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Remarks on the Preservation of the Almost Fixed Point Property Involving Several Types of Digitizations

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Abstract: This paper explores a certain relationship between the almost fixed point property (*AFPP* for short) of a compact and n-dimensional Euclidean space and that of its digitized space. Based on several types of digitizations, we prove that the *AFPP* of a compact and n-dimensional Euclidean cube is preserved by each of the $U(k)$, the $L(k)$ and the Khalimsky digitizations, $k \in 3^n - 1, n \in \mathbb{N}$.

Keywords: digital space; U-and L-digitization; fixed point property; almost fixed point property; Khalimsky topology; digital topology

MSC: 55N35; 55M10; 68R10; 68U05

1. Introduction

In order to study the fixed point property (*FPP* for brevity) and the almost (or approximate) fixed point property (*AFPP* for short) for Euclidean topological spaces and digital spaces, we need to recall some terminology from digital topology and fixed point theory. Hereafter, let $\mathbb{N}$, $\mathbb{Z}^n$ and $\mathbb{R}$ represent the sets of natural numbers, points in the Euclidean n-dimensional space with integer coordinates and real numbers, respectively. In addition, for distinct integers $a, b \in \mathbb{Z}$, we often use the notation $[a, b]_{\mathbb{Z}} := \{t \in \mathbb{Z} \,|\, a \leq t \leq b\}$ called a digital interval [1]. We say that a digital image (X, k) (see Equation (2)) is k-connected if, for any two points $x, y \in X$, there is a finite sequence $< x_i >_{i \in [0,l]_{\mathbb{Z}}} \subset X$ such that $x_0 = x$ and $x_l = y$ and, furthermore, x_i and x_j are k-adjacent (see Equation (1) and (2) in Section 2) if $|i - j| = 1, i, j \in [0, l]_{\mathbb{Z}}$ [1]. We say that a non-empty and k-connected digital image (X, k) has the *FPP* [2] if every k-continuous map $f : (X, k) \to (X, k)$ has a point $x \in X$ such that $f(x) = x$ (see Section 2 for more details). In addition, we say that a non-empty digital image (X, k) has the *AFPP* [2] if every k-continuous map $f : (X, k) \to (X, k)$ has a point $x \in X$ such that $f(x) = x$ or $f(x)$ is k-adjacent to x [2]. In general, a non-empty object Y of a category has the *FPP* if every morphism $h : Y \to Y$ has a point $y \in Y$ such that $h(y) = y$. It is obvious that the *AFPP* is weaker than the *FPP* [2].

Recently, many works relating to the *FPP* and the *AFPP* for digital spaces have been proceeded [2–11]. Furthermore, given a Euclidean subspace X, several types of digitizations of X were also developed [6,12,13]. These approaches indeed play important roles in applied topology and computer science, e.g., image processing, image analysis and so on. Hereafter, a compact and n-dimensional Euclidean space means a certain bounded and closed (or compact) n-dimensional Euclidean topological space $(\prod_{i \in \{1,2,\cdots,n\}}[-l_i, l_i] := X, E^n_X)$,$l_i \in \mathbb{N}$. Then, we naturally wonder if there is a certain relationship between the *AFPP* of the above (X, E^n_X) and the *AFPP* of a space obtained by its digitization (or a digitized space for short). Furthermore, based on the study of the *AFPP* of a finite digital picture, e.g., $[a, b]_{\mathbb{Z}} \times [c, d]_{\mathbb{Z}}$ with 8-adjacency [2], we may ask if the n-dimensional digital cube $(([-1, 1]_{\mathbb{Z}})^n := [-1, 1]^n_{\mathbb{Z}} := X, k)$ on $\mathbb{Z}^n$ has the *AFPP*. Regarding this issue, we need to recall the notion of a digital space. For a nonempty binary symmetric relation set (X, π), we recall that X is

π-connected [11] if for any two elements x and y of X there is a finite sequence $< x_i >_{i\in[0,l]_{\mathbb{Z}}}$ of elements in X such that $x = x_0$, $y = x_l$ and $(x_j, x_{j+1}) \in \pi$ for $j \in [0, l-1]_{\mathbb{Z}}$. We say that a *digital space* is a nonempty, π-connected, symmetric relation set, denoted by (X, π) [11]. It is well known that a digital space [11] includes a digital image (X, k) with digital k-connectivity (i.e., Rosenfeld model) [2,14], a Khalimsky (K-, for brevity) topological space with Khalimsky adjacency [15], a Marcus-Wyse (M-, for short) topological space with Marcus-Wyse adjacency [16], and so forth [5,9,10] (see Section 2 in details).

Based on the several kinds of digitizations of a Euclidean space in [6,12,13], the present paper explores a certain relationship between the *AFPP* for Euclidean topological subspaces in $\mathbb{R}^n$ and that for their U-, L-, K-, or M-digitized spaces in $\mathbb{Z}^n$ from the viewpoint of digital topology, where U-, L-, K- and M- means the upper limit, the lower limit, Khalimsky and Marcus-Wyse topology, respectively.

In fixed point theory for digital spaces, we also assume that every digital space (X, π) is π-connected and non-empty.

The rest of the paper is organized as follows: Section 2 provides basic notions from digital topology. Section 3 investigates some properties of digitizations in a K-, an M-, a U-, or an L-topological approach. Section 4 develops a link between the *AFPP* from the viewpoint of *ETC* and the *AFPP* from the viewpoint of *DTC*, *KTC*, or *MTC*, where *ETC*, *DTC*, *KTC* and *MTC* are a Euclidean topological, a digital topological, a Khalimsky topological and a Marcus-Wyse topological category, respectively (for more details, see Section 2).

2. Several Kinds of Digital Topological Categories, *DTC*, *KTC* and *MTC*

To study the *FPP* or the *AFPP* for digital spaces from the viewpoint of digital topology, we first need to recall the k-adjacency relations of n-dimensional integer grids (see Equation (2)), a digital k-neighborhood, digital continuity, and so forth [2,14,17]. To study n-dimensional digital images, $n \in \mathbb{N}$, as a generalization of the k-adjacency relations of $\mathbb{Z}^n, n \in \{1, 2, 3\}$, we will take the following approach [17] (see also [18]).

For a natural number m, $1 \le m \le n$, distinct points

$$p = (p_1, p_2, \cdots, p_n) \text{ and } q = (q_1, q_2, \cdots, q_n) \in \mathbb{Z}^n, \tag{1}$$

are $k(m, n)$-adjacent if at most m of their coordinates differ by ± 1, and all others coincide.

According to the operator of Equation (1), the $k(m, n)$-adjacency relations of $\mathbb{Z}^n, n \in \mathbb{N}$, are obtained [17] (see also [18]) as follows:

$$\left\{ \begin{array}{l} \text{(a) } k := k(t,n) = \sum\limits_{i=n-t}^{n-1} 2^{n-i} C_i^n, \text{where } C_i^n = \dfrac{n!}{(n-i)!\; i!} \\ \text{or, equivalently,} \\ \text{(b) } k := k(t,n) = \sum\limits_{i=1}^{t} 2^i C_i^n, \text{where } C_i^n = \dfrac{n!}{(n-i)!\; i!}. \end{array} \right\} \tag{2}$$

A. Rosenfeld [14] called a set $X(\subset \mathbb{Z}^n)$ with a k-adjacency a digital image, denoted by (X, k). Indeed, to study digital images on $\mathbb{Z}^n$ in the graph-theoretical approach [2,14], using the k-adjacency relations of $\mathbb{Z}^n$ of Equation (2), we say that a digital k-neighborhood of p in $\mathbb{Z}^n$ is the set [14]

$$N_k(p) := \{q \mid p \text{ is } k\text{-adjacent to } q\} \cup \{p\}.$$

In addition, for a k-adjacency relation of $\mathbb{Z}^n$, a simple k-path with $l+1$ elements on $(\mathbb{Z}^n, k)$ is assumed to be a finite sequence $< x_i >_{i\in[0,l]_{\mathbb{Z}}} \subset \mathbb{Z}^n$ (or k-path) such that x_i and x_j are k-adjacent if and only if $|\, i-j \,| = 1$. If $x_0 = x$ and $x_l = y$, then the length of the simple k-path, denoted by $l_k(x, y)$, is the number l. A simple closed k-curve with l elements on $(\mathbb{Z}^n, k)$, denoted by

$SC_k^{n,l} :=< x_i >_{i \in [0,l]_{\mathbb{Z}}}$ [17], is a simple k-path $< x_i >_{i \in [0,l-1]_{\mathbb{Z}}}$ on $(\mathbb{Z}^n, k)$, where x_i and x_j are k-adjacent if and only if $|i - j| = \pm 1 (mod\, l)$.

For a digital image (X, k), for $X \subset \mathbb{Z}^n$, we put [17]

$$N_k(x, 1) := N_k(x) \cap X. \tag{3}$$

As a generalization of $N_k(x, 1)$ of Equation (3), for a digital image (X, k) let us recall a digital k-neighborhood [17]. Namely, the digital k-neighborhood of $x_0 \in X$ with radius ε is defined in X to be the following subset of X [17]

$$N_k(x_0, \varepsilon) := \{x \in X \,|\, l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\}, \tag{4}$$

where $l_k(x_0, x)$ is the length of a shortest simple k-path from x_0 to x and $\varepsilon \in \mathbb{N}$.

Given a digital image (X, k) on $\mathbb{Z}^n$ and for two points $x, y \in X$, if there is no k-path connecting between these points, then we define $l_k(x, y) = \infty$. In addition, we may represent the notion of "k-connected" as follows: a digital image (X, k) on $\mathbb{Z}^n$ is k-connected if, for any distinct points $x, y \in X$. there is a k-path connecting these two points.

Definition 1. *We say that a k-connected digital image* (X, k) *on* $\mathbb{Z}^n$ *is bounded if for some point* $x_0 \in X$*, there is an* $N_k(x_0, \varepsilon)$ *that is equal to the set X, where* $\varepsilon \lneq \infty$*.*

In general, we say that a digital image (X, k) on $\mathbb{Z}^n$ is bounded if there is a finite set $\{x_i \in X \,|\, i \in M : \text{finite}\}$ such that $X = \cup_{i \in M} N_k(x_i, \varepsilon_i)$, where $\varepsilon_i \lneq \infty$.

The author in [2] established the notion of digital continuity of a map $f : (X, k_0) \to (Y, k_1)$ by saying that f maps every k_0-connected subset of (X, k_0) into a k_1-connected subset of (Y, k_1) (see Theorem 2.4 of [2]). Motivated by this approach, the digital continuity of maps between digital images was represented in terms of the neighborhood of Equation (3), as follows:

Proposition 1 ([17])**.** *Let* (X, k_0) *and* (Y, k_1) *be digital images in* $\mathbb{Z}^{n_0}$ *and* $\mathbb{Z}^{n_1}$*, respectively. A function* $f : (X, k_0) \to (Y, k_1)$ *is* (k_0, k_1)*-continuous if and only if for every* $x \in X$, $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.

In Proposition 1, in case $k_0 = k_1$, the map f is called a k_1*-continuous map*. Using digitally continuous maps, we establish the category of digital images, denoted by *DTC*, consisting of the following two data [17] (see also [5]):

- The set of objects (X, k), denoted by $Ob(DTC)$;
- For every ordered pair of objects (X, k_1) and (Y, k_2), the set of all (k_1, k_2)-continuous maps $f : (X, k_1) \to (Y, k_2)$ as morphisms.

In *DTC*, in case $k_0 = k_1 := k$, we will particularly use the notation *DTC(k)*.

The authors in [2] initiated the study of the *FPP* and the *AFPP* for digital pictures (see Proposition 2). Based on the approach, many works explored the properties for several types of digital spaces, such as Khalimsky, Marcus-Wyse topological spaces, and digital metric spaces associated with some typical fixed point theorems.

Proposition 2 ([2])**.** *Consider a bounded digital plane (or finite digital picture)* $(X, k), k \in \{4, 8\}$*, i.e.,* $([a, b]_{\mathbb{Z}} \times [c, d]_{\mathbb{Z}} := X, k)$*.*

Then, it does not have the FPP. However, $(X, 8)$ *has the AFPP.*

Motivated by Proposition 2, we obtain the following:

Theorem 1. *For* $n \in \mathbb{N}$*, the n-dimensional digital cube with k-adjacency* $([-1, 1]_{\mathbb{Z}}^n := X, k)$ *on* $\mathbb{Z}^n$ *has the AFPP if and only if* $k = 3^n - 1$*.*

Proof. Consider $[-1,1]^n_{\mathbb{Z}} := X$ with a certain k-adjacency of $\mathbb{Z}^n$ (see Equation (2)), i.e., a digital image (X,k). Motivated by Proposition 2, it is obvious that any k-adjacency of $\mathbb{Z}^n$ (X,k) does not have the *FPP*. With the given hypothesis, in case (X,k) has the *AFPP*, for any k-continuous self-map of (X,k), there is a point $x \in X$ such that $f(x) = x$ or $f(x)$ is k-adjacent to x. For any k-connectivity of (X,k), since any k-continuity of f implies $(3^n - 1)$-continuity of f (see Equations (1) and (2)), we may take the $(3^n - 1)$-connectivity of X for supporting the given *AFPP* of (X,k).

Conversely, if $k \neq 3^n - 1$, then we first prove that (X,k) does not have the *AFPP*. For instance, in $\mathbb{Z}^2$, consider the digital image $([-1,1]^2_{\mathbb{Z}} := X, 4)$ instead of $(X,8)$. Let us consider a self-map of $(X,4)$. To be precise, assume $f : (X,4) \to (X,4)$ as the composite of the following two 4-continuous maps f_1 and f_2 (see Figure 1(1)).

$$\left\{\begin{array}{l} f_1(X_1) = \{(0,-1)\}, X_1 = \{(0,-1),(1,-1)\}, \\ f_1(X_2) = \{(0,0)\}, X_2 = \{(0,0),(1,0),(0,1),(1,1)\}, \\ f_1(X_3) = \{(-1,0)\}, X_3 = \{(-1,0),(-1,1)\}, \text{and} \\ f_1((-1,-1)) = (-1,-1). \end{array}\right\} \tag{5}$$

Then, we obtain $f_1(X) = \{(0,0),(0,-1),(-1,-1),(-1,0)\}$ (see Figure 1(2)). Let us further consider the map $f_2 : f_1(X) \to f_1(X)$ such that

$$(0,0) \leftrightarrow (-1,-1), \text{ and } (0,-1) \leftrightarrow (-1,0). \tag{6}$$

Owing to the 4-continuous maps f_1 and f_2, the composite $f = f_2 \circ f_1 : (X,4) \to (X,4)$ is also a 4-continuous map. Although this map f is a 4-continuous self-map of $(X,4)$, it is not a map for supporting the *AFPP* of $(X,4)$.

As a generalization of the non-*AFPP* of $([-1,1]^2_{\mathbb{Z}} := X, 4)$, using a method similar to the Equations (5) and (6), we obtain that a digital image $(X := [-1,1]^n_{\mathbb{Z}}, k), k \neq 3^n - 1$ does not have the *AFPP* either. For instance, on $\mathbb{Z}^3$, consider $(Y := [-1,1]^3_{\mathbb{Z}}, 18 := k(2,3))$. Using the notion of 18-continuity of any self-map of $(Y,18)$ (see Proposition 1), we prove that the digital image $(Y,18)$ does not have the *AFPP*. To be precise, consider a self-map g of $(Y,18)$ in the following way: For $t \in [-1,1]_{\mathbb{Z}}$,

$$\left\{\begin{array}{l} g(1,1,t) = (-1,-1,t), \\ g(1,0,t) = (0,0,t) = g(0,1,t), \\ g(-1,1,t) = (-1,0,t), g(1,-1,t) = (0,-1,t), \text{ and} \\ g(Y_1) = 1_{Y_1}, \text{ where } Y_1 = [-1,0]^2_{\mathbb{Z}} \times [-1,1]_{\mathbb{Z}}. \end{array}\right\} \tag{7}$$

According to this map g, we obtain

$$g(Y) = [-1,0]^2_{\mathbb{Z}} \times [-1,1]_{\mathbb{Z}} := Z(\subset Y).$$

Let us now consider the self-map h of Z such that

$$\left\{\begin{array}{l} h(Z_1) = 1_{Z_1}, Z_1 = \{(0,0),(-1,0),(0,-1),(-1,-1)\} \times [0,1]_{\mathbb{Z}}, \\ h(s,-1) = (s,0), \text{where } s \in \{(0,0),(-1,0),(0,-1),(-1,-1)\}. \end{array}\right\} \tag{8}$$

Then, we obtain

$$h(Z) = [-1,0]^2_{\mathbb{Z}} \times [0,1]_{\mathbb{Z}}(\subset Y) := W.$$

Let us now further consider the self-map r of W such that

$$\left\{\begin{array}{l} (0,0,0) \leftrightarrow (-1,-1,1), (0,-1,0) \leftrightarrow (-1,0,1), \\ (-1,-1,0) \leftrightarrow (0,0,1), (-1,0,0) \leftrightarrow (0,-1,1). \end{array}\right\} \tag{9}$$

Then, it is obvious that each of the maps h and r is a 6-continuous map and the map g is an 18-continuous map (see Equations (7)–(9)). Hence, the composite $r \circ h \circ g : (Y, 18) \to (Y, 18)$ is an 18-continuous map. However, this composite does not have the *AFPP* of $(Y, 18)$ (see the map r of Equation (9)).

Finally, in case of $(X := [-1, 1]^n_{\mathbb{Z}}, 3^n - 1)$, according to the notion of $(3^n - 1)$-continuity of any self-map of $(X, 3^n - 1)$ (see Proposition 1), it is obvious that the digital image $(X, 3^n - 1)$ has the *AFPP*. Indeed, to obtain a contradiction, suppose the digital image $(X, 3^n - 1)$ does not have the *AFPP*. Then, any self-map of $(X, 3^n - 1)$ is not a $(3^n - 1)$-continuous map (see the point $0_3 := (0, 0, 0)$). $\square$

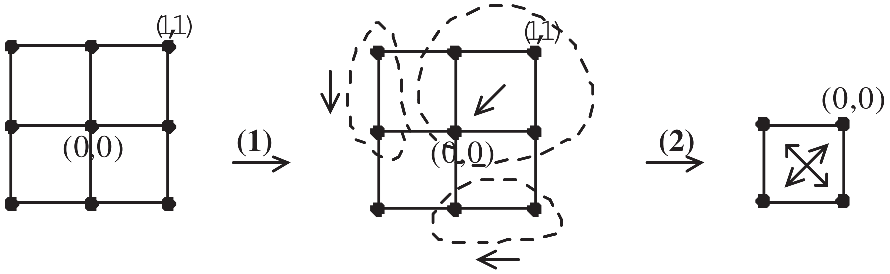

Figure 1. The non-*AFPP* of the digital 2-cube with 4-adjacency, $([-1, 1]^2_{\mathbb{Z}} := X, 4)$. (1) Configuration of the map f_1; (2) Explanation of the map f_2.

Let us now briefly recall some basic facts and terminology involving the K-topology. The *Khalimsky line topology* on $\mathbb{Z}$, denoted by $(\mathbb{Z}, \kappa)$, is induced by the set $\{[2n - 1, 2n + 1]_{\mathbb{Z}} : n \in \mathbb{Z}\}$ as a subbase [15]. Furthermore, the product topology on $\mathbb{Z}^n$ induced by $(\mathbb{Z}, \kappa)$ is called the *Khalimsky product topology* on $\mathbb{Z}^n$ (or *Khalimsky n-dimensional space*), which is denoted by $(\mathbb{Z}^n, \kappa^n)$. Based on this approach, for a point p in $(\mathbb{Z}^n, \kappa^n)$, its smallest open neighborhood $SN_K(p)$ is obtained [19].

Hereafter, for a subset $X \subseteq \mathbb{Z}^n, n \geq 1$, we will denote by (X, κ^n_X) a subspace induced by $(\mathbb{Z}^n, \kappa^n)$, and it is called a *K-topological space*. For a point x in (X, κ^n_X), we often call $SN_K(x)$ the *smallest open neighborhood* of x in (X, κ^n_X).

For (X, κ^n_X), we say that distinct points x and y in X are *K-adjacent* in (X, κ^n_X) if $y \in SN_K(x)$ or $x \in SN_K(y)$ [19]. According to this K-adjacency, it is obvious that a K-topological space (X, κ^n_X) is a digital space.

A simple closed K-curve with l elements on $(\mathbb{Z}^n, \kappa^n)$, denoted by $SC^{n,l}_K$, is defined as a finite sequence $< x_i) >_{i \in [0, l-1]_{\mathbb{Z}}}$ in $\mathbb{Z}^n$ [20], where x_i and x_j are K-adjacent if and only if $|i - j| = \pm 1 (mod\, l)$.

Using the set of K-topological spaces (X, κ^n_X) and that of K-continuous maps for every ordered pair objects of K-topological spaces, we obtain the category of K-topological spaces, denoted by KTC [4].

Let us now recall basic concepts on M-topology. The *M-topology* on $\mathbb{Z}^2$, denoted by $(\mathbb{Z}^2, \gamma)$, is induced by the set $\{U(p) \,|\, p \in \mathbb{Z}^2\}$ in Equation (10) below as a base [16], where, for each point $p = (x, y) \in \mathbb{Z}^2$,

$$U(p) := \left\{ \begin{array}{l} N_4(p) \text{ if } x + y \text{ is even, and} \\ \{p\} : \text{ otherwise.} \end{array} \right\} \tag{10}$$

Owing to Equation (10), the set $U(p)$ is the smallest open neighborhood of the point p in $\mathbb{Z}^2$, denoted by $SN_M(p)$. Hereafter, for a subset $X \subseteq \mathbb{Z}^2$, we will denote by (X, γ_X) a subspace induced by $(\mathbb{Z}^2, \gamma)$, and it is called an *M-topological space*. For a point x in (X, γ_X), we denote by $SN_M(x)$ the *smallest open neighborhood* of x in (X, γ_X). For (X, γ_X), we say that distinct points x and y in X are *M-adjacent* in (X, γ_X) if $y \in SN_M(x)$ or $x \in SN_M(y)$ [10], where $SN_M(p)$ is the smallest open set containing the point p in (X, γ_X). According to this M-adjacency, it turns out that an M-topological space (X, γ_X) is a digital space [9].

A simple closed M-curve with l elements on $(\mathbb{Z}^2, \gamma)$, denoted by $SC^{2,l}_M$, is defined as a finite sequence $< x_i >_{i \in [0, l-1]_{\mathbb{Z}}}$ in $\mathbb{Z}^2$ [8], where x_i and x_j are M-adjacent if and only if $|i - j| = \pm 1 (mod\, l)$.

Using the set of M-topological spaces (X, γ_X) and that of M-continuous maps for every ordered pair of objects of M-topological spaces, we obtain the category of M-topological spaces, denoted by *MTC* [10].

Remark 1. *It is obvious that $SC_K^{n,l}$ [4], $SC_M^{2,l}$ [7] and $SC_k^{n,l}$[3] do not have the AFPP in the categories KTC, MTC and DTC, respectively. For instance, for $SC_K^{n,l} := (x_i)_{i\in[0,l-1]_{\mathbb{Z}}}$, consider a self-map of $SC_K^{n,l}$ such that $f(x_i) = x_{i+2(mod\, l)}$. Whereas f is a K-continuous map, there is no point $x \in SC_K^{n,l}$ such that $f(x) = x$ or $f(x)$ is K-adjacent to x [5]. By using a method similar to this approach for $SC_K^{n,l}$, it is obvious that $SC_M^{2,l}$ and $SC_k^{n,l}$ do not have the AFPP in DTC and MTC, respectively (see also [7]).*

3. Some Properties of a *K*-, an *M*-, a *U*- or an *L*-Digitization

Regarding several types of digitizations of $X(\subseteq \mathbb{R}^n)$ into a certain digital space, first of all we need to examine if given a digitization preserves the typical connectedness of X into the digital connectedness of the corresponding digitized space associated with a digital space structure. Indeed, the authors in [13] intensively studied this property. To combine this approach with the study of a preservation of the *AFPP* of a compact Euclidean topological space into that of its digitized space, we need to study a K-, an M-, a U- or an L-digitization [6,12,13]. Hence, this section recalls four types of local rules being used to formulate special kinds of neighborhoods of a given point $p \in \mathbb{Z}^n$.

Definition 2 ([6])**.** *In $\mathbb{R}^n$, for each point $p := (p_i)_{i\in[1,n]_{\mathbb{Z}}} \in \mathbb{Z}^n$, we define the set $N_K(p) := \{(x_i)_{i\in[1,n]_{\mathbb{Z}}}\}$, which is called the local K-neighborhood of p associated with $(\mathbb{Z}^n, \kappa^n)$, where $t \in \mathbb{Z}$ and*

$$\begin{Bmatrix} \text{if } p_i = 2t, \ \text{ then } x_i \in [2t - \frac{1}{2}, 2t + \frac{1}{2}], \\ \text{if } p_i = 2t+1, \ \text{ then } x_i \in (2t + \frac{1}{2}, 2t + \frac{3}{2}). \end{Bmatrix}$$

It is obvious [6] that the set $\{N_K(p) \mid p \in \mathbb{Z}^n\}$ is a partition of $\mathbb{R}^n$.

Remark 2. *In view of Definition 2, for each point $p \in \mathbb{Z}^n$, $N_K(p)$ can be substantially used to digitize $(\mathbb{R}^n, E^n)$ onto the K-topological space $(\mathbb{Z}^n, \kappa^n)$ by using the following map [6]: For each $N_K(p)$, $p \in \mathbb{Z}^n$*

$$N_K(p)(\subset \mathbb{R}^n) \to p(\in \mathbb{Z}^n).$$

Using $N_K(p)$ of Definition 2 and the method given in Remark 2, let us recall the K-digitization of a non-empty space (X, E_X^n).

Definition 3 ([6])**.** *For a nonempty space (X, E_X^n), we define a K-digitization of X, denoted by $D_K(X)$, to be the space with K-topology*

$$D_K(X) := \{p \in \mathbb{Z}^n \mid N_K(p) \cap X \neq \emptyset\}.$$

Let us now recall the M-digitization. For a point $p \in \mathbb{Z}^2$, the authors in [12,13] used an M-localized neighborhood of the given point p, denoted by $N_M(p)$, associated with $(\mathbb{Z}^2, \gamma)$.

Definition 4 ([12,13])**.** *In $\mathbb{R}^2$, for a point $p := (p_1, p_2) \in \mathbb{Z}^2$, we define the following neighborhood of p:*

$$N_M(p) := \left\{ \begin{array}{l} \{(t_1, t_2) \mid t_i \in [p_i - \frac{1}{2}, p_i + \frac{1}{2}],\ i \in \{1, 2\}\} \\ \textit{if } p = (p_1, p_2) \in \{(2m, 2n) \mid m, n \in \mathbb{Z}\}; \\ \{(t_1, t_2) \mid t_i \in [p_i - \frac{1}{2}, p_i + \frac{1}{2}],\ i \in \{1, 2\}\} \setminus \{(p_1 \pm \frac{1}{2}, p_2 \pm \frac{1}{2})\} \\ \textit{if } p = (p_1, p_2) \in \{(2m+1, 2n+1) \mid m, n \in \mathbb{Z}\} \textit{ and;} \\ \{(t_1, t_2) \mid t_i \in (p_i - \frac{1}{2}, p_i + \frac{1}{2}),\ i \in \{1, 2\}\} \\ \textit{if } p = (p_1, p_2) \in \{(2m, 2n+1), (2m+1, 2n) \mid m, n \in \mathbb{Z}\}, \end{array} \right\}$$

which is called an M-localized neighborhood of p associated with $(\mathbb{Z}^2, \gamma)$.

It is obvious [12] that the set $\{N_M(p) \mid p \in \mathbb{Z}^2\}$ is a partition of $\mathbb{R}^2$.

Remark 3. *In view of Definition 4, for each point $p \in \mathbb{Z}^2$, $N_M(p)$ can be substantially used to digitized $(\mathbb{R}^2, E^2)$ onto the M-topological space $(\mathbb{Z}^2, \gamma)$ via the following map. For each $N_M(p), p \in \mathbb{Z}^2$*

$$N_M(p)(\subset \mathbb{R}^2) \to p(\in \mathbb{Z}^2).$$

Using $N_K(p)$ of Definition 4 and the method given in Remark 3, we can define an M-digitization of a non-empty space (X, E^2_X), as follows.

Definition 5 ([12,13])**.** *For a nonempty 2-dimensional Euclidean topological space (X, E^2_X) in $\mathbb{R}^2$, we define an M-digitization of X, denoted by $D_M(X)$, to be the set in $\mathbb{Z}^2$ with M-topology*

$$D_M(X) := \{p \in \mathbb{Z}^2 \mid N_M(p) \cap X \neq \emptyset\}.$$

Remark 4. *In view of Definition 5, for each point $p \in \mathbb{Z}^2$, $N_M(p)$ can be substantially used to digitize the spaces (X, E^2_X) in $Ob(ETC)$ into M-topological spaces $D_M(X)$ in $Ob(MTC)$.*

Using Definitions 3 and 5 and Remarks 1, 2 and 3, for $X \subseteq \mathbb{R}^n$, we obtain the following:

Proposition 3. *For $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^2$, there are K- and M-digitizations*

$$D_K : P(\mathbb{R}^n) \to (\mathbb{Z}^n, \kappa^n) \text{ and } D_M : P(\mathbb{R}^2) \to (\mathbb{Z}^2, \gamma)$$

defined by

$$D_K(X) = (D_K(X), \kappa^n_{D_K(X)}) \text{ and } D_M(Y) = (D_M(Y), \gamma_{D_M(Y)}).$$

In Proposition 3, $P(T)$ means the power set of the set T.

Let us now recall the so-called U-digitization of (X, U_X). The upper limit topology (U-topology, for brevity) on $\mathbb{R}$, denoted by $(\mathbb{R}, E_U)$, is induced by the set $\{(a, b] \mid a, b \in \mathbb{R} \text{ and } a < b\}$ as a base [21]. Based on the U-topology on $\mathbb{R}$, we obtain the product topology on $\mathbb{R}^n$, denoted by $(\mathbb{R}^n, E^n_U)$, induced by $(\mathbb{R}, E_U)$. Based on $(\mathbb{R}^n, E^n_U)$, we use a U-local rule [13] that is used to digitize $(\mathbb{R}^n, E^n_U)$ into $(\mathbb{Z}^n, D^n)$, where $(\mathbb{Z}^n, D^n)$ is a discrete topological space.

Definition 6 ([13])**.** *Under $(\mathbb{R}^n, E^n_U)$, for a point $p := (p_i)_{i \in [1,n]_\mathbb{Z}} \in \mathbb{Z}^n$, we define $N_U(p) := \{(x_i)_{i \in [1,n]_\mathbb{Z}} \mid x_i \in (p_i - \frac{1}{2}, p_i + \frac{1}{2}]\}$, and we call $N_U(p)$ the U-localized neighborhood of p associated with $(\mathbb{R}^n, E^n_U)$.*

Using the U-local rule of Definition 6, we define the following:

Definition 7 ([13])**.** *Let $D_{U(k)} : (\mathbb{R}^n, E^n) \to (\mathbb{Z}^n, k)$ be the map defined by $D_{U(k)}(x) = p$, where $x \in N_U(p)$, $p \in \mathbb{Z}^n$ and the k-adjacency is taken according to the situation. Then, we say that $D_{U(k)}$ is a $U(k)$-digitization operator.*

Using the method similar to the establishment of $(\mathbb{R}^n, E^n_U)$ and the above U-local rule, let us now consider the L-local rule associated with L-topology and its product topology, where the lower limit topology (L-topology, for brevity) on $\mathbb{R}$, denoted by $(\mathbb{R}, E_L)$, is induced by the set $\{[a, b) \mid a, b \in \mathbb{R}$ and $a < b\}$ as a base [21].

Definition 8 ([13])**.** *Under $(\mathbb{R}^n, E^n_L)$, for a point $p := (p_i)_{i\in[1,n]_{\mathbb{Z}}} \in \mathbb{Z}^n$, we define $N_L(p) := \{(x_i)_{i\in[1,n]_{\mathbb{Z}}} \mid x_i \in [p_i - \frac{1}{2}, p_i + \frac{1}{2})\}$. We call $N_L(p)$ the L-localized neighborhood of p associated with $(\mathbb{R}^n, E^n_L)$.*

It is obvious [13] that the set $\{N_L(p) \mid p \in \mathbb{Z}^n\}$ is a partition of $\mathbb{R}^n$.

Using the L-local rule of Definition 8, we define the following:

Definition 9 ([13])**.** *Let $D_{L(k)} : (\mathbb{R}^n, E^n) \to (\mathbb{Z}^n, k)$ be the map defined by $D_{L(k)}(x) = p$, where $x \in N_L(p)$, $p \in \mathbb{Z}^n$ and the k-adjacency determined according to the situation. Then, we say that $D_{L(k)}$ is an $L(k)$-digitization operator.*

For a non-empty set $X \subset \mathbb{R}^n$, let us now recall a $U(k)$- and an $L(k)$-digitization, as follows.

Definition 10 ([13])**.** *Let X be a subspace in $(\mathbb{R}^n, E^n_U)$ (resp. $(\mathbb{R}^n, E^n_L)$). The U- (resp. L-) digitization of X, denoted by $D_U(X)$ (resp. $D_L(X)$), is defined as follows:*

$$\begin{cases} D_U(X) = \{p \in \mathbb{Z}^n \mid N_U(p) \cap X \neq \emptyset\}; \\ D_L(X) = \{p \in \mathbb{Z}^n \mid N_L(p) \cap X \neq \emptyset\} \end{cases}$$

with a k-adjacency of $\mathbb{Z}^n$ of (2) depending on the situation.

Using Definition 10, for $X \subseteq \mathbb{R}^n$, we obtain the following:

Proposition 4. *Given a k-adjacency of $\mathbb{Z}^n$ and $X \subseteq \mathbb{R}^n$, there are $U(k)$- and $L(k)$-digitizations*

$$D_{U(k)}, D_{L(k)} : P(\mathbb{R}^n) \to (\mathbb{Z}^n, k)$$

defined by

$$D_{U(k)}(X) = (D_U(X), k) \text{ and } D_{L(k)}(X) = (D_L(X), k).$$

In Proposition 4, $P(\mathbb{R}^n)$ means the power set of of the set $\mathbb{R}^n$.

4. Explorations of the Preservation of the *AFPP* of a Compact Plane into the *AFPP* of a *K*-, an *M*-, a *U(k)*-, or an *L(k)*-Digitized Space

The author in [8,10] proved the *FPP* of the smallest open neighborhood of $(\mathbb{Z}^n, \kappa^n)$ [10] and the non-*FPP* of a compact M-topological plane in $(\mathbb{Z}^2, \gamma)$ [8]. Thus, we may now pose the following queries about the *AFPP* of compact M-topological plane X and the preservation of the *AFPP* of a compact n-dimensional Euclidean space (or cube) into that of each of K-, M-, U- and L-digitization, as follows:

Question 1 Let X be the set $\prod_{i\in\{1,2,\cdots,n\}}[-l_i, l_i]_{\mathbb{Z}}$. How about the *FPP* or the *AFPP* of the K-topological space (X, κ^n_X) ?

Question 2 Let Y be the set $\prod_{i\in\{1,2\}}[-l_i, l_i]_{\mathbb{Z}}$. What about the *AFPP* of the M-topological space (Y, γ_Y) ?

Question 3 How about the preservation of the *AFPP* of a compact n-dimensional Euclidean cube into the *AFPP* of its $U(k)$-, or $L(k)$-digitized space ?

To address these queries, we first compare the *FPP* among a compact n-dimensional Euclidean space, a compact and n-dimensional K-topological space and a compact M-topological plane as follows:

Lemma 1. *The smallest open neighborhood of* $(\mathbb{Z}^2, \gamma)$ *has the FPP.*

Proof. As the smallest open set $SN_M(p)$ of $(\mathbb{Z}^2, \gamma), p \in \mathbb{Z}^2$, we may consider $U(p)$ (see (10)), where $p \in \{(2m, 2n), (2m+1, 2n+1) \mid m, n \in \mathbb{Z}\}$ or a singleton $\{p\}$, where $p \in \{(2m+1, 2n), (2m, 2n+1) \mid m, n \in \mathbb{Z}\}$.

- Case 1 Consider $U(p)$, where $p \in \{(2m, 2n), (2m+1, 2n+1) \mid m, n \in \mathbb{Z}\}$. Then, assume any M-continuous self-map f of $(U(p), \gamma_{U(p)})$. If p is mapped by f onto a point $q \in U(p) \setminus \{p\}$, then the map should be a constant map with $f(U(p)) = \{q\}$ according to the M-continuity of f, which implies that $(U(p), \gamma_{U(p)})$ has the *FPP* with a fixed point q associated with the map f. In addition, in case $f(p) = p$, the assertion is trivial.
- Case 2 Assume that $U(p)$ is a singleton. Then, it is obvious that $(U(p), \gamma_{U(p)})$ has the *FPP*.

□

In *MTC*, we say that an M-homeomorphic invariant is a property of an M-topological space which is invariable under M-homeomorphism [9].

Proposition 5 ([9]). *Each of the FPP and the AFPP from the viewpoint of MTC is an M-homeomorphic invariant.*

Indeed, in Lemma 1, the shape of $U(p)(\neq \{p\})$ is a diamond. Then, we may pose a query about the *FPP* of another shape of a diamond, as follows:

Corollary 1. *Consider an M-topological space* (X, γ_X) *which is M-homeomorphic to* (Y, γ_Y)*, where* $Y = \{(0,1) := y_1, (1,1) := y_2, (0,2) := y_3, (-1,1) := y_4, (0,0) := y_5\}$*. Then,* (X, γ_X) *has the FPP.*

Proof. According to Proposition 5, since the *FPP* in *MTC* is an M-topological invariant property [8], we may prove that (Y, γ_Y) has the *FPP*. For any M-continuous self-map f of (Y, γ_Y), we prove that there is always a point $y \in Y$ such that $f(y) = y$. To be precise, consider any M-continuous self-map f of (Y, γ_Y). In case $f(y_1) = y_1$, y_1 is a fixed point of f. In case $f(y_1) \neq y_1$, i.e., we may assume $f(y_1) \in \{y_2, y_3, y_4, y_5\}$. Then, according to the M-continuity of f, f should have the fixed point $f(y_1) \in Y$, which implies that there is a point $y_i \in \{y_2, y_3, y_4, y_5\}$ satisfying $f(y_i) = y_i$. Thus, (Y, γ_Y) is proved to have the *FPP*. □

The notion of an M-retract is used to study both the *FPP* and the *AFPP* of M-topological spaces [8]. Thus, let us recall it.

Definition 11 ([8]). *In MTC, we say that an M-continuous map* $r : (X', \gamma_{X'}) \to (X, \gamma_X)$ *is an M-retraction if*

(1) (X, γ_X) *is a subspace of* $(X', \gamma_{X'})$*, and*
(2) $r(a) = a$ *for all* $a \in (X, \gamma_X)$*.*

Then, we say that (X, γ_X) *is an M-retract of* $(X', \gamma_{X'})$*.*

The author in [8] proved that a compact M-topological plane does not have the *FPP*. Hence, as a more generalized version, we need to study the following:

Lemma 2 ([8]). *For* (X, γ_X) *let* (A, γ_A) *be an M-retract of* (X, γ_X)*. If* (X, γ_X) *has the AFPP, then* (A, γ_A) *also has the AFPP.*

Using this property, unlike the shape of a diamond in Lemma 1 and Corollary 1, as a generalization of the non-*FPP* of a compact M-topological plane [7], we now prove the non-*AFPP* of a compact M-topological plane, as follows:

Theorem 2. *A compact M-topological plane does not have the AFPP.*

Proof. Consider a compact M-topological plane (X, γ_X) containing the set $X_1 \in \{[2m, 2m+1]_{\mathbb{Z}} \times [2n, 2n+1]_{\mathbb{Z}}, [2m+1, 2m+2]_{\mathbb{Z}} \times [2n+1, 2n+2]_{\mathbb{Z}} \,|\, m, n \in \mathbb{Z}\}$. Then, we first prove that (X_1, γ_{X_1}) is an M-retract of (X, γ_X). Furthermore, we second permutate (X_1, γ_{X_1}) as an M-continuous self-map of (X_1, γ_{X_1}). After combining these two processes, we obtain an M-continuous self-map of (X, γ_X) which does not support the *AFPP* of (X, γ_X).

For instance, let us consider the compact M-topological plane $([-1, 1]^2_{\mathbb{Z}} := X, \gamma_X)$. Then, further consider two self-maps f_1 (see Figure 2a(1)), f_2 (see Figure 2a(2)) of X such that

$$\left\{\begin{array}{l} f_1(X_1) = \{(-1,0)\}, \text{ where } X_1 = \{(-1,0), (-1,1), (0,1)\}, \\ f_1(X_2) = \{(0,-1)\}, \text{ where } X_2 = \{(1,0), (1,-1), (0,-1)\}, \\ f_1(X_3) = \{(0,0)\}, \text{ where } X_3 = \{(0,0), (1,1)\}, \text{and} \\ f_1((-1,-1)) = (-1,-1). \end{array}\right\} \tag{11}$$

Furthermore, f_2 is defined as follows:

$$(0,0) \leftrightarrow (-1,-1) \text{ and } (-1,0) \leftrightarrow (0,-1). \tag{12}$$

Since the two maps f_1 and f_2 are M-continuous self-maps of X (see Equations (11) and (12)), the composite $f_2 \circ f_1$ is also an M-continuous self-map of X. However, owing to this composite $f_2 \circ f_1$, (X, γ_X) does not have the *AFPP*.

In general, let us consider a compact M-topological plane $([2m, 2m+l_1]_{\mathbb{Z}} \times [2n, 2m+l_2]_{\mathbb{Z}} := X, \gamma_X), l_i \in \mathbb{N}, i \in \{1,2\}$ (see Figure 2b) or $([2m, 2m+l_1]_{\mathbb{Z}} \times [2n+1, 2m+l_2]_{\mathbb{Z}} := X, \gamma_X), l_i \in \mathbb{N}, i \in \{1,2\}$ (see Figure 2c). Without loss of generality, we may assume $X := [0,5]_{\mathbb{Z}} \times [0,5]_{\mathbb{Z}}$ (see Figure 2b) or $X := [0,5]_{\mathbb{Z}} \times [1,5]_{\mathbb{Z}}$ (see Figure 2c) because the other cases are obviously similar to these cases. Then, consider the following two M-continuous self-maps g_1 (see Figure 2b(1)), g_2 (see Figure 2b(2)) of (X, γ_X) such that

$$\left\{\begin{array}{l} g_1(X_5) = \{(0,1)\}, \text{ where } X_5 = (\{0\} \times [2,5]_{\mathbb{Z}}) \cup \{(1,2), (1,4)\}, \\ g_1(X_6) = \{(1,1)\}, \text{ where } X_6 = ([2,5]_{\mathbb{Z}} \times [2,5]_{\mathbb{Z}}) \cup \{(1,3), (1,5), (3,1), (5,1)\}, \\ g_1(X_7) = \{(1,0)\}, \text{ where } X_7 = ([2,5]_{\mathbb{Z}} \times \{0\}) \cup \{(2,1), (4,1)\}, \text{ and} \\ g_1(X_8) = 1_{X_8}, \text{ where } X_8 = [0,1]_{\mathbb{Z}} \times [0,1]_{\mathbb{Z}}. \end{array}\right\} \tag{13}$$

Furthermore, g_2 is defined as follows:

$$(0,0) \leftrightarrow (1,1) \text{ and } (1,0) \leftrightarrow (0,1). \tag{14}$$

Then, the maps g_1 and g_2 are M-continuous maps (see Equations (13) and (14)) so that the composite $g_2 \circ g_2$ is also an M-continuous map. However, there is no point in X supporting the *AFPP* of (X, γ_X).

Similarly, let us consider another case such as $X := [0,5]_{\mathbb{Z}} \times [1,5]_{\mathbb{Z}}$ (see Figure 2c). Then, consider the following two M-continuous self-maps h_1, h_2 of (X, γ_X) such that

$$\left.\begin{cases} h_1(X_9) = \{(1,2)\}, \text{ where } X_9 = (\{0\} \times [1,5]_{\mathbb{Z}}) \cup (\{1\} \times [3,5]_{\mathbb{Z}}) \cup \{(2,3),(2,5)\}, \\ h_1(X_{10}) = \{(2,2)\}, \text{ where } X_{10} = ([3,5]_{\mathbb{Z}} \times [3,5]_{\mathbb{Z}}) \cup \{(2,4),(4,2)\}, \\ h_1(X_{11}) = \{(2,1)\}, \text{ where } X_{11} = ([3,5]_{\mathbb{Z}} \times \{1\}) \cup \{(3,2),(5,2)\}, \text{ and} \\ h_1(X_{12}) = 1_{X_{12}}, \text{ where } X_{12} = [1,2]_{\mathbb{Z}} \times [1,2]_{\mathbb{Z}}. \end{cases}\right\} \tag{15}$$

Furthermore, h_2 is defined as follows:

$$(1,1) \leftrightarrow (2,2) \text{ and } (2,1) \leftrightarrow (1,2). \tag{16}$$

Then, the maps h_1 and h_2 are M-continuous maps (see Equations (15) and (16)) so that the composite $h_2 \circ h_2$ is also an M-continuous map. However, there is no point in X supporting the *AFPP* of (X, γ_X). □

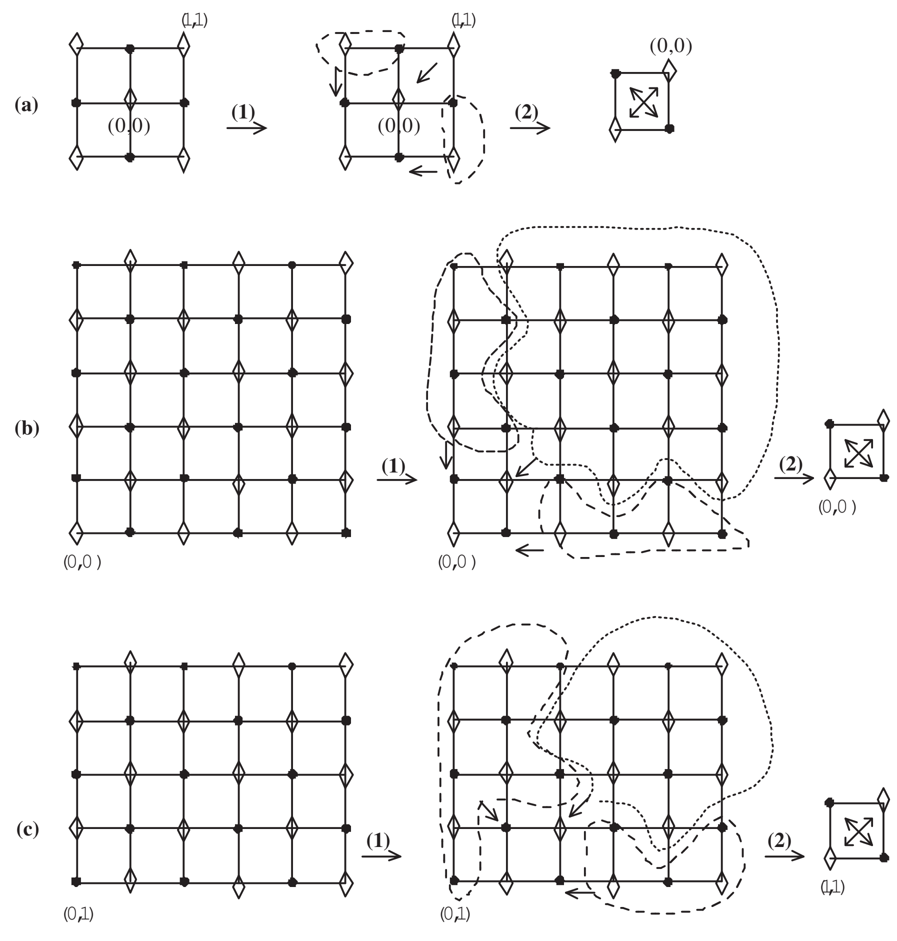

Figure 2. The non-*AFPP* of an compact M-topological plane.

Based on Propositions 2 and 3, 4 and Theorem 1, we have the following:

Theorem 3. *Let X be a compact and two-dimensional Euclidean topological plane, i.e., $(\prod_{i\in\{1,2\}}[-l_i, l_i] := X, E_X^2)$, $l_i \in \mathbb{N}$. Then, we obtain the following:*

(1) The functor D_M does not preserve the AFPP,
(2) The functor $D_{U(k)}$ preserves the AFPP if $k = 8$,

(3) *The functor $D_{L(k)}$ preserves the AFPP if $k = 8$*

Let X be a compact and n-dimensional Euclidean topological cube, i.e., $([-1,1]^n := X, E^n_X)$. Then, we obtain the following:

(4) *The functor $D_{U(k)}$ preserves the AFPP if $k = 3^n - 1$,*
(5) *The functor $D_{L(k)}$ preserves the AFPP if $k = 3^n - 1$.*

Proof. Based on Theorem 1 and Propositions 3 and 4, we consider the following digitizations:

$$\begin{Bmatrix} (1)\, D_M : ETC \to MTC, \\ (2)\, D_{U(k)} : ETC \to DTC \text{ in terms of the } U\text{-digitization, and} \\ (3)\, D_{L(k)} : ETC \to DTC \text{ via an } L\text{-digitization.} \end{Bmatrix}$$

(1) For $(X, E^2_X)(\subset (\mathbb{R}^2, E^2)$, since $D_M(X)$ is also M-connected [13] and furthermore that $(D_M(X), \gamma_{D_M(X)})$ is a compact M-topological plane, by Theorem 2, we obtain that $(D_M(X), \gamma_{D_M(X)})$ does not have the *AFPP*, which completes the proof.
(2) Using Propositions 2 and 4, the proof is completed.
(3) Using the method similar to the proof (2), we complete the proof.
(4) For $(X := [-1,1]^n, E^n_X)(\subset (\mathbb{R}^n, E^n)$, it is obvious that $(D_{U(k)}(X), k)$ is k-connected, $k = 3^n - 1$. Hence, by Theorem 1, the digital image $(D_{U(k)}(X), k), k = 3^n - 1$ has the *AFPP*. Hence, $D_{U(k)}$ preserves the *AFPP* if $k = 3^n - 1$.

Indeed, in case $k \neq 3^n - 1$, $(D_{U(k)}(X), k)$ does not have the *AFPP*. For instance, consider the compact Euclidean topological plane $([0,1] \times [0,1] := X, E^2_X)$. Since $([0,1] \times [0,1] := X, E^2_X)$ has the *FPP* [21], it obviously has the *AFPP*. Apparently, according to Theorem 1, the 4-connected digital image $(D_{U(4)}(X), 4)$ does not have the *AFPP* because $D_{U(4)}(X) = [0,1]^2_{\mathbb{Z}}$ is equal to $SC^{2,4}_4$. By Remark 1, $(D_{U(4)}(X), 4)$ does not have the *AFPP*.

(5) It is obvious that $(D_{L(k)}(X), k)$ is k-connected, $k = 3^n - 1$. Hence, by Theorem 1, the digital image $(D_{L(k)}(X), k), k = 3^n - 1$ has the *AFPP*.

Indeed, in case $k \neq 3^n - 1$, by using a method similar to the case of (2) above, we prove that $(D_{L(k)}(X), k)$ does not have the *AFPP*. □

Regarding Questions 1 and 3, the author in [10] proved the *FPP* of $SN_K(p)$ in $(\mathbb{Z}^n, \kappa^n)$. Moreover, the authors in [13] proved that the functor D_K preserves the connectedness of (X, κ^n_X) into its K-digitized space $(D_K(X), \kappa^n_{D_K(X)})$. Based on this situation, we can conclude that $D_K : ([-1,1]^n := X, E^n_X) \to (D_K(X), \kappa^n_{D_K(X)})$ preserves the *FPP* and furthermore the *AFPP*. As a general case of this case, we have the following conjecture.

The author in [10] proved that a smallest open set of $(\mathbb{Z}^n, \kappa^n)$ has the *FPP*, and the authors in [22] proved that $\prod_{i \in \{1,2,\cdots,n\}} [-l_i, l_i]_{\mathbb{Z}} := Y, \kappa^n_Y)$ has the *FPP*, and, using these results, we obtain the following:

Remark 5. *Let X be the compact and n-dimensional Euclidean space $\prod_{i \in \{1,2,\cdots,n\}} [-l_i, l_i] \subset \mathbb{R}^n, l_i \in \mathbb{N}$. Then, $(D_K(X), \kappa^n_{D_K(X)})$ has the AFPP because it has the FPP.*

5. Conclusions

We have studied the *AFPP* of an n-dimensional digital cube $(X, 3^n - 1)$ and also investigated the preservation of the *AFPP* via each of K-, $U(k)$- and $L(k)$-digitizations if $k = 3^n - 1$. In addition, based on the non-*FPP* of a compact M-topological plane, we also explored the non-preservation of the *AFPP* via an M-digitization. Furthermore, based on the *FPP* of $SN_K(p)$, we also proved the preservation of the *FPP* of $([-1,1]^n := X, E^n_X)$ via a K-digitization. This approach can facilitate the study of applied sciences such as object classifications, image processing, pattern recognition, artificial intelligence, and so on.

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