0.484 & 0.36 & 0.434 \end{pmatrix}.$$

Consequently, $\mathcal{M}_2$ *is a* $\mathbb{R}^3_+$*-monotone matrix. Furthermore, for all* $t \in [0,1]$ *we have*

$$\limsup_{\widetilde{a}_{11}x_1+\widetilde{a}_{12}x_2+\widetilde{a}_{13}x_3 \to +\infty} \frac{F_1(t,x_1,x_2,x_3)}{\widetilde{a}_{11}x_1+\widetilde{a}_{12}x_2+\widetilde{a}_{13}x_3} = \limsup_{\widetilde{a}_{11}x_1+\widetilde{a}_{12}x_2+\widetilde{a}_{13}x_3 \to +\infty} \frac{13Me^{1.6M}e^{-\widetilde{a}_{11}x_1-\widetilde{a}_{12}x_2-\widetilde{a}_{13}x_3}}{\widetilde{a}_{11}x_1+\widetilde{a}_{12}x_2+\widetilde{a}_{13}x_3} = 0,$$

$$\limsup_{\widetilde{a}_{21}x_1+\widetilde{a}_{22}x_2+\widetilde{a}_{23}x_3 \to +\infty} \frac{F_2(t,x_1,x_2,x_3)}{\widetilde{a}_{21}x_1+\widetilde{a}_{22}x_2+\widetilde{a}_{23}x_3} = \limsup_{\widetilde{a}_{21}x_1+\widetilde{a}_{22}x_2+\widetilde{a}_{23}x_3 \to +\infty} \frac{14Me^{2.4M}e^{-\widetilde{a}_{21}x_1-\widetilde{a}_{22}x_2-\widetilde{a}_{23}x_3}}{\widetilde{a}_{21}x_1+\widetilde{a}_{22}x_2+\widetilde{a}_{23}x_3} = 0,$$

$$\limsup_{\widetilde{a}_{31}x_1+\widetilde{a}_{32}x_2+\widetilde{a}_{33}x_3 \to +\infty} \frac{F_3(t,x_1,x_2,x_3)}{\widetilde{a}_{31}x_1+\widetilde{a}_{32}x_2+\widetilde{a}_{33}x_3} = \limsup_{\widetilde{a}_{31}x_1+\widetilde{a}_{32}x_2+\widetilde{a}_{33}x_3 \to +\infty} \frac{15Me^{2.08M}e^{-\widetilde{a}_{31}x_1-\widetilde{a}_{32}x_2-\widetilde{a}_{33}x_3}}{\widetilde{a}_{31}x_1+\widetilde{a}_{32}x_2+\widetilde{a}_{33}x_3} = 0.$$

As a result, (C5)–(C6) hold.

4. Conclusions

In this paper, we utilize the index of fixed point to research the positive solutions for the system of Riemann–Liouville type fractional boundary value problems (1). We first investigate corresponding operator equations for (1), and then establish some coupling behaviors for our nonlinearities $f_i (i = 1,2,3)$ by virtue of nonnegative matrix theory, which ensure that our nonlinearities can grow superlinearly and sublinearly at ∞.

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