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Article

Matching Extendabilities of $G = C_m \vee P_n$

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Abstract: A graph is considered to be induced-matching extendable (bipartite matching extendable) if every induced matching (bipartite matching) of G is included in a perfect matching of G. The induced-matching extendability and bipartite-matching extendability of graphs have been of interest. By letting $G = C_m \vee P_n$ ($m \geq 3$ and $n \geq 1$) be the graph join of C_m (the cycle with m vertices) and P_n (the path with n vertices) contains a perfect matching, we find necessary and sufficient conditions for G to be induced-matching extendable and bipartite-matching extendable.

Keywords: perfect matching; k-extendable; induced matching extendable; bipartite matching extendable graph

1. Introduction

Throughout this paper we follow traditional graph theoretical terminologies and only consider simple connected graphs.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $S \subseteq V(G)$ we define $E(S) = \{uv \in E(G) \mid u, v \in S\}$. Similarly for $N \subseteq E(G)$ we have $V(N) = \{v \in V(G) \mid \exists x \in V(G),\ vx \in N\}$.

A collection of edges $M \subseteq E(G)$ is a matching of G if no two edges in M are adjacent in G. If $V(M) = V(G)$, then M is a perfect matching of G [1]. A matching M is an induced matching of G if no two edges of M are joined by an edge of G [2].

The problem of matching extendability asks if a matching of G is included in a perfect matching of G. First, the concept of k-extendable graphs (Definition 1) was introduced by Plummer [3]. The family of k-extendable graphs has been studied extensively [4–9].

Definition 1. *A connected graph G is called k-extendable if every matching of size k ($1 \leq k \leq \frac{1}{2}(|V(G)| - 2)$) extends to a perfect matching in G.*

Along this line the following definitions are also introduced.

Definition 2 ([10])**.** *A graph G is called k-factor-critical if $G - S$ has a perfect matching for any $S \subseteq V(G)$ with $|S| = k$.*

Definition 3 ([11])**.** *A connected graph G is called induced-matching extendable if any induced matching of G is included in a perfect matching of G.*

Furthermore, A matching M is a bipartite matching if $G[V(M)]$ is a bipartite graph [12]. From the research of k-extendable graphs, induced-matching extendable graphs and k-factor-critical graphs, the important roles of bipartite matching and non-bipartite matching were noticed. Wang et al. proposed the novel concept of bipartite-matching extendable graph in 2008 [12]. More recently, in 2017, Chiarelli et al. presented the sufficient conditions for graphs G and H, under which the lexicographic product $G[H]$ is 2-extendable [8].

Definition 4 ([12])**.** *A connected graph G is bipartite-matching extendable if every bipartite matching of G is included in a perfect matching of G.*

It is easy to see that a graph G is induced-matching extendable if it is bipartite-matching extendable. We also note that bipartite-matching extendability is the same as regular extendability when G itself is bipartite.

In general, matching extendable graphs frequently appear in applications and have been well studied. See, for instance, resonance circle theory in chemical graphs [13,14]. Other work on matching extendable graph can be found in [12,15–17].

In this note we will consider induced-matching extendability and bipartite-matching extendability for another specific class of graphs. Given two graphs G and H, $G \vee H$ is the graph join of G and H, with every vertex of G connected to every vertex of H by an edge. For general notations and facts on graph join one may see [1,4].

Graph joins have been considered in many different topics including edge-colouring [18], the chromatic index [19,20], the total chromatic number [21], the Laplacian spectrum [22], the skewness [23], the thickness [24]. In the remaining of this paper we consider the induced-matching extendability and bipartite-matching extendability of graph join $C_m \vee P_n$ $(m \geq 3,\ n \geq 1)$. In particular, we will show the following main results.

Theorem 1. *Let m and n be positive integers with $m + n \equiv 0 \pmod 2$:*

(i) If $m > n$, then $C_m \vee P_n$ is k-extendable if and only if

$$k \leq \left\lfloor \frac{n+1}{2} \right\rfloor .$$

(ii) If $n \geq m$, then $C_m \vee P_n$ is k-extendable if and only if

$$k \leq \left\lfloor \frac{m}{2} \right\rfloor .$$

Theorem 2. *Let $m \geq 3$ and $n \geq 1$ be two positive integers, then $C_m \vee P_n$ is induced-matching extendable if and only if*

$$m + n \equiv 0 \pmod 2$$

and

$$\frac{1-(-1)^r}{2} + \left\lfloor \frac{n}{3} \right\rfloor \leq m \leq 3n + 5$$

where $n \equiv r \pmod 3$ for some $0 \leq r \leq 2$.

Theorem 3. *Let $m \geq 3$ and $n \geq 1$ be two positive integers, then $C_m \vee P_n$ is bipartite-matching extendable if and only if*

$$m + n \equiv 0 \pmod 2$$

and

$$\left\lfloor \frac{m}{3} \right\rfloor - \frac{1-(-1)^r}{2} \leq n \leq 3m + 2$$

where $m \equiv r \pmod 3$ *for some* $0 \leq r \leq 2$.

First, in Section 2, we introduce some previously established facts on matching extendability. We then establish Theorem 1 on k-extendability of the graph join $C_m \vee P_n$ in Section 3. Lastly we prove Theorem 2 in Section 4 and Theorem 3 in Section 5.

2. Some Preliminaries

In this section, we list some interesting and useful previous results on matching extendabilities. They build the foundation for our study.

Lemma 1 ([25]). *A graph G has a perfect matching if and only if* $o(G-S) \leq |S|$ *for every* $S \subseteq V(G)$.

Here, $o(G)$ is the number of components of G with an odd number of vertices. From Lemma 1 and the definition of bipartite-matching extendability, Wang et al. obtained the following [12]:

Lemma 2 ([12]). *A graph G is bipartite-matching extendable if and only if* $o(G - V(M) - S) \leq |S|$ *for every bipartite matching M of G and every* $S \subseteq V(G) \setminus V(M)$.

Lemma 3 ([12]). *A graph G is bipartite-matching extendable if and only if* $o(G-S) \leq |S| - 2m_b(S)$ *for any* $S \subseteq V(G)$, *where* $m_b(S)$ *is the number of edges in a maximum bipartite matching of* $G[S]$.

A matching M is called a *forbidden matching* if it is a bipartite matching and $V(M)$ is a vertex cut such that $G - V(M)$ has an odd component [15]. Consequently a graph G is not bipartite-matching extendable if there exists a forbidden matching. The following is also shown in [15].

Lemma 4 ([15]). *If G is bipartite-matching extendable, then*

- *G is* 2*-connected;*
- *G does not have a forbidden matching;*
- *if* $\{u, v\}$ *is a vertex cut of G and* $uv \notin E(G)$, *then* $G - \{u, v\}$ *has exactly two components and both of them are odd ;*
- *for a bipartite matching M of G and an independent set X of* $G - V(M)$, $|N_{G-V(M)}(X)| \geq |S|$.

It is easy to see, from the definitions and properties of induced-matching extendable and bipartite-matching extendable graphs, that $W_n = C_n \vee K_1 (n \geq 3)$ is bipartite-matching extendable if and only if $n = 3, 5, 7$. We now move on to consider the generalization of K_1 to P_n.

3. On k-Extendable Graph Joins $C_m \vee P_n$

In this section, we examine when is the graph join $C_m \vee P_n$ k-extendable.

First, let $G = C_m \vee P_n$ have a perfect matching. Then we must have $m + n \equiv 0 \pmod 2$. In the rest of this section we only need to consider m and n being both even or both odd. We now consider Case (i) of Theorem 1. Case (ii) is similar.

Case (i) of Theorem 1. For convenience we let

$$C_m = x_1 x_2 \cdots x_{m-1} x_m$$

(with the edge $x_1 x_m$) and

$$P_n = y_1 y_2 \cdots y_{n-1} y_n.$$

We will first show that $C_m \vee P_n$ is not k-extendable for $k > \lfloor \frac{n+1}{2} \rfloor$. By the Definition 1, it suffices to show that $C_m \vee P_n - V(M)$ dose not have a perfect matching, for some matching M of $C_m \vee P_n$ with $|M| > \lfloor \frac{n+1}{2} \rfloor$.

- If m and n are odd:
  - If $n = 1$, note that for the matching $M_0 = \{x_1y_m, x_2x_3\}$ of G, $G - V(M_0)$ has an isolated vertex x_1. Consequently $G - V(M_0)$ does not have a perfect matching. Hence $C_m \vee P_n$ is not 2-extendable.
  - If $n \geq 3$, from the structural characteristics of graph $C_m \vee P_n$, there must exist a matching, say

    $$M_1 = \{y_4y_5, y_6y_7, \cdots, y_{n-1}y_n, x_1y_1, x_3y_2, x_4y_3\}$$

    of $C_m \vee P_n$ with size $\frac{n+3}{2}$, such that $C_m \vee P_n - V(M_1)$ has an isolated vertex x_2, which imply that $C_m \vee P_n - V(M_1)$ does not have perfect matching, and thus the $C_m \vee P_n$ is not $\frac{n+3}{2}$-extendable. Consequently, $C_m \vee P_n$ is k ($k \leq \frac{n+3}{2} - 1 = \left\lfloor \frac{n+1}{2} \right\rfloor$) extendable follows from the known conclusion that G is k-extendable, it must be a $k-1$ extendable.
- If m and n are even:
  - If $n = 2$, consider the matching $M_0' = \{x_1y_1, x_3y_2\}$. Then $G - V(M_0')$ has an isolated vertex x_2 and consequently without a perfect matching. Hence $C_m \vee P_n$ is not 2-extendable.
  - If $n \geq 4$, consider the matching

    $$M_1' = \{y_3y_4, y_5y_6, \cdots, y_{n-1}y_n, x_my_1, x_2y_2\}$$

    of size $\frac{n+2}{2}$. Then $C_m \vee P_n - V(M_1')$ has an isolated vertex x_1 and consequently without a perfect matching. Hence $C_m \vee P_n$ is not $\frac{n+2}{2}$-extendable.

We will now show that $C_m \vee P_n$ is indeed k-extendable for smaller values of k. The key idea in the following argument lies in the fact that $C_m \vee P_n$ is highly connected and $C_m \vee P_n - V(M)$ contains a Hamiltonian path of even order for any "small" matching M.

- If m and n are both odd, with $m > n \geq 1$, note that $C_m \vee P_n$ is $(n+2)$-connected. For every matching M of size $\frac{n+1}{2}$ in $C_m \vee P_n$, $C_m \vee P_n - V(M)$ is connected. By the definition of $C_m \vee P_n$, it is not difficult to see that $C_m \vee P_n - V(M)$ is not only connected but also containing a Hamiltonian path of even order. This implies that $C_m \vee P_n - V(M)$ has a perfect matching. With Definition 1, we have $C_m \vee P_n$ is k-extendable for $k \leq \frac{n+1}{2}$.
- Similarly, for even $m > n \geq 2$ and any matching M of size $\frac{n}{2}$, $C_m \vee P_n - V(M)$ contains a Hamiltonian path of even length. This implies that $C_m \vee P_n - V(M)$ has a perfect matching. Hence $C_m \vee P_n$ is k-extendable for $k \leq \frac{n}{2}$.

□

4. Proof of Theorem 2

We note that the $m + n \equiv 0 \pmod 2$ is obvious for the same reason as before. Also as before we let

$$C_m = x_1x_2 \cdots x_{m-1}x_m$$

(with the edge x_1x_m) and

$$P_n = y_1y_2 \cdots y_{n-1}y_n.$$

First we show the bounds for m are necessary for induced-matching extendability. Suppose that G is induced-matching extendable. For the lower bound:

- If $n = 3s$ for some s, consider an induced matching

  $$M = \{y_{3t+1}y_{3t+2} : 0 \leq t \leq s-1 = \lfloor \frac{n}{3} \rfloor - 1\}.$$

If $m \leq \lfloor \frac{n}{3} \rfloor - 1$, then

$$o(G - V(M) - V(C_m)) = o(P_n - V(M)) = \lfloor \frac{n}{3} \rfloor > m = |V(C_m)|.$$

This implies, by Lemma 1, that $G - V(M)$ does not have a perfect matching. This is a contradiction with the assumption G is induced-matching extendable. Therefore, $\lfloor \frac{n}{3} \rfloor \leq m$.

- If $n = 3s + 1$, consider an induced matching

$$M = \{y_{3t+2}y_{3t+3} : 0 \leq t \leq s - 1 = \lfloor \frac{n}{3} \rfloor - 1\}.$$

If $m \leq \lfloor \frac{n}{3} \rfloor$, then

$$o(G - V(M) - V(C_m)) = o(P_n - V(M)) = \lfloor \frac{n}{3} \rfloor + 1 > m = |V(C_m)|,$$

again implying, with Lemma 1, that $G - V(M)$ does not have a perfect matching, a contradiction. Therefore, $\lfloor \frac{n}{3} \rfloor + 1 \leq m$.

- If $n = 3s + 2$, consider the induced matching

$$M = \{y_{3t+1}y_{3t+2} : 0 \leq t \leq s - 1 = \lfloor \frac{n}{3} \rfloor - 1\}.$$

If $m \leq \lfloor \frac{n}{3} \rfloor - 1$, then

$$o(G - V(M) - V(C_m)) = o(P_n - V(M)) = \lfloor \frac{n}{3} \rfloor > m = |V(C_m)|,$$

yielding a contradiction as before. Hence $\lfloor \frac{n}{3} \rfloor \leq m$.

For the upper bound, consider the induced matching

$$M = \{x_{3i+1}x_{3i+2} : 0 \leq i \leq \lfloor \frac{m}{3} \rfloor - 1\}.$$

Then we have

$$o(C_m - V(M)) = \begin{cases} \lfloor \frac{m}{3} \rfloor - 1, & m \equiv 1 \pmod 3 \\ \lfloor \frac{m}{3} \rfloor, & m \equiv 0, 2 \pmod 3. \end{cases}$$

Since G is induced-matching extendable, $G - V(M)$ has a perfect matching. By Lemma 1, we have

$$o(G - V(M) - V(P_n)) = o(C_m - V(M)) \leq |V(P_n)| = n.$$

Thus

$$m \leq \begin{cases} 3n, & m \equiv 0 \pmod 3 \\ 3n + 4, & m \equiv 1 \pmod 3 \\ 3n + 5, & m \equiv 2 \pmod 3 \end{cases} \tag{1}$$

We now show that $G = C_m \vee P_n$ is indeed induced-matching extendable under these conditions. Let M be an induced matching:

- If $V(M) \cap V(C_m) = \emptyset$, then $M \subseteq E(P_n)$ and components of $P_n - V(M)$ are either paths or isolated vertices. Assume $M = \{x_2x_3, x_5x_6, \cdots, x_{3s-4}x_{3s-3}, x_{3s-1}x_{3s}\}$ be the induced matching that maximizes the number of odd components of $P_n - V(M)$, therefore the x_1, x_4, x_7, x_{3s-2} and x_{3s+1} are $\lfloor \frac{n}{3} \rfloor + 1$ isolated vertices of $P_n - V(M)$. It is easy to see that

$$o(P_n - V(M)) \leq \lfloor \frac{n}{3} \rfloor + 1$$

if $n \equiv 1 \pmod 3$, and

$$o(P_n - V(M)) \leq \lfloor \frac{n}{3} \rfloor$$

otherwise.

Now let N be a maximum matching of $P_n - V(M)$:

- if $P_n - V(M) \cup V(N)$ has no vertices left, then $G - V(M) \cup V(N)$ is isomorphic to C_m with m being even. Consequently, $G - V(M) \cup V(N)$ has a perfect matching N_1. Now $N \cup N_1$ is a perfect matching of $G - V(M)$.
- if $P_n - V(M) \cup V(N)$ has some vertices left, then $G - V(M) \cup V(N)$ is isomorphic to the join of C_m and some isolated vertices. It is easy to see that $G - V(M) \cup V(N)$ has a perfect matching, say N_2. We now have $N \cup N_2$ as a perfect matching of $G - V(M)$.

- If $V(M) \cap V(C_m) \neq \emptyset$, we consider two cases: $M \subseteq E(G) - E(C_m) \cup E(P_n)$ or $M \subseteq E(C_m)$:
  - If $M \subseteq E(G) - E(C_m) \cup E(P_n)$, then $|M| = 1$ and it is easy to find a perfect matching for $G - V(M)$.
  - If $M \subseteq E(C_m)$, then the components of $C_m - V(M)$ are either paths or isolated vertices. Let N be a maximum matching of $C_m - V(M)$:
    - If $C_m - V(M) \cup V(N)$ has no vertex left, then $G - V(M) \cup V(N)$ is isomorphic to P_n with even number of vertices. Thus $G - V(M) \cup V(N)$ has a perfect matching, say N_1. Consequently, $N \cup N_1$ is a perfect matching of $G - V(M)$.
    - If $C_m - V(M) \cup V(N)$ has some isolated vertices left, then $G - V(M) \cup V(N)$ is isomorphic to the join of P_n and some isolated vertices. With (1) we know $G - V(M) \cup V(N)$ has a perfect matching, say N_2. Now $N \cup N_2$ is a perfect matching of $G - V(M)$.

Therefore $G - V(M)$ has a perfect matching in all cases, implying that G is induced-matching extendable under the given conditions.

5. Proof of Theorem 3

Some of our arguments here are very similar to those of the previous section. Again we note that $m + n \equiv 0 \pmod 2$ is obvious, and label C_m and P_n the same way.

First we show the only if part. Let $G = C_m \vee P_n$ be bipartite-matching extendable. For the lower bound:

- If $m = 3s$ for some s, consider the bipartite matching

$$M = \{x_{3t+1}x_{3t+2} : 0 \leq t \leq s - 1 = \lfloor \frac{m}{3} \rfloor - 1\}.$$

  If $n \leq \lfloor \frac{m}{3} \rfloor - 1$, then

$$o(G - V(M) - V(P_n)) = o(C_m - V(M)) = \lfloor \frac{m}{3} \rfloor > n = |V(P_n)|,$$

  contradiction to Lemma 2 and the bipartite-matching extendability. Therefore $n \geq \lfloor \frac{m}{3} \rfloor$.
- If $m = 3s + 1$, consider the bipartite matching

$$M = \{x_{3t+1}x_{3t+2} : 0 \leq t \leq s - 1 = \lfloor \frac{m}{3} \rfloor - 1\}.$$

If $n \leq \lfloor \frac{m}{3} \rfloor - 2$, then

$$o(G - V(M) - V(P_n)) = o(C_m - V(M)) = \lfloor \frac{m}{3} \rfloor - 1 > n = |V(P_n)|,$$

a contradiction. Hence $n \geq \lfloor \frac{m}{3} \rfloor - 1$.

- If $m = 3s + 2$, consider the bipartite matching

$$M = \{x_{3t+1}x_{3t+2} : 0 \leq t \leq s = \lfloor \frac{m}{3} \rfloor\}.$$

If $n \leq \lfloor \frac{m}{3} \rfloor - 1$, then

$$o(G - V(M) - V(P_n)) = o(C_m - V(M)) = \lfloor \frac{m}{3} \rfloor - 1 > n = |V(P_n)|,$$

a contradiction. Thus $n \geq \lfloor \frac{m}{3} \rfloor$.

For the upper bound, consider the bipartite matching M with $E(M) \subset E(P_n)$. We have

$$o(P_n - V(M)) \leq \begin{cases} \lfloor \frac{n}{3} \rfloor + 1, & n \equiv 1 \pmod 3 \\ \lfloor \frac{n}{3} \rfloor, & n \equiv 0, 2 \pmod 3 \end{cases}$$

Suppose, for comparison, that $n > 3m + 2$. Consider now a bipartite matching

$$M' = \{y_{3i+2}y_{3i+3} : 0 \leq i \leq \lfloor \frac{n}{3} \rfloor - 1\}.$$

Then we have

$$o(G - V(M) - V(C_m)) = o(P_n - V(M)) \geq m = |V(C_m)|,$$

contradicting to Lemma 2 and the bipartite-matching extendability. Therefore we have $n \leq 3m + 2$.

Next we show that $G = C_m \vee P_n$ is indeed bipartite-matching extendable under these conditions. For this purpose we let M be a bipartite matching and we consider two cases:

- If $V(M) \cap V(P_n) = \emptyset$, then $M \subseteq E(C_m)$. Let N be a maximum matching of $C_m - V(M)$:
  - If $C_m - V(M) \cup V(N)$ has no vertices, then $G - V(M) \cup V(N)$ is isomorphic to P_n with even n. Consequently, $G - V(M) \cup V(N)$ has a perfect matching N_1. Now $N \cup N_1$ is a perfect matching of $G - V(M)$.
  - If $C_m - V(M) \cup V(N)$ has some isolated vertices, then $G - V(M) \cup V(N)$ is isomorphic to the join of P_n and some isolated vertices. It is easy to see that $G - V(M) \cup V(N)$ has a perfect matching, say N_2. Then $N \cup N_2$ is a perfect matching of $G - V(M)$.
- If $V(M) \cap V(P_n) \neq \emptyset$, we have $M \subseteq E(G) - E(C_m) \cup E(P_n)$ or $M \subseteq E(P_n)$:
  - If $M \subseteq E(G) - E(C_m) \cup E(P_n)$, we know that the $M-$ saturated vertices in P_n and C_m are not adjacent to each other in P_n, C_m, respectively. Moreover, the components of $P_n - V(M)$ and $C_m - V(M)$ are either path or isolated vertex. Further note that each vertex in components of $P_n - V(M)$ is adjacent to each vertex in components of $C_m - V(M)$, $G - V(M)$, resulting an odd length Hamiltonian path as before. Hence $G - V(M)$ has a perfect matching.
  - If $M \subseteq E(P_n)$, then

$$o(P_n - V(M)) \leq \begin{cases} \lfloor \frac{n}{3} \rfloor + 1, & n \equiv 1 \pmod 3 \\ \lfloor \frac{n}{3} \rfloor, & n \equiv 0, 2 \pmod 3 \end{cases}$$

Let N' be a maximum matching of $P_n - V(M)$. We then argue as before that $G - V(M) \cup V(N')$ has a perfect matching, say N'_1. Then $N' \cup N'_1$ is a perfect matching of $G - V(M)$.

Thus $G - V(M)$ has a perfect matching in all cases. Hence $G = C_m \vee P_n$ is bipartite-matching extendable under the given conditions.

6. Concluding Remarks

Through searching the Hamiltonian path or cycle for the auxiliary substructure of graph join $G = C_m \vee P_n$, we presented the necessary and sufficient conditions for G to be induced-matching extendable and bipartite-matching extendable. Our results provide a fundamental basis that helps study the induced and bipartite matching extendability for general graphs, and will probably be used to analyze the resonance circle properties of the chemical graphs.

As for future work, we plan to explore the correlations between the k-extendable and forbidden subgraphs of graphs. It is also interesting to investigate which of the graphs $C_m \vee C_n (m, n \geq 3)$ are k-extendable, induced-matching extendable or bipartite-matching extendable.

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