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# Stability Results for a Coupled System of Impulsive Fractional Differential Equations

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**Abstract:** In this paper, we establish sufficient conditions for the existence, uniqueness and Ulam–Hyers stability of the solutions of a coupled system of nonlinear fractional impulsive differential equations. The existence and uniqueness results are carried out via Banach contraction principle and Schauder’s fixed point theorem. The main theoretical results are well illustrated with the help of an example.

**Keywords:** caputo fractional derivative; coupled system; impulses; existence theory; stability theory

## 1. Introduction

Fractional differential equations (FDEs) provide an excellent tool for the description of memory and hereditary properties of different processes and materials. Thus, contrary to the classical derivative, the fractional derivative is nonlocal. Fractional calculus has played a very important role in enhancing the mathematical modeling of several phenomena occurring in engineering and scientific disciplines, such as blood flow systems, control theory, aerodynamics, the nonlinear oscillation of earthquake, the fluid-dynamic traffic model, polymer rheology, regular variation in thermodynamics, etc. FDEs are more accurate than the integer-order derivatives. Therefore, in the last few decades, fractional calculus has received great attention from researchers [1–13]. On the other hand, it is impossible to describe the complicated systems and processes with a single differential equation. Therefore, the coupled systems involving FDEs have also received incredible attention; consequently, many results are devoted to them [14–31].

It is well known that the effects of a pulse cannot be ignored in many processes and phenomena. For example, in biological systems such as heart beats, blood flows, mechanical systems with impact, population dynamical systems and so on. Thus, researchers used differential equations with impulses to describe the aforesaid kinds of phenomena. Therefore, many mathematicians studied impulsive FDEs with different boundary conditions; see [32–40] and references cited therein.

In fields such as numerical analysis, optimization theory, and nonlinear analysis, we mostly deal with the approximate solutions and hence we need to check how close these solutions are to the actual solutions of the related system. For this purpose, many approaches can be used, but the approach of Ulam–Hyers stability is a simple and easy one. The aforesaid stability was first initiated by Ulam in 1940 and then was confirmed by Hyers in 1941 [41,42]. That’s why this stability is known as Ulam–Hyers stability. In 1978 [43], Rassias generalized the Ulam–Hyers stability by considering

variables. Thereafter, mathematicians extended the work mentioned above to functional, differential, integrals and FDEs; for more information about the topic, the reader is recommended to [44–59].

Inspired from the above discussion, in this article, we study the existence, uniqueness and stability analysis of a coupled system of nonlinear FDEs with impulses of the form:

$$\begin{cases} {}^c\mathbf{D}^\alpha x(t) + h(t, {}^c\mathbf{D}^a x(t), {}^c\mathbf{D}^b y(t)) = 0, & t \neq t_m, m = 1, 2, \dots, n, \\ {}^c\mathbf{D}^\beta y(t) + w(t, {}^c\mathbf{D}^a x(t), {}^c\mathbf{D}^b y(t)) = 0, & t \neq t_m, m = 1, 2, \dots, n, \\ \Delta x|_{t=t_m} = M_{1m}(x(t_m)), \Delta x'|_{t=t_m} = N_{1m}(x(t_m)), \Delta x''|_{t=t_m} = O_{1m}(x(t_m)), \\ \Delta y|_{t=t_m} = M_{2m}(y(t_m)), \Delta y'|_{t=t_m} = N_{2m}(y(t_m)), \Delta y''|_{t=t_m} = O_{2m}(y(t_m)), \\ x(0) = x'(0) = 0, {}^c\mathbf{D}^\varepsilon x(\Omega) = x''(1), \\ y(0) = y'(0) = 0, {}^c\mathbf{D}^\rho y(\Phi) = y''(1), \end{cases} \quad (1)$$

where  $t \in J = [0, 1]$ ,  $2 < \alpha, \beta \leq 3$ ,  $0 < a, b, \varepsilon, \Omega, \rho, \Phi < 1$ .  ${}^c\mathbf{D}$  stands for Caputo fractional derivative and  $h, w : J \times \mathbb{R}^3 \rightarrow \mathbb{R}$  are continuous functions.  $M_{1m}, M_{2m}, N_{1m}, N_{2m}, O_{1m}, O_{2m} \in C(\mathbb{R}, \mathbb{R})$  and  $t_m$  satisfied  $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = 1$ ,  $\Delta x|_{t=t_m} = x(t_m^+) - x(t_m^-)$ ,  $\Delta x'|_{t=t_m} = x'(t_m^+) - x'(t_m^-)$ ,  $\Delta x''|_{t=t_m} = x''(t_m^+) - x''(t_m^-)$ ,  $\Delta y|_{t=t_m} = y(t_m^+) - y(t_m^-)$ ,  $\Delta y'|_{t=t_m} = y'(t_m^+) - y'(t_m^-)$ ,  $\Delta y''|_{t=t_m} = y''(t_m^+) - y''(t_m^-)$ ,  $x(t_m^+), y(t_m^+)$ , and  $x(t_m^-), y(t_m^-)$  represent the right and left limits of  $x(t), y(t)$ , respectively, at  $t = t_m$ .

The remaining article is organized as follows: In Section 2, we give some definitions and lemmas related to fractional calculus. In Section 3, we establish our main results about the existence and uniqueness of solutions for the proposed system (1). In Section 4, we study the Ulam–Hyers stability. In Section 5, we provide an example to support our main results.

## 2. Background Materials

In this section, we give some basic definitions of fractional calculus that will be used throughout the article.

**Definition 1.** (see [60]) If  $x : (0, \infty) \rightarrow \mathbb{R}$  and  $\alpha > 0$ , then the Caputo fractional derivative of order  $\alpha$  is defined as

$${}^c\mathbf{D}^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s) ds, \quad n-1 < \alpha < n, \quad n = [\alpha] + 1,$$

where  $[\alpha]$  denotes the integer part of real number  $\alpha$ , provided that the right side is pointwise defined on  $(0, \infty)$ .

**Definition 2.** (see [60]) The Riemann–Liouville fractional integral of order  $\alpha > 0$  for a function  $x : (0, \infty) \rightarrow \mathbb{R}$  is defined as

$$\mathbf{I}^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds, \quad t > 0,$$

provided that the right side is pointwise defined on  $(0, \infty)$ , where  $\Gamma$  is the Euler Gamma function.

**Lemma 1.** (see [60]) The solution of the differential equations involving Caputo derivative  ${}^c\mathbf{D}^\alpha x(t) = f(t)$ ,  $t \in J$ , has the form:

$$\mathbf{I}^\alpha {}^c\mathbf{D}^\alpha x(t) = \mathbf{I}^\alpha f(t) + e_0 + e_1 t + \dots + e_{n-1} t^{n-1},$$

for some  $e_i \in \mathbb{R}$ ,  $i = 0, 1, \dots, n-1$ ,  $n = [\alpha] + 1$ .

**Lemma 2.** (see [60]) If  $\alpha, \beta > 0$ ,  $t \in J$ , then, for  $x(t)$ , we have

$${}^c\mathbf{D}^\alpha \mathbf{I}^\alpha x(t) = x(t), \quad \mathbf{I}^\alpha \mathbf{I}^\beta x(t) = \mathbf{I}^{\alpha+\beta} x(t).$$

**Lemma 3.** (*Banach contraction principle*, see [59]) If  $X$  is real Banach space and  $\mathbf{W} : X \rightarrow X$  is a contraction mapping, then  $\mathbf{W}$  has a unique fixed point in  $X$ .

**Theorem 1.** (*Schauder fixed point theorem*, see [59]) If  $\omega$  is a closed bounded convex subset of a Banach space  $X$  and  $\mathbf{W} : \omega \rightarrow \omega$  is completely continuous, then  $\mathbf{W}$  has at least one fixed point in  $\omega$ .

For the sake of convenience, we introduce the Banach space as follows:  
Let  $J = [0, 1]$ ,  $J' = J/\{t_1, t_2, \dots, t_n\}$ . Define the set by

$$X = PC(J) = \{x(t) : x(t), x'(t), x''(t), {}^cD^{\alpha}x(t), {}^cD^{\beta}x(t) \in C(J'), x(t_m^+) \text{ and } x(t_m^-) \text{ exists and satisfying } x(t_m^-) = x(t_m), 1 \leq m \leq n\}.$$

It is easy to verify that  $X$  is a Banach space equipped with the norm:

$$\|x\|_0 = \max \left\{ \sup_{t \in J} |x(t)|, \sup_{t \in J} |x'(t)|, \sup_{t \in J} |x''(t)|, \sup_{t \in J} |{}^cD^{\alpha}x(t)|, \sup_{t \in J} |{}^cD^{\beta}x(t)| \right\}, \forall x(t) \in PC(J).$$

Similarly, we can define a set  $Y = PC(J)$ , which is a Banach space endowed with the defined norm:

$$\|y\|_0 = \max \left\{ \sup_{t \in J} |y(t)|, \sup_{t \in J} |y'(t)|, \sup_{t \in J} |y''(t)|, \sup_{t \in J} |{}^cD^{\alpha}y(t)|, \sup_{t \in J} |{}^cD^{\beta}y(t)| \right\}, \forall y(t) \in PC(J).$$

Furthermore, we define the Banach space  $Y' = X \times Y$  with the norms  $\|(x, y)\| = \|x\|_0 + \|y\|_0$  and  $\|(x, y)\| = \max \{\|x\|_0, \|y\|_0\}$ .

**Definition 3.** A pair of functions  $(x(t), y(t)) \in Y'$  is called a solution of (1) if  $(x(t), y(t))$  satisfy all the equations and boundary value conditions of the system (1).

**Lemma 4.** Assume that  $f \in C(J, \mathbb{R})$ . A function  $x \in PC(J)$  is a solution of the boundary value system

$$\begin{cases} {}^cD^{\alpha}x(t) + f(t) = 0, 2 < \alpha \leq 3, \\ \Delta x|_{t=t_m} = M_{1m}(x(t_m)), m = 1, 2, \dots, n, \\ \Delta x'|_{t=t_m} = N_{1m}(x(t_m)), m = 1, 2, \dots, n, \\ \Delta x''|_{t=t_m} = O_{1m}(x(t_m)), m = 1, 2, \dots, n, \\ x(0) = x'(0) = 0, {}^cD^{\varepsilon}x(\Omega) = x''(1), 0 < \varepsilon, \Omega < 1, \end{cases} \quad (2)$$

if and only if  $x \in PC(J)$  is the solution of integral equation

$$x(t) = \begin{cases} -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds + e^* t^2, t \in [0, t_1], \\ -\frac{1}{\Gamma(\alpha)} \int_{t_m}^t (t-s)^{\alpha-1} f(s) ds - \frac{1}{\Gamma(\alpha)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-1} f(s) ds \\ -\frac{1}{\Gamma(\alpha-1)} \sum_{j=1}^m (t-t_j) \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-2} f(s) ds - \frac{1}{2\Gamma(\alpha-2)} \sum_{j=1}^m (t-t_j)^2 \\ \times \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-3} f(s) ds + \sum_{j=1}^m M_{1j}(x(t_j)) + \sum_{j=1}^m (t-t_j) N_{1j}(x(t_j)) \\ + \sum_{j=1}^m \frac{(t-t_j)^2}{2} O_{1j}(x(t_j)) + e^* t^2, t \in (t_m, t_{m+1}], 1 \leq m \leq n, \end{cases} \quad (3)$$

where  $t \in (t_m, t_{m+1}]$ ,  $1 \leq m \leq n$ ,

$$\begin{aligned}
e^* = & \frac{\Gamma(3-\varepsilon)}{2(\Gamma(3-\varepsilon)-\Omega^{2-\varepsilon})} \left[ -\frac{1}{\Gamma(\alpha-\varepsilon)} \int_{t_j}^{\Omega} (\Omega-s)^{\alpha-\varepsilon-1} f(s) ds - \frac{\Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} \right. \\
& \times \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-2} f(s) ds - \frac{\Omega^{2-\varepsilon}}{\Gamma(\alpha-2)\Gamma(3-\varepsilon)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-3} f(s) ds \\
& + \frac{\Omega^{1-\varepsilon}}{\Gamma(\alpha-2)\Gamma(2-\varepsilon)} \sum_{j=1}^m t_j \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-3} f(s) ds + \frac{\Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m N_{1j}(x(t_j)) \\
& + \frac{\Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m O_{1j}(x(t_j)) - \frac{\Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m t_j O_{1j}(x(t_j)) + \frac{1}{\Gamma(\alpha-2)} \int_{t_m}^1 (1-s)^{\alpha-3} f(s) ds \\
& \left. + \frac{1}{\Gamma(\alpha-2)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-3} f(s) ds - \sum_{j=1}^m O_{1j}(x(t_j)) \right].
\end{aligned}$$

**Proof.** Applying Lemma 1, for some constants  $e_0, e_1, e_2 \in \mathbb{R}$ , we have

$$\begin{aligned}
x(t) &= -I^\alpha f(s) ds + e_0 + e_1 t + e_2 t^2 \\
&= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds + e_0 + e_1 t + e_2 t^2, \quad t \in [0, t_1].
\end{aligned} \tag{4}$$

Then, we obtain

$$\begin{cases} x'(t) = -\frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f(s) ds + e_1 + 2e_2 t, \\ x''(t) = -\frac{1}{\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-3} f(s) ds + 2e_2. \end{cases} \tag{5}$$

When  $t \in (t_1, t_2)$ , we have

$$\begin{cases} x(t) = -\frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} f(s) ds + e_3 + e_4(t-t_1) + e_5(t-t_1)^2, \\ x'(t) = -\frac{1}{\Gamma(\alpha-1)} \int_{t_1}^t (t-s)^{\alpha-2} f(s) ds + e_4 + 2e_5(t-t_1), \\ x''(t) = -\frac{1}{\Gamma(\alpha-2)} \int_{t_1}^t (t-s)^{\alpha-3} f(s) ds + 2e_5, \end{cases} \tag{6}$$

where  $e_3, e_4, e_5$  are arbitrary constants, from (4)–(6), we can find

$$x(t_1^-) = -\frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} f(s) ds + e_0 + e_1 t_1 + e_2 t_1^2, \quad x(t_1^+) = e_3, \tag{7}$$

$$x'(t_1^-) = -\frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1-s)^{\alpha-2} f(s) ds + e_1 + 2e_2 t_1, \quad x'(t_1^+) = e_4, \tag{8}$$

$$x''(t_1^-) = -\frac{1}{\Gamma(\alpha-2)} \int_0^{t_1} (t_1-s)^{\alpha-3} f(s) ds + 2e_2, \quad x''(t_1^+) = 2e_5. \tag{9}$$

Furthermore,  $\Delta x|_{t=t_1} = M_{11}(x(t_1))$ ,  $\Delta x'|_{t=t_1} = N_{11}(x(t_1))$ ,  $\Delta x''|_{t=t_1} = O_{11}(x(t_1))$ , and (7)–(9) give us:

$$\begin{cases} e_3 = -\frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} f(s) ds + e_0 + e_1 t_1 + e_2 t_1^2 + M_{11}(x(t_1)), \\ e_4 = -\frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1-s)^{\alpha-2} f(s) ds + e_1 + 2e_2 t_1 + N_{11}(x(t_1)), \\ e_5 = -\frac{1}{2\Gamma(\alpha-2)} \int_0^{t_1} (t_1-s)^{\alpha-3} f(s) ds + e_2 + \frac{1}{2} O_{11}(x(t_1)). \end{cases}$$

Plugging  $e_3$ ,  $e_4$ , and  $e_5$  into the first equation of (6) for  $t \in (t_1, t_2]$ , we have

$$\begin{aligned} x(t) = & -\frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} f(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s) f(s) ds - \frac{(t-t_1)}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1-s)^{\alpha-2} f(s) ds \\ & - \frac{(t-t_1)^2}{2\Gamma(\alpha-2)} \int_0^{t_1} (t_1-s)^{\alpha-3} f(s) ds + M_{11}(x(t_1)) + (t-t_1) N_{11}(x(t_1)) + \frac{(t-t_1)^2}{2} O_{11}(x(t_1)) \\ & + e_0 + e_1 t + e_2 t^2. \end{aligned}$$

Repeating the same process for  $t \in (t_m, t_{m+1}]$  such that ( $m = 1, 2, \dots, n$ ), then we can write

$$\begin{aligned} x(t) = & -\frac{1}{\Gamma(\alpha)} \int_{t_m}^t (t-s)^{\alpha-1} f(s) ds - \frac{1}{\Gamma(\alpha)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-1} f(s) ds - \frac{1}{\Gamma(\alpha-1)} \sum_{j=1}^m (t-t_j) \\ & \times \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-2} f(s) ds - \frac{1}{2\Gamma(\alpha-2)} \sum_{j=1}^m (t-t_j)^2 \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-3} f(s) ds + \sum_{j=1}^m M_{1j}(x(t_j)) \quad (10) \\ & + \sum_{j=1}^m (t-t_j) N_{1j}(x(t_j)) + \sum_{j=1}^m \frac{(t-t_j)^2}{2} O_{1j}(x(t_j)) + e_0 + e_1 t + e_2 t^2. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} x''(t) = & -\frac{1}{\Gamma(\alpha-2)} \int_{t_m}^t (t-s)^{\alpha-3} f(s) ds - \frac{1}{\Gamma(\alpha-2)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-3} f(s) ds + O_{1j}(x(t_j)) \quad (11) \\ & + 2e_2. \end{aligned}$$

By utilizing conditions  $x(0) = x'(0) = 0$  in (4), we get  $e_0 = e_1 = 0$ . In addition, it follows from (11) that

$$\begin{aligned} x''(1) = & -\frac{1}{\Gamma(\alpha-2)} \int_{t_m}^1 (1-s)^{\alpha-3} f(s) ds - \frac{1}{\Gamma(\alpha-2)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-3} f(s) ds + O_{1j}(x(t_j)) \quad (12) \\ & + 2e_2. \end{aligned}$$

In view of  $j \in \{0, 1, \dots, n\}$  such that  $\Omega \in (t_j, t_{j+1}]$ , we have

$$\begin{aligned} x(\Omega) = & -\frac{1}{\Gamma(\alpha)} \int_{t_j}^{\Omega} (\Omega-s)^{\alpha-1} f(s) ds - \frac{1}{\Gamma(\alpha)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-1} f(s) ds - \frac{1}{\Gamma(\alpha-1)} \sum_{j=1}^m (\Omega-t_j) \\ & \times \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-2} f(s) ds - \frac{1}{2\Gamma(\alpha-2)} \sum_{j=1}^m (\Omega-t_j)^2 \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-3} f(s) ds \quad (13) \\ & + \sum_{j=1}^m M_{1j}(x(t_j)) + \sum_{j=1}^m (\Omega-t_j) N_{1j}(x(t_j)) + \frac{(\Omega-t_j)^2}{2} O_{1j}(x(t_j)) + e_2 \Omega^2. \end{aligned}$$

By applying result (5), we get

$$\begin{aligned} {}^cD_{0+}^{\varepsilon} x(\Omega) = & -\frac{1}{\Gamma(\alpha-\varepsilon)} \int_{t_j}^{\Omega} (\Omega-s)^{\alpha-\varepsilon-1} f(s) ds - \frac{\Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-2} f(s) ds \\ & - \frac{\Omega^{2-\varepsilon}}{\Gamma(\alpha-2)\Gamma(3-\varepsilon)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-3} f(s) ds + \frac{\Omega^{1-\varepsilon}}{\Gamma(\alpha-2)\Gamma(2-\varepsilon)} \sum_{j=1}^m t_j \\ & \times \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-3} f(s) ds + \frac{\Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m N_{1j}(x(t_j)) + \frac{\Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m O_{1j} \quad (14) \\ & - \frac{\Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m t_j O_{1j} + \frac{2\Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} e_2. \end{aligned}$$

Since  ${}^cD_{0+}^\varepsilon x(\Omega) = x''(1)$ , thus (13) and (14) gives

$$\begin{aligned} e_2 &= \frac{\Gamma(3-\varepsilon)}{2(\Gamma(3-\varepsilon)-\Omega^{2-\varepsilon})} \left[ -\frac{1}{\Gamma(\alpha-\varepsilon)} \int_{t_j}^{\Omega} (\Omega-s)^{\alpha-\varepsilon-1} f(s) ds - \frac{\Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} \right. \\ &\quad \times \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-2} f(s) ds - \frac{\Omega^{2-\varepsilon}}{\Gamma(\alpha-2)\Gamma(3-\varepsilon)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-3} f(s) ds \\ &\quad + \frac{\Omega^{1-\varepsilon}}{\Gamma(\alpha-2)\Gamma(2-\varepsilon)} \sum_{j=1}^m t_j \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-3} f(s) ds + \frac{\Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m N_{1j}(x(t_j)) \\ &\quad + \frac{\Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m O_{1j} - \frac{\Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m t_j O_{1j} + \frac{1}{\Gamma(\alpha-2)} \int_{t_m}^1 (1-s)^{\alpha-3} f(s) ds \\ &\quad \left. + \frac{1}{\Gamma(\alpha-2)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-3} f(s) ds - \sum_{j=1}^m O_{1j}(x(t_j)) \right]. \end{aligned}$$

Plugging the values of  $e_0, e_1$  and  $e_2$  into (4) and (10), (3) can thus be obtained. Conversely, we consider that  $x(t)$  is a solution of (3). Then, it is obvious that (3) satisfies (2).  $\square$

Similarly as in Lemma 4, we can prove the following:

**Lemma 5.** Let  $\vartheta \in C(J, \mathbb{R})$ . A function  $y \in PC(J)$  is the solution of

$$\begin{cases} {}^cD^\beta y(t) + \vartheta(t) = 0, \quad 2 < \beta \leq 3, \\ \Delta y|_{t=t_m} = M_{2m}(y(t_m)), \quad m = 1, 2, \dots, n, \\ \Delta y'|_{t=t_m} = N_{2m}(y(t_m)), \quad m = 1, 2, \dots, n, \\ \Delta y''|_{t=t_m} = O_{2m}(y(t_m)), \quad m = 1, 2, \dots, n, \\ y(0) = y'(0) = 0, \quad {}^cD_{0+}^\rho y(\Phi) = y''(1), \quad 0 < \rho, \Phi < 1, \end{cases} \quad (15)$$

if and only if  $y \in PC(J)$  is the solution of the integral equation

$$y(t) = \begin{cases} -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \vartheta(s) ds + c^* t^2, \quad t \in [0, t_1], \\ -\frac{1}{\Gamma(\beta)} \int_{t_m}^t (t-s)^{\beta-1} \vartheta(s) ds - \frac{1}{\Gamma(\beta)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\beta-1} \vartheta(s) ds \\ -\frac{1}{\Gamma(\beta-1)} \sum_{j=1}^m (t-t_j) \int_{t_{j-1}}^{t_j} (t_j-s)^{\beta-2} \vartheta(s) ds - \frac{1}{2\Gamma(\beta-2)} \sum_{j=1}^m (t-t_j)^2 \\ \times \int_{t_{j-1}}^{t_j} (t_j-s)^{\beta-3} \vartheta(s) ds + \sum_{j=1}^m M_{2j}(y(t_j)) + \sum_{j=1}^m (t-t_j) N_{2j}(y(t_j)) \\ + \sum_{j=1}^m \frac{(t-t_j)^2}{2} O_{2j}(y(t_j)) + c^* t^2, \quad t \in (t_m, t_{m+1}], \quad 1 \leq m \leq n, \end{cases} \quad (16)$$

where  $c \in \{0, 1, \dots, n\}$  such that  $\Phi \in (t_c, t_{c+1}]$ , and

$$\begin{aligned}
c^* = & \frac{\Gamma(3-\rho)}{2(\Gamma(3-\rho)-\Phi^{2-\rho})} \left[ -\frac{1}{\Gamma(\beta-\rho)} \int_{t_j}^{\Phi} (\Phi-s)^{\beta-\rho-1} \vartheta(s) ds - \frac{\Phi^{1-\rho}}{\Gamma(\beta-1)\Gamma(2-\rho)} \right. \\
& \times \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\beta-2} \vartheta(s) ds - \frac{\Phi^{2-\rho}}{\Gamma(\beta-2)\Gamma(3-\rho)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\beta-3} \vartheta(s) ds \\
& + \frac{\Phi^{1-\rho}}{\Gamma(\beta-2)\Gamma(2-\rho)} \sum_{j=1}^m t_j \int_{t_{j-1}}^{t_j} (t_j-s)^{\beta-3} \vartheta(s) ds + \frac{\Phi^{1-\rho}}{\Gamma(2-\rho)} \sum_{j=1}^m N_{2j}(y(t_j)) \\
& + \frac{\Phi^{2-\rho}}{\Gamma(3-\rho)} \sum_{j=1}^m O_{2j}(y(t_j)) - \frac{\Phi^{1-\rho}}{\Gamma(2-\rho)} \sum_{j=1}^m t_j O_{2j}(y(t_j)) + \frac{1}{\Gamma(\beta-2)} \int_{t_m}^1 (1-s)^{\beta-3} \vartheta(s) ds \\
& \left. + \frac{1}{\Gamma(\beta-2)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\beta-3} \vartheta(s) ds - \sum_{j=1}^m O_{2j}(y(t_j)) \right].
\end{aligned}$$

### 3. Main Results

In this section, we use fixed point theorems to prove the existence of solutions to problem (1). According to Lemmas 4 and 5, we define operator  $\mathbf{W}: Y' \rightarrow Y'$  by

$$\mathbf{W}(x, y)(t) = (\mathbf{W}_1(x, y)(t), \mathbf{W}_2(x, y)(t))^T, \forall (x, y) \in Y', t \in [0, 1], \quad (17)$$

where

$$\mathbf{W}_1(x, y)(t) = \begin{cases} -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s) h(s, {}^c D^\alpha x(s), {}^c D^\beta y(s)) ds + e^* t^2, & t \in [0, t_1], \\ -\frac{1}{\Gamma(\alpha)} \int_{t_m}^t (t-s)^{\alpha-1} h(s, {}^c D^\alpha x(s), {}^c D^\beta y(s)) ds \\ -\frac{1}{\Gamma(\alpha)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-1} h(s, {}^c D^\alpha x(s), {}^c D^\beta y(s)) ds \\ -\frac{1}{\Gamma(\alpha-1)} \sum_{j=1}^m (t-t_j) \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-2} h(s, {}^c D^\alpha x(s), {}^c D^\beta y(s)) ds \\ -\frac{1}{2\Gamma(\alpha-2)} \sum_{j=1}^m (t-t_j)^2 \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-3} h(s, {}^c D^\alpha x(s), {}^c D^\beta y(s)) ds \\ + \sum_{j=1}^m M_{1j}(x(t_j)) + \sum_{j=1}^m (t-t_j) N_{1j}(x(t_j)) + \sum_{j=1}^m \frac{(t-t_j)^2}{2} O_{1j}(x(t_j)) + e^* t^2, \\ t \in (t_m, t_{m+1}], 1 \leq m \leq n, \end{cases}$$

and

$$\mathbf{W}_2(x, y)(t) = \begin{cases} -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} w(s, {}^c D^\alpha x(s), {}^c D^\beta y(s)) ds + c^* t^2, & t \in [0, t_1], \\ -\frac{1}{\Gamma(\beta)} \int_{t_m}^t (t-s)^{\beta-1} w(s, {}^c D^\alpha x(s), {}^c D^\beta y(s)) ds \\ -\frac{1}{\Gamma(\beta)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\beta-1} w(s, {}^c D^\alpha x(s), {}^c D^\beta y(s)) ds \\ -\frac{1}{\Gamma(\beta-1)} \sum_{j=1}^m (t-t_j) \int_{t_{j-1}}^{t_j} (t_j-s)^{\beta-2} w(s, {}^c D^\alpha x(s), {}^c D^\beta y(s)) ds \\ -\frac{1}{2\Gamma(\beta-2)} \sum_{j=1}^m (t-t_j)^2 \int_{t_{j-1}}^{t_j} (t_j-s)^{\beta-3} w(s, {}^c D^\alpha x(s), {}^c D^\beta y(s)) ds \\ + \sum_{j=1}^m M_{2j}(y(t_j)) + \sum_{j=1}^m (t-t_j) N_{2j}(y(t_j)) + \sum_{j=1}^m \frac{(t-t_j)^2}{2} O_{2j}(y(t_j)) + c^* t^2, \\ t \in (t_m, t_{m+1}], 1 \leq m \leq n, \end{cases}$$

with

$$\begin{aligned}
e^* = & Z_\alpha \left[ -\frac{1}{\Gamma(\alpha-\varepsilon)} \int_{t_j}^\Omega (\Omega-s)^{\alpha-\varepsilon-1} h(s, {}^c D^\alpha x(s), {}^c D^\beta y(s)) ds \right. \\
& - \frac{\Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-2} h(s, {}^c D^\alpha x(s), {}^c D^\beta y(s)) ds \\
& \left. - \frac{\Omega^{2-\varepsilon}}{\Gamma(\alpha-2)\Gamma(3-\varepsilon)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-3} h(s, {}^c D^\alpha x(s), {}^c D^\beta y(s)) ds \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{\Omega^{1-\varepsilon}}{\Gamma(\alpha-2)\Gamma(2-\varepsilon)} \sum_{j=1}^m t_j \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} h(s, {}^c D^a x(s), {}^c D^b y(s)) ds \\
& + \frac{\Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m N_{1j}(x(t_j)) + \frac{\Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m O_{1j}(x(t_j)) - \frac{\Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m t_j O_{1j}(x(t_j)) \\
& + \frac{1}{\Gamma(\alpha-2)} \int_{t_m}^1 (1-s)^{\alpha-3} h(s, {}^c D^a x(s), {}^c D^b y(s)) ds \\
& + \frac{1}{\Gamma(\alpha-2)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} h(s, {}^c D^a x(s), {}^c D^b y(s)) ds - \sum_{j=1}^m O_{1j}(x(t_j)) \Big], \\
c^* = & Z_\beta \left[ - \frac{1}{\Gamma(\beta-\rho)} \int_{t_j}^\Phi (\Phi - s)^{\beta-\rho-1} w(s, {}^c D^a x(s), {}^c D^b y(s)) ds \right. \\
& - \frac{\Phi^{1-\rho}}{\Gamma(\beta-1)\Gamma(2-\rho)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta-2} w(s, {}^c D^a x(s), {}^c D^b y(s)) ds \\
& - \frac{\Phi^{2-\rho}}{\Gamma(\beta-2)\Gamma(3-\rho)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta-3} w(s, {}^c D^a x(s), {}^c D^b y(s)) ds \\
& + \frac{\Phi^{1-\rho}}{\Gamma(\beta-2)\Gamma(2-\rho)} \sum_{j=1}^m t_j \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta-3} w(s, {}^c D^a x(s), {}^c D^b y(s)) ds \\
& + \frac{\Phi^{1-\rho}}{\Gamma(2-\rho)} \sum_{j=1}^m N_{2j}(y(t_j)) + \frac{\Phi^{2-\rho}}{\Gamma(3-\rho)} \sum_{j=1}^m O_{2j}(y(t_j)) - \frac{\Phi^{1-\rho}}{\Gamma(2-\rho)} \sum_{j=1}^m t_j O_{2j}(y(t_j)) \\
& + \frac{1}{\Gamma(\beta-2)} \int_{t_m}^1 (1-s)^{\beta-3} w(s, {}^c D^a x(s), {}^c D^b y(s)) ds \\
& \left. + \frac{1}{\Gamma(\beta-2)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta-3} w(s, {}^c D^a x(s), {}^c D^b y(s)) ds - \sum_{j=1}^m O_{2j}(y(t_j)) \right],
\end{aligned}$$

where  $Z_\alpha = \frac{\Gamma(3-\varepsilon)}{2(\Gamma(3-\varepsilon)-\Omega^{2-\varepsilon})}$  and  $Z_\beta = \frac{\Gamma(3-\rho)}{2(\Gamma(3-\rho)-\Phi^{2-\rho})}$ . Thus, solving problem (1) is equivalent to obtain a fixed point of the operator  $\mathbf{W}$ . Next, we have to prove the uniqueness of solutions of problem (1).

**Theorem 2.** Let the following conditions  $(\mathbf{M}_1)$ – $(\mathbf{M}_3)$  hold, and then the boundary value problem (1) has a unique solution.

$(\mathbf{M}_1)$ : For all  $t \in J$  and  $x_j, y_j \in \mathbb{R}$  ( $j = 1, 2$ ) there exists some positive constants  $\mu_j, \mu'_j$  ( $j = 1, 2$ ) such that

$$\begin{aligned}
|h(t, x_1, y_1) - h(t, x_2, y_2)| &\leq \mu_1 |x_1 - x_2| + \mu_2 |y_1 - y_2|, \\
|w(t, x_1, y_1) - w(t, x_2, y_2)| &\leq \mu'_1 |x_1 - x_2| + \mu'_2 |y_1 - y_2|.
\end{aligned}$$

$(\mathbf{M}_2)$ : For all  $x, y \in \mathbb{R}$ , there exist some positive constants  $I_{jk}, \hat{I}_{jk}, \check{I}_{jk}$  ( $j = 1, 2$ ;  $m = 1, 2, \dots, n$ ) such that

$$\begin{aligned}
|M_{jm}(x) - M_{jm}(y)| &\leq I_{jm} |x - y|, \\
|N_{jm}(x) - N_{jm}(y)| &\leq \hat{I}_{jm} |x - y|, \\
|O_{jm}(x) - O_{jm}(y)| &\leq \check{I}_{jm} |x - y|.
\end{aligned}$$

$(\mathbf{M}_3)$ :

$$\left[ (\mu_1 + \mu_2) \left( \frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha-1)} + \frac{z_\alpha}{\Gamma(\alpha-\varepsilon+1)} + \frac{z_\alpha\Omega^{1-\varepsilon}}{\Gamma(\alpha)\Gamma(2-\varepsilon)} + \frac{z_\alpha\Omega^{2-\varepsilon}}{\Gamma(\alpha-1)\Gamma(3-\varepsilon)} \right. \right. \\ \left. \left. + \frac{z_\alpha\Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} + \frac{2z_\alpha}{\Gamma(\alpha-1)} \right) + \sum_{j=1}^m I_{1j} + \sum_{j=1}^m \hat{I}_{1j} + \frac{z_\alpha\Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \hat{I}_{1j} + \frac{z_\alpha\Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m \check{I}_{1j} \right. \\ \left. + \frac{z_\alpha\Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \check{I}_{1j} + z_\alpha \sum_{j=1}^m \check{I}_{1j} + \frac{1}{2} \sum_{j=1}^m \check{I}_{1j} \right] < 1$$

and

$$\left[ (\mu'_1 + \mu'_2) \left( \frac{2}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta)} + \frac{1}{2\Gamma(\beta-1)} + \frac{z_\beta}{\Gamma(\beta-\rho+1)} + \frac{z_\beta\Phi^{1-\rho}}{\Gamma(\beta)\Gamma(2-\rho)} + \frac{z_\beta\Phi^{2-\rho}}{\Gamma(\beta-1)\Gamma(3-\rho)} \right. \right. \\ \left. \left. + \frac{z_\beta\Phi^{1-\rho}}{\Gamma(\beta-1)\Gamma(2-\rho)} + \frac{2z_\beta}{\Gamma(\beta-1)} \right) + \sum_{j=1}^m I_{2j} + \sum_{j=1}^m \hat{I}_{2j} + \frac{z_\beta\Phi^{1-\rho}}{\Gamma(2-\rho)} \sum_{j=1}^m \hat{I}_{2j} + \frac{z_\beta\Phi^{2-\rho}}{\Gamma(3-\rho)} \sum_{j=1}^m \check{I}_{2j} \right. \\ \left. + \frac{z_\beta\Phi^{1-\rho}}{\Gamma(2-\rho)} \sum_{j=1}^m \check{I}_{2j} + z_\beta \sum_{j=1}^m \check{I}_{2j} + \frac{1}{2} \sum_{j=1}^m \check{I}_{2j} \right] < 1.$$

**Proof.** By using the Banach contraction principle, we can prove that  $\mathbf{W}$ , defined by (17), has a fixed point. Before proving the main result first, we will prove the contraction. When  $t \in J$ , from (17) and conditions  $(\mathbf{M}_1) - (\mathbf{M}_2)$ , for all  $(x_1, y_1), (x_2, y_2) \in Y'$ , we have

$$\begin{aligned} & |\mathbf{W}_1(x_1, y_1)(t) - \mathbf{W}_1(x_2, y_2)(t)| \\ & \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |h(s, {}^c D^\alpha x_1(s), {}^c D^\beta y_1(s)) - h(s, {}^c D^\alpha x_2(s), {}^c D^\beta y_2(s))| ds \\ & \quad + \frac{z_\alpha t^2}{\Gamma(\alpha-\varepsilon)} \int_{t_j}^{\Omega} (\Omega-s)^{\alpha-\varepsilon-1} |h(s, {}^c D^\alpha x_1(s), {}^c D^\beta y_1(s)) - h(s, {}^c D^\alpha x_2(s), {}^c D^\beta y_2(s))| ds \\ & \quad + \frac{z_\alpha t^2 \Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-2} |h(s, {}^c D^\alpha x_1(s), {}^c D^\beta y_1(s)) - h(s, {}^c D^\alpha x_2(s), {}^c D^\beta y_2(s))| ds \\ & \quad + \frac{z_\alpha t^2 \Omega^{2-\varepsilon}}{\Gamma(\alpha-2)\Gamma(3-\varepsilon)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-3} |h(s, {}^c D^\alpha x_1(s), {}^c D^\beta y_1(s)) - h(s, {}^c D^\alpha x_2(s), {}^c D^\beta y_2(s))| ds \\ & \quad + \frac{z_\alpha t^2 \Omega^{1-\varepsilon}}{\Gamma(\alpha-2)\Gamma(2-\varepsilon)} \sum_{j=1}^m t_j \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-3} |h(s, {}^c D^\alpha x_1(s), {}^c D^\beta y_1(s)) - h(s, {}^c D^\alpha x_2(s), {}^c D^\beta y_2(s))| ds \\ & \quad + \frac{z_\alpha t^2 \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m |N_{1j}(x_1(t_j)) - N_{1j}(x_2(t_j))| + \frac{z_\alpha t^2 \Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m |O_{1j}(x_1(t_j)) - O_{1j}(x_2(t_j))| \\ & \quad + \frac{z_\alpha t^2}{\Gamma(\alpha-2)} \int_{t_m}^1 (1-s)^{\alpha-3} |h(s, {}^c D^\alpha x_1(s), {}^c D^\beta y_1(s)) - h(s, {}^c D^\alpha x_2(s), {}^c D^\beta y_2(s))| ds \\ & \quad + \frac{z_\alpha t^2}{\Gamma(\alpha-2)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-3} |h(s, {}^c D^\alpha x_1(s), {}^c D^\beta y_1(s)) - h(s, {}^c D^\alpha x_2(s), {}^c D^\beta y_2(s))| ds \\ & \quad + \frac{z_\alpha t^2 \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m |t_j| |O_{1j}(x_1(t_j)) - O_{1j}(x_2(t_j))| + z_\alpha t^2 \sum_{j=1}^m |O_{1j}(x_1(t_j)) - O_{1j}(x_2(t_j))|, \\ & \leq \frac{1}{\Gamma(\alpha+1)} [\mu_1 |{}^c D^\alpha x_1(s) - {}^c D^\alpha x_2(s)| + \mu_2 |{}^c D^\beta y_1(s) - {}^c D^\beta y_2(s)|] \end{aligned}$$

$$\begin{aligned}
& + \frac{z_\alpha}{\Gamma(\alpha - \varepsilon + 1)} [\mu_1 |^c \mathbf{D}^a x_1(s) - ^c \mathbf{D}^a x_2(s)| + \mu_2 |^c \mathbf{D}^b y_1(s) - ^c \mathbf{D}^b y_2(s)|] \\
& + \frac{z_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha) \Gamma(2-\varepsilon)} [\mu_1 |^c \mathbf{D}^a x_1(s) - ^c \mathbf{D}^a x_2(s)| + \mu_2 |^c \mathbf{D}^b y_1(s) - ^c \mathbf{D}^b y_2(s)|] \\
& + \frac{z_\alpha \Omega^{2-\varepsilon}}{\Gamma(\alpha-1) \Gamma(3-\varepsilon)} [\mu_1 |^c \mathbf{D}^a x_1(s) - ^c \mathbf{D}^a x_2(s)| + \mu_2 |^c \mathbf{D}^a y_1(s) - ^c \mathbf{D}^a y_2(s)|] \\
& + \frac{z_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha-1) \Gamma(2-\varepsilon)} [\mu_1 |^c \mathbf{D}^a x_1(s) - ^c \mathbf{D}^a x_2(s)| + \mu_2 |^c \mathbf{D}^b y_1(s) - ^c \mathbf{D}^b y_2(s)|] \\
& + \frac{z_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m |N_{1j}(x_1(t_j)) - N_{1j}(x_2(t_j))| + \frac{z_\alpha \Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m |O_{1j}(x_1(t_j)) - O_{1j}(x_2(t_j))| \\
& + \frac{z_\alpha}{\Gamma(\alpha-1)} [\mu_1 |^c \mathbf{D}^a x_1(s) - ^c \mathbf{D}^a x_2(s)| + \mu_2 |^c \mathbf{D}^b y_1(s) - ^c \mathbf{D}^b y_2(s)|] \\
& + \frac{z_\alpha}{\Gamma(\alpha-1)} [\mu_1 |^c \mathbf{D}^a x_1(s) - ^c \mathbf{D}^a x_2(s)| + \mu_2 |^c \mathbf{D}^b y_1(s) - ^c \mathbf{D}^b y_2(s)|] \\
& + \frac{z_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m t_j |O_{1j}(x_1(t_j)) - O_{1j}(x_2(t_j))| + z_\alpha \sum_{j=1}^m |O_{1j}(x_1(t_j)) - O_{1j}(x_2(t_j))|, \\
& \leq \frac{1}{\Gamma(\alpha+1)} [\mu_1 \|x_1 - x_2\|_0 + \mu_2 \|y_1 - y_2\|_0] + \frac{z_\alpha}{\Gamma(\alpha - \varepsilon + 1)} [\mu_1 \|x_1 - x_2\|_0 + \mu_2 \|y_1 - y_2\|_0] \\
& + \frac{z_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha) \Gamma(2-\varepsilon)} [\mu_1 \|x_1 - x_2\|_0 + \mu_2 \|y_1 - y_2\|_0] + \frac{z_\alpha \Omega^{2-\varepsilon}}{\Gamma(\alpha-1) \Gamma(3-\varepsilon)} [\mu_1 \|x_1 - x_2\|_0 + \mu_2 \|y_1 - y_2\|_0] \\
& + \frac{z_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha-1) \Gamma(2-\varepsilon)} [\mu_1 \|x_1 - x_2\|_0 + \mu_2 \|y_1 - y_2\|_0] + \frac{z_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \hat{I}_{1j} \|x_1 - x_2\|_0 + \frac{z_\alpha \Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m \check{I}_{1j} \quad (18) \\
& \times \|x_1 - x_2\|_0 + \frac{z_\alpha}{\Gamma(\alpha-1)} [\mu_1 \|x_1 - x_2\|_0 + \mu_2 \|y_1 - y_2\|_0] + \frac{z_\alpha}{\Gamma(\alpha-1)} [\mu_1 \|x_1 - x_2\|_0 + \mu_2 \|y_1 - y_2\|_0] \\
& + \frac{z_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \check{I}_{1j} \|x_1 - x_2\|_0 + z_\alpha \sum_{j=1}^m \check{I}_{1j} \|x_1 - x_2\|_0, \\
& \leq \frac{1}{\Gamma(\alpha+1)} [\mu_1 + \mu_2] \|(x_1 - x_2, y_1 - y_2)\| + \frac{z_\alpha}{\Gamma(\alpha - \varepsilon + 1)} [\mu_1 + \mu_2] \|(x_1 - x_2, y_1 - y_2)\| \\
& + \frac{z_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha) \Gamma(2-\varepsilon)} [\mu_1 + \mu_2] \|(x_1 - x_2, y_1 - y_2)\| + \frac{z_\alpha \Omega^{2-\varepsilon}}{\Gamma(\alpha-1) \Gamma(3-\varepsilon)} [\mu_1 + \mu_2] \|(x_1 - x_2, y_1 - y_2)\| \\
& + \frac{z_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha-1) \Gamma(2-\varepsilon)} [\mu_1 + \mu_2] \|(x_1 - x_2, y_1 - y_2)\| + \frac{z_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \hat{I}_{1j} \|(x_1 - x_2, y_1 - y_2)\| \\
& + \frac{z_\alpha \Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m \check{I}_{1j} \|(x_1 - x_2, y_1 - y_2)\| + \frac{z_\alpha}{\Gamma(\alpha-1)} [\mu_1 + \mu_2] \|(x_1 - x_2, y_1 - y_2)\| \\
& + \frac{z_\alpha}{\Gamma(\alpha-1)} [\mu_1 + \mu_2] \|(x_1 - x_2, y_1 - y_2)\| + \frac{z_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \check{I}_{1j} \|(x_1 - x_2, y_1 - y_2)\| \\
& + z_\alpha \sum_{j=1}^m \check{I}_{1j} \|(x_1 - x_2, y_1 - y_2)\|, \\
& \leq z_\alpha \left[ (\mu_1 + \mu_2) \left( \frac{1}{z_\alpha \Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha - \varepsilon + 1)} + \frac{\Omega^{1-\varepsilon}}{\Gamma(\alpha) \Gamma(2-\varepsilon)} + \frac{\Omega^{2-\varepsilon}}{\Gamma(\alpha-1) \Gamma(3-\varepsilon)} + \frac{\Omega^{1-\varepsilon}}{\Gamma(\alpha-1) \Gamma(2-\varepsilon)} \right. \right. \\
& \left. \left. + \frac{2}{\Gamma(\alpha-1)} \right) + \frac{\Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \hat{I}_{1j} + \frac{\Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m \check{I}_{1j} + \frac{\Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \check{I}_{1j} + \sum_{j=1}^m \check{I}_{1j} \right] \\
& \times \|(x_1 - x_2, y_1 - y_2)\|, \quad t \in [0, t_1].
\end{aligned}$$

When  $t \in (t_m, t_{m+1}]$ , then

$$\begin{aligned}
& |\mathbf{W}_1(x_1, y_1)(t) - \mathbf{W}_1(x_2, y_2)(t)| \\
& \leq \frac{1}{\Gamma(\alpha)} \int_{t_m}^t (t-s)^{\alpha-1} |h(s, {}^c D^a x_1(s), {}^c D^b y_1(s)) - h(s, {}^c D^a x_2(s), {}^c D^b y_2(s))| ds \\
& + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-1} |h(s, {}^c D^a x_1(s), {}^c D^b y_1(s)) - h(s, {}^c D^a x_2(s), {}^c D^b y_2(s))| ds \\
& + \frac{1}{\Gamma(\alpha-1)} \sum_{j=1}^m |(t - t_j)| \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-2} |h(s, {}^c D^a x_1(s), {}^c D^b y_1(s)) - h(s, {}^c D^a x_2(s), {}^c D^b y_2(s))| ds \\
& + \frac{1}{2\Gamma(\alpha-2)} \sum_{j=1}^m |(t - t_j)^2| \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} |h(s, {}^c D^a x_1(s), {}^c D^b y_1(s)) - h(s, {}^c D^a x_2(s), {}^c D^b y_2(s))| ds \\
& + \sum_{j=1}^m |\mathcal{M}_{1j}(x_1(t_j)) - \mathcal{M}_{1j}(x_2(t_j))| + \sum_{j=1}^m |(t - t_j)| |\mathcal{N}_{1j}(x_1(t_j)) - \mathcal{N}_{1j}(x_2(t_j))| \\
& + \frac{z_\alpha t^2}{\Gamma(\alpha-\varepsilon)} \int_{t_j}^\Omega (\Omega - s)^{\alpha-\varepsilon-1} |h(s, {}^c D^a x_1(s), {}^c D^b y_1(s)) - h(s, {}^c D^a x_2(s), {}^c D^b y_2(s))| ds \\
& + \frac{z_\alpha t^2 \Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-2} |h(s, {}^c D^a x_1(s), {}^c D^b y_1(s)) - h(s, {}^c D^a x_2(s), {}^c D^b y_2(s))| ds \quad (19) \\
& + \frac{z_\alpha t^2 \Omega^{2-\varepsilon}}{\Gamma(\alpha-2)\Gamma(3-\varepsilon)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} |h(s, {}^c D^a x_1(s), {}^c D^b y_1(s)) - h(s, {}^c D^a x_2(s), {}^c D^b y_2(s))| ds \\
& + \frac{z_\alpha t^2 \Omega^{1-\varepsilon}}{\Gamma(\alpha-2)\Gamma(2-\varepsilon)} \sum_{j=1}^m |t_j| \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} |h(s, {}^c D^a x_1(s), {}^c D^b y_1(s)) - h(s, {}^c D^a x_2(s), {}^c D^b y_2(s))| ds \\
& + \frac{z_\alpha t^2 \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m |\mathcal{N}_{1j}(x_1(t_j)) - \mathcal{N}_{1j}(x_2(t_j))| + \frac{z_\alpha t^2 \Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m |\mathcal{O}_{1j}(x_1(t_j)) - \mathcal{O}_{1j}(x_2(t_j))| \\
& + \frac{z_\alpha t^2}{\Gamma(\alpha-2)} \int_{t_m}^1 (1-s)^{\alpha-3} |h(s, {}^c D^a x_1(s), {}^c D^b y_1(s)) - h(s, {}^c D^a x_2(s), {}^c D^b y_2(s))| ds \\
& + \frac{z_\alpha t^2}{\Gamma(\alpha-2)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} |h(s, {}^c D^a x_1(s), {}^c D^b y_1(s)) - h(s, {}^c D^a x_2(s), {}^c D^b y_2(s))| ds \\
& + \frac{z_\alpha t^2 \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m |t_j| |\mathcal{O}_{1j}(x_1(t_j)) - \mathcal{O}_{1j}(x_2(t_j))| + z_\alpha t^2 \sum_{j=1}^m |\mathcal{O}_{1j}(x_1(t_j)) - \mathcal{O}_{1j}(x_2(t_j))| \\
& + \sum_{j=1}^m \frac{|(t - t_j)^2|}{2} |\mathcal{O}_{1j}(x_1(t_j)) - \mathcal{O}_{1j}(x_2(t_j))|.
\end{aligned}$$

Utilizing **(M<sub>1</sub>)** and **(M<sub>2</sub>)** in (19) and taking the maximum, we get

$$\begin{aligned}
& \leq \frac{1}{\Gamma(\alpha+1)} [\mu_1 \|x_1 - x_2\|_0 + \mu_2 \|y_1 - y_2\|_0] + \frac{1}{\Gamma(\alpha+1)} [\mu_1 \|x_1 - x_2\|_0 + \mu_2 \|y_1 - y_2\|_0] \\
& + \frac{1}{\Gamma(\alpha)} [\mu_1 \|x_1 - x_2\|_0 + \mu_2 \|y_1 - y_2\|_0] + \frac{1}{2\Gamma(\alpha-1)} [\mu_1 \|x_1 - x_2\|_0 + \mu_2 \|y_1 - y_2\|_0] \\
& + \sum_{j=1}^m I_{1j} \|x_1 - x_2\|_0 + \sum_{j=1}^m \hat{I}_{1j} \|x_1 - x_2\|_0 + \frac{z_\alpha}{\Gamma(\alpha-\varepsilon+1)} [\mu_1 \|x_1 - x_2\|_0 + \mu_2 \|y_1 - y_2\|_0] \\
& + \frac{z_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha)\Gamma(2-\varepsilon)} [\mu_1 \|x_1 - x_2\|_0 + \mu_2 \|y_1 - y_2\|_0] + \frac{z_\alpha \Omega^{2-\varepsilon}}{\Gamma(\alpha-1)\Gamma(3-\varepsilon)} [\mu_1 \|x_1 - x_2\|_0 + \mu_2 \|y_1 - y_2\|_0]
\end{aligned}$$

$$\begin{aligned}
& + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} [\mu_1 \|x_1 - x_2\|_0 + \mu_2 \|y_1 - y_2\|_0] + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \hat{I}_{1j} \|x_1 - x_2\|_0 \\
& + \frac{\mathcal{Z}_\alpha \Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m \check{I}_{1j} \|x_1 - x_2\|_0 + \frac{\mathcal{Z}_\alpha}{\Gamma(\alpha-1)} [\mu_1 \|x_1 - x_2\|_0 + \mu_2 \|y_1 - y_2\|_0] \\
& + \frac{\mathcal{Z}_\alpha}{\Gamma(\alpha-1)} [\mu_1 \|x_1 - x_2\|_0 + \mu_2 \|y_1 - y_2\|_0] + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \check{I}_{1j} \|x_1 - x_2\|_0 + \mathcal{Z}_\alpha \sum_{j=1}^m \check{I}_{1j} \|x_1 - x_2\|_0 \\
& + \frac{1}{2} \sum_{j=1}^m \check{I}_{1j} \|x_1 - x_2\|_0, \\
& \leq \frac{1}{\Gamma(\alpha+1)} [\mu_1 + \mu_2] \|(x_1 - x_2, y_1 - y_2)\| + \frac{1}{\Gamma(\alpha+1)} [\mu_1 + \mu_2] \|(x_1 - x_2, y_1 - y_2)\| \\
& + \frac{1}{\Gamma(\alpha)} [\mu_1 + \mu_2] \|(x_1 - x_2, y_1 - y_2)\| + \frac{1}{2\Gamma(\alpha-1)} [\mu_1 + \mu_2] \|(x_1 - x_2, y_1 - y_2)\| \\
& + \sum_{j=1}^m I_{1j} \|(x_1 - x_2, y_1 - y_2)\| + \sum_{j=1}^m \hat{I}_{1j} \|(x_1 - x_2, y_1 - y_2)\| + \frac{\mathcal{Z}_\alpha}{\Gamma(\alpha-\varepsilon+1)} [\mu_1 + \mu_2] \|(x_1 - x_2, y_1 - y_2)\| \\
& + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha)\Gamma(2-\varepsilon)} [\mu_1 + \mu_2] \|(x_1 - x_2, y_1 - y_2)\| + \frac{\mathcal{Z}_\alpha \Omega^{2-\varepsilon}}{\Gamma(\alpha-1)\Gamma(3-\varepsilon)} [\mu_1 + \mu_2] \|(x_1 - x_2, y_1 - y_2)\| \quad (20) \\
& + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} [\mu_1 + \mu_2] \|(x_1 - x_2, y_1 - y_2)\| + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \hat{I}_{1j} \|(x_1 - x_2, y_1 - y_2)\| \\
& + \frac{\mathcal{Z}_\alpha \Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m \check{I}_{1j} \|(x_1 - x_2, y_1 - y_2)\| + \frac{\mathcal{Z}_\alpha}{\Gamma(\alpha-1)} [\mu_1 + \mu_2] \|(x_1 - x_2, y_1 - y_2)\| \\
& + \frac{\mathcal{Z}_\alpha}{\Gamma(\alpha-1)} [\mu_1 + \mu_2] \|(x_1 - x_2, y_1 - y_2)\| + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \check{I}_{1j} \|(x_1 - x_2, y_1 - y_2)\| \\
& + \mathcal{Z}_\alpha \sum_{j=1}^m \check{I}_{1j} \|(x_1 - x_2, y_1 - y_2)\| + \frac{1}{2} \sum_{j=1}^m \check{I}_{1j} \|(x_1 - x_2, y_1 - y_2)\| \\
& \leq \left[ (\mu_1 + \mu_2) \left( \frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha-1)} + \frac{\mathcal{Z}_\alpha}{\Gamma(\alpha-\varepsilon+1)} + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha)\Gamma(2-\varepsilon)} + \frac{\mathcal{Z}_\alpha \Omega^{2-\varepsilon}}{\Gamma(\alpha-1)\Gamma(3-\varepsilon)} \right. \right. \\
& + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} + \frac{2\mathcal{Z}_\alpha}{\Gamma(\alpha-1)} \left. \left. \right) + \sum_{j=1}^m I_{1j} + \sum_{j=1}^m \hat{I}_{1j} + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \hat{I}_{1j} + \frac{\mathcal{Z}_\alpha \Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m \check{I}_{1j} \right. \\
& \left. + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \check{I}_{1j} + \mathcal{Z}_\alpha \sum_{j=1}^m \check{I}_{1j} + \frac{1}{2} \sum_{j=1}^m \check{I}_{1j} \right] \|(x_1 - x_2, y_1 - y_2)\|.
\end{aligned}$$

In the same fashion, we can obtain

$$\begin{aligned}
& |\mathbf{W}_2(x_1, y_1)(t) - \mathbf{W}_2(x_2, y_2)(t)| \\
& \leq \mathcal{Z}_\beta \left[ (\mu'_1 + \mu'_2) \left( \frac{1}{\mathcal{Z}_\beta \Gamma(\beta+1)} + \frac{1}{\Gamma(\beta-\rho+1)} + \frac{\Phi^{1-\rho}}{\Gamma(\beta)\Gamma(2-\rho)} + \frac{\Phi^{2-\rho}}{\Gamma(\beta-1)\Gamma(3-\rho)} + \frac{\Phi^{1-\rho}}{\Gamma(\beta-1)\Gamma(2-\rho)} \right. \right. \\
& + \frac{2}{\Gamma(\beta-1)} \left. \left. \right) + \frac{\Phi^{1-\rho}}{\Gamma(2-\rho)} \sum_{j=1}^m \hat{I}_{2j} + \frac{\Phi^{2-\rho}}{\Gamma(3-\rho)} \sum_{j=1}^m \check{I}_{2j} + \frac{\Phi^{1-\rho}}{\Gamma(2-\rho)} \sum_{j=1}^m \check{I}_{2j} + \sum_{j=1}^m \check{I}_{2j} \right] \\
& \times \|(x_1 - x_2, y_1 - y_2)\|, \quad t \in [0, t_1], \quad (21)
\end{aligned}$$

and

$$\begin{aligned}
& |\mathbf{W}_2(x_1, y_1)(t) - \mathbf{W}_2(x_2, y_2)(t)| \\
\leq & \left[ (\mu'_1 + \mu'_2) \left( \frac{2}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta)} + \frac{1}{2\Gamma(\beta-1)} + \frac{z_\beta}{\Gamma(\beta-\rho+1)} + \frac{z_\beta \Phi^{1-\rho}}{\Gamma(\beta)\Gamma(2-\rho)} + \frac{z_\beta \Phi^{2-\rho}}{\Gamma(\beta-1)\Gamma(3-\rho)} \right. \right. \\
& + \frac{z_\beta \Phi^{1-\rho}}{\Gamma(\beta-1)\Gamma(2-\rho)} + \frac{2z_\beta}{\Gamma(\beta-1)} \Big) + \sum_{j=1}^m I_{2j} + \sum_{j=1}^m \hat{I}_{2j} + \frac{z_\beta \Phi^{1-\rho}}{\Gamma(2-\rho)} \sum_{j=1}^m \hat{I}_{2j} + \frac{z_\beta \Phi^{2-\rho}}{\Gamma(3-\rho)} \sum_{j=1}^m \check{I}_{2j} \\
& \left. \left. + \frac{z_\beta \Phi^{1-\rho}}{\Gamma(2-\rho)} \sum_{j=1}^m \check{I}_{2j} + z_\beta \sum_{j=1}^m \check{I}_{2j} + \frac{1}{2} \sum_{j=1}^m \check{I}_{2j} \right] \|(x_1 - x_2, y_1 - y_2)\|, \quad t \in (t_m, t_{m+1}]. \tag{22}
\end{aligned}$$

Thus, from (18)–(22) and  $(M_3)$ , we infer that  $\mathbf{W}$  is a contraction mapping. According to Lemma 3,  $\mathbf{W}$  has a fixed point  $(x^*(t), y^*(t)) \in Y'$ , which is unique. Therefore, problem (1) has a unique solution  $(x^*(t), y^*(t))$ .  $\square$

**Theorem 3.** Let  $(M_1) – (M_2)$ ,  $(M_4)$  and for all  $t \in J$  such that  $h(t, 0, 0) = w(t, 0, 0) = 0$ ,  $M_{ik} = N_{ik} = O_{ik} = 0$ , ( $i = 1, 2$ ;  $k = 1, 2, \dots, n$ ) hold. Then, (1) has at least one solution  $(x^*(t), y^*(t))$ .

**Proof.** For the sake of simplicity, let us denote

$$\begin{aligned}
\omega = & \left[ (\mu_1 + \mu_2) \left( \frac{3}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha-1)} + \frac{z_\alpha}{\Gamma(\alpha-\varepsilon+1)} + \frac{z_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha)\Gamma(2-\varepsilon)} + \frac{z_\alpha \Omega^{2-\varepsilon}}{\Gamma(\alpha-1)\Gamma(3-\varepsilon)} \right. \right. \\
& + \frac{z_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} + \frac{2z_\alpha}{\Gamma(\alpha-1)} \Big) + \sum_{j=1}^m I_{1j} + \sum_{j=1}^m \hat{I}_{1j} + \frac{z_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \hat{I}_{1j} + \frac{z_\alpha \Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m \check{I}_{1j} \\
& \left. \left. + \frac{z_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \check{I}_{1j} + z_\alpha \sum_{j=1}^m \check{I}_{1j} + \frac{1}{2} \sum_{j=1}^m \check{I}_{1j} \right], \\
\xi = & \left[ (\mu'_1 + \mu'_2) \left( \frac{3}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta)} + \frac{1}{2\Gamma(\beta-1)} + \frac{z_\beta}{\Gamma(\beta-\rho+1)} + \frac{z_\beta \Phi^{1-\rho}}{\Gamma(\beta)\Gamma(2-\rho)} + \frac{z_\beta \Phi^{2-\rho}}{\Gamma(\beta-1)\Gamma(3-\rho)} \right. \right. \\
& + \frac{z_\beta \Phi^{1-\rho}}{\Gamma(\beta-1)\Gamma(2-\rho)} + \frac{2z_\beta}{\Gamma(\beta-1)} \Big) + \sum_{j=1}^m I_{2j} + \sum_{j=1}^m \hat{I}_{2j} + \frac{z_\beta \Phi^{1-\rho}}{\Gamma(2-\rho)} \sum_{j=1}^m \hat{I}_{2j} + \frac{z_\beta \Phi^{2-\rho}}{\Gamma(3-\rho)} \sum_{j=1}^m \check{I}_{2j} \\
& \left. \left. + \frac{z_\beta \Phi^{1-\rho}}{\Gamma(2-\rho)} \sum_{j=1}^m \check{I}_{2j} + z_\beta \sum_{j=1}^m \check{I}_{2j} + \frac{1}{2} \sum_{j=1}^m \check{I}_{2j} \right],
\end{aligned}$$

and  $\mathcal{R}_v = \max\{(\frac{1}{\omega} + 1, \frac{1}{\xi} + 1)\}$ . Define the operator  $\mathbf{W}$ , as in (17), and a closed ball of Banach space  $Y'$  as follows:

$$v = \{(x, y) \in Y' : \|(x, y)\| \leq \mathcal{R}_v\}. \tag{23}$$

Similar to (18)–(22), we easily show that  $\mathbf{W}(v) \subset v$  by applying  $(M_4)$ .  $\mathbf{W}(v) \subset v$  indicates that  $\mathbf{W}(v)$  is uniformly bounded in  $Y'$ . The continuity of the operator  $\mathbf{W}$  is follows from the continuity of  $h, w, M_{im}, N_{im}$  and  $O_{im}$ . Now, we need to prove that  $\mathbf{W} : v \rightarrow v$  is equicontinuous. Let  $(x, y) \in v$  and  $\ell_1, \ell_2 \in [0, 1]$  with  $\ell_1 < \ell_2$ . When  $0 \leq \ell_1 < \ell_2 \leq t_1$ , similar to Equation (18), we have

$$\begin{aligned}
& |\mathbf{W}_1(x, y)(\ell_2) - \mathbf{W}_1(x, y)(\ell_1)| \\
= & \left| \frac{1}{\Gamma(\alpha)} \int_0^{\ell_1} [(\ell_2 - s)^{\alpha-1} - (\ell_1 - s)^{\alpha-1}] h(s, {}^c D^a x(s), {}^c D^b y(s)) ds + \frac{1}{\Gamma(\alpha)} \int_{\ell_1}^{\ell_2} (\ell_2 - s)^{\alpha-1} \right. \\
& \times h(s, {}^c D^a x(s), {}^c D^b y(s)) ds - e^*(\ell_2 - \ell_1)^2 \Big|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{\ell_1} [(\ell_2 - s)^{\alpha-1} - (\ell_1 - s)^{\alpha-1}] |h(s, {}^c D^{\alpha}x(s), {}^c D^{\beta}y(s))| ds + \frac{1}{\Gamma(\alpha)} \int_{\ell_1}^{\ell_2} (\ell_2 - s)^{\alpha-1} \\
&\quad \times |h(s, {}^c D^{\alpha}x(s), {}^c D^{\beta}y(s))| ds + |e^*|(\ell_2 - \ell_1)^2, \\
&\leq \frac{\mu_1 + \mu_2}{\Gamma(\alpha)} \|x, y\| \int_0^{\ell_1} [(\ell_2 - s)^{\alpha-1} - (\ell_1 - s)^{\alpha-1}] ds + \frac{\mu_1 + \mu_2}{\Gamma(\alpha+1)} \|x, y\| (\ell_2 - \ell_1)^{\alpha} + |e^*|(\ell_2 - \ell_1)^2, \\
&\leq \frac{\mu_1 + \mu_2}{\Gamma(\alpha+1)} \|x, y\| [(\ell_2 - \ell_1)^{\alpha} - (\ell_2^{\alpha} - \ell_1^{\alpha})] + \frac{\mu_1 + \mu_2}{\Gamma(\alpha+1)} \|x, y\| (\ell_2 - \ell_1)^{\alpha} + |e^*|(\ell_2 - \ell_1)^2, \\
&\leq Z_{\alpha} \left[ (\mu_1 + \mu_2) \left( \frac{2}{Z_{\alpha} \Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-\varepsilon+1)} + \frac{\Omega^{1-\varepsilon}}{\Gamma(\alpha) \Gamma(2-\varepsilon)} + \frac{\Omega^{2-\varepsilon}}{\Gamma(\alpha-1) \Gamma(3-\varepsilon)} + \frac{\Omega^{1-\varepsilon}}{\Gamma(\alpha-1) \Gamma(2-\varepsilon)} \right. \right. \\
&\quad \left. \left. + \frac{2}{\Gamma(\alpha-1)} \right) + \frac{\Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \hat{I}_{1j} + \frac{\Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m \check{I}_{1j} + \frac{\Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \check{I}_{1j} + \sum_{j=1}^m \check{I}_{1j} \right] \|x, y\| (\ell_2 - \ell_1)^2, \\
&\leq \varpi R_v (\ell_2 - \ell_1)^2.
\end{aligned} \tag{24}$$

In the same fashion, we obtain

$$|\mathbf{W}_2(x, y)(\ell_2) - \mathbf{W}_2(x, y)(\ell_1)| \leq \xi R_v (\ell_2 - \ell_1)^2. \tag{25}$$

In addition, we obtain the same result when  $t_m < \ell_1 < \ell_2 \leq t_{m+1}$ ,  $1 \leq m \leq n$ , similar to (20)

$$|\mathbf{W}_1(x, y)(\ell_2) - \mathbf{W}_1(x, y)(\ell_1)| \leq \varpi R_v (\ell_2 - \ell_1)^2 \tag{26}$$

and

$$|\mathbf{W}_2(x, y)(\ell_2) - \mathbf{W}_2(x, y)(\ell_1)| \leq \xi R_v (\ell_2 - \ell_1)^2. \tag{27}$$

Thus, it follows from (24)–(27) that, for any  $\epsilon > 0$ , there exists a positive constant  $\sigma = \frac{\epsilon}{R_v} \min\{\frac{1}{\varpi}, \frac{1}{\xi}\}$  independent of  $\ell_1, \ell_2$  and  $(x, y)$  such that  $\|\mathbf{W}(x, y)(\ell_2) - \mathbf{W}(x, y)(\ell_1)\| < \epsilon$ , whenever  $|\ell_2 - \ell_1| \leq \sigma$ . Thereby,  $\mathbf{W} : Y' \rightarrow Y'$  is equicontinuous. By the Arzela–Ascoli theorem, we know that  $\mathbf{W} : Y' \rightarrow Y'$  is completely continuous. In view of Theorem 1,  $\mathbf{W}$  has a unique fixed point  $(x^*(t), y^*(t)) \in \bar{V}$ , which is a solution of system (1).  $\square$

#### 4. Ulam–Hyers Stability

In this section, we are interested in Ulam–Hyers stability and its types for the solution of (1).

**Definition 4.** [61] Problem (1) is Ulam–Hyers stable if there exists a constant  $\mathbf{K}_{\alpha, \beta} = (\mathbf{K}_{\alpha}, \mathbf{K}_{\beta}) > 0$  such that, for any  $\epsilon = (\epsilon_{\alpha}, \epsilon_{\beta}) > 0$ , and  $m = 1, 2, \dots, n$ , there exists a solution  $(x, y) \in Y'$  of:

$$\begin{cases} |{}^c D^{\alpha}x(t) - h(t, {}^c D^{\alpha}x(t), {}^c D^{\beta}y(t))| \leq \epsilon_{\alpha}, \\ |\Delta x(t_m) - M_{1m}(x(t_m))| \leq \epsilon_{\alpha}, \\ |\Delta x'(t_m) - N_{1m}(x(t_m))| \leq \epsilon_{\alpha}, \\ |\Delta x''(t_m) - O_{1m}(x(t_m))| \leq \epsilon_{\alpha}, \\ |{}^c D^{\beta}y(t) - w(t, {}^c D^{\alpha}x(t), {}^c D^{\beta}y(t))| \leq \epsilon_{\beta}, \\ |\Delta y(t_m) - M_{2m}(y(t_m))| \leq \epsilon_{\beta}, \\ |\Delta y'(t_m) - N_{2m}(y(t_m))| \leq \epsilon_{\beta}, \\ |\Delta y''(t_m) - O_{2m}(y(t_m))| \leq \epsilon_{\beta}, \end{cases} \tag{28}$$

corresponding to a solution  $(\zeta, \chi) \in Y'$  of (1) such that

$$|(x, y)(t) - (\zeta, \chi)(t)| \leq \mathbf{K}_{\alpha, \beta} \epsilon.$$

**Definition 5.** [61] Problem (1) is generalized Ulam–Hyers stable if there exists a function  $\Theta_{\alpha,\beta} \in C(\mathbb{R}_+, \mathbb{R}_+)$ ,  $\Theta_{\alpha,\beta}(0) = 0$  for each  $\epsilon > 0$ , such that for every solution  $(x, y) \in Y'$  of the inequality (28), there is a solution  $(\zeta, \chi) \in Y'$  of (1) such that

$$|(x, y)(t) - (\zeta, \chi)(t)| \leq \Theta_{\alpha,\beta}(\epsilon).$$

**Definition 6.** [61] Problem (1) is Ulam–Hyers–Rassias stable with respect to  $(\Psi_{\alpha,\beta}, \varphi_{\alpha,\beta})$ , where  $\Psi_{\alpha,\beta} = (\Psi_\alpha, \Psi_\beta) \in C(J, \mathbb{R})$  and  $\varphi_{\alpha,\beta} = (\varphi_\alpha, \varphi_\beta) \in C(J, \mathbb{R})$ , if, for every  $\epsilon = (\epsilon_\alpha, \epsilon_\beta) > 0$ , there exists a real number  $K_{\Psi,\varphi} > 0$ , such that for  $m = 1, 2, \dots, n$  and for a solution  $(x, y) \in Y'$  of:

$$\left\{ \begin{array}{l} |{}^cD^\alpha x(t) - h(t, {}^cD^\alpha x(t), {}^cD^\beta y(t))| \leq \Psi_\alpha(t), \\ |\Delta x(t_m) - M_{1m}(x(t_m))| \leq \varphi_\alpha, \\ |\Delta x'(t_m) - N_{1m}(x(t_m))| \leq \varphi_\alpha, \\ |\Delta x''(t_m) - O_{1m}(x(t_m))| \leq \varphi_\alpha, \\ |{}^cD^\beta y(t) - w(t, {}^cD^\alpha x(t), {}^cD^\beta y(t))| \leq \Psi_\beta(t), \\ |\Delta y(t_m) - M_{2m}(y(t_m))| \leq \varphi_\beta, \\ |\Delta y'(t_m) - N_{2m}(y(t_m))| \leq \varphi_\beta, \\ |\Delta y''(t_m) - O_{2m}(y(t_m))| \leq \varphi_\beta, \end{array} \right. \quad (29)$$

there is a solution  $(\zeta, \chi) \in Y'$  of (1) such that

$$|(x, y)(t) - (\zeta, \chi)(t)| \leq K_{\Psi,\varphi}(\Psi_{\alpha,\beta}(t) + \varphi_{\alpha,\beta})\epsilon.$$

**Definition 7.** [61] Problem (1) is generalized Ulam–Hyers–Rassias stable with respect to  $(\Psi_{\alpha,\beta}, \varphi_{\alpha,\beta}) \in C(J, \mathbb{R})$ , if there exists a real number  $K_{\Psi,\varphi} > 0$ , such that for  $m = 1, 2, \dots, n$  and for every solution  $(x, y) \in Y'$  of the following:

$$\left\{ \begin{array}{l} |{}^cD^\alpha x(t) - h(t, {}^cD^\alpha x(t), {}^cD^\beta y(t))| \leq \Psi_\alpha(t)\epsilon_\alpha, \\ |\Delta x(t_m) - M_{1m}(x(t_m))| \leq \varphi_\alpha\epsilon_\alpha, \\ |\Delta x'(t_m) - N_{1m}(x(t_m))| \leq \varphi_\alpha\epsilon_\alpha, \\ |\Delta x''(t_m) - O_{1m}(x(t_m))| \leq \varphi_\alpha\epsilon_\alpha, \\ |{}^cD^\beta y(t) - w(t, {}^cD^\alpha x(t), {}^cD^\beta y(t))| \leq \Psi_\beta(t)\epsilon_\beta, \\ |\Delta y(t_m) - M_{2m}(y(t_m))| \leq \varphi_\beta\epsilon_\beta, \\ |\Delta y'(t_m) - N_{2m}(y(t_m))| \leq \varphi_\beta\epsilon_\beta, \\ |\Delta y''(t_m) - O_{2m}(y(t_m))| \leq \varphi_\beta\epsilon_\beta, \end{array} \right. \quad (30)$$

there is a solution  $(\zeta, \chi) \in Y'$  of (1) such that

$$|(x, y)(t) - (\zeta, \chi)(t)| \leq K_{\Psi,\varphi}(\Psi_{\alpha,\beta}(t) + \varphi_{\alpha,\beta}).$$

**Remark 1.** A function  $(x, y) \in Y'$  is a solution of the inequality (28), if and only if there exist functions  $F_h, F_w \in Y'$  and a sequence  $F_m, \hat{F}_m$ ,  $m = 1, 2, \dots, n$  depending on  $(x, y)$ , such that

- $|F_h(t)| \leq \epsilon_\alpha$ ,  $|F_w(t)| \leq \epsilon_\beta$ ,  $|F_m| \leq \epsilon_\alpha$ ,  $|\hat{F}_m| \leq \epsilon_\beta$ ,  $t \in J_m$ ,  $m = 1, \dots, n$ ;
- ${}^cD^\alpha x(t) = -h(t, {}^cD^\alpha x(t), {}^cD^\beta y(t)) + F_h(t)$ ;
- $\Delta x|_{t=t_m} = M_{1m}(x(t_m)) + F_m$ ;
- $\Delta x'|_{t=t_m} = N_{1m}(x(t_m)) + F_m$ ;
- $\Delta x''|_{t=t_m} = O_{1m}(x(t_m)) + F_m$ ;
- ${}^cD^\beta y(t) = -w(t, {}^cD^\alpha x(t), {}^cD^\beta y(t)) + F_w(t)$ ;
- $\Delta y|_{t=t_m} = M_{2m}(y(t_m)) + \hat{F}_m$ ;
- $\Delta y'|_{t=t_m} = N_{2m}(y(t_m)) + \hat{F}_m$ ;
- $\Delta y''|_{t=t_m} = O_{2m}(y(t_m)) + \hat{F}_m$ .

**Remark 2.** A function  $(x, y) \in Y'$  is a solution of the inequality (29), if and only if there exist functions  $F_h, F_w \in Y'$  and a sequence  $F_m, \hat{F}_m$ ,  $m = 1, 2, \dots, n$  depending on  $(x, y)$ , such that

- $|F_h(t)| \leq \Psi_\alpha$ ,  $|\hat{F}_w(t)| \leq \Psi_\beta$ ,  $|F_m| \leq \varphi_\alpha$ ,  $|F_m| \leq \varphi_\beta$ ,  $t \in J_m$ ,  $m = 1, \dots, n$ ;
- ${}^cD^\alpha x(t) = -h(t, {}^cD^\alpha x(t), {}^cD^b y(t)) + F_h(t)$ ;
- $\Delta x|_{t=t_m} = M_{1m}(x(t_m)) + F_m$ ;
- $\Delta x|_{t=t_m} = N_{1m}(x(t_m)) + F_m$ ;
- $\Delta x|_{t=t_m} = O_{1m}(x(t_m)) + F_m$ ;
- ${}^cD^\beta x(t) = -w(t, {}^cD^\alpha x(t), {}^cD^b y(t)) + \hat{F}_w(t)$ ;
- $\Delta y|_{t=t_m} = M_{2m}(y(t_m)) + \hat{F}_m$ ;
- $\Delta y|_{t=t_m} = N_{2m}(y(t_m)) + \hat{F}_m$ ;
- $\Delta y|_{t=t_m} = O_{2m}(y(t_m)) + \hat{F}_m$ .

Similarly, one can easily state such a remark for the inequality (30).

**Theorem 4.** If the assumptions  $(M_1) – (M_2)$  hold with

$$\Lambda_0 = 1 - \frac{\Lambda_2^* \Lambda_4^*}{(1 - \Lambda_1^*)(1 - \Lambda_3^*)} > 0, \quad (31)$$

then (1) is Ullam–Hyers and generalized Ullam–Hyers stable.

**Proof.** Let  $(x, y) \in Y'$  be any solution of the inequality (28) and let  $(\zeta, \chi) \in Y'$  be the unique solution of the following:

$$\begin{cases} {}^cD^\alpha \zeta(t) + h(t, {}^cD^\alpha \zeta(t), {}^cD^b \chi(t)) = 0, t \neq t_m, m = 1, 2, \dots, n, \\ {}^cD^\beta \chi(t) + w(t, {}^cD^\alpha \zeta(t), {}^cD^b \chi(t)) = 0, t \neq t_m, m = 1, 2, \dots, n, \\ \Delta \zeta|_{t=t_m} = M_{1m}(\zeta(t_m)), \Delta \zeta'|_{t=t_m} = N_{1m}(\zeta(t_m)), \Delta \zeta''|_{t=t_m} = O_{1m}(\zeta(t_m)), \\ \Delta \chi|_{t=t_m} = M_{2m}(\chi(t_m)), \Delta \chi'|_{t=t_m} = N_{2m}(\chi(t_m)), \Delta \chi''|_{t=t_m} = O_{2m}(\chi(t_m)), \\ \zeta(0) = \zeta'(0) = 0, {}^cD^\varepsilon \zeta(\Omega) = \zeta''(1), \\ \chi(0) = \chi'(0) = 0, {}^cD^\rho \chi(\Phi) = \chi''(1). \end{cases} \quad (32)$$

By Lemma 2.4, we have

$$\begin{cases} {}^cD^\alpha x(t) + h(t, {}^cD^\alpha x(t), {}^cD^b y(t)) = F_h(t), t \neq t_m, m = 1, 2, \dots, n, \\ {}^cD^\beta y(t) + w(t, {}^cD^\alpha x(t), {}^cD^b y(t)) = F_w(t), t \neq t_m, m = 1, 2, \dots, n, \\ \Delta x|_{t=t_m} = M_{1m}(x(t_m)) + F_m, \Delta x'|_{t=t_m} = N_{1m}(x(t_m)) + F_m, \Delta x''|_{t=t_m} = O_{1m}(x(t_m)) + F_m, \\ \Delta y|_{t=t_m} = M_{2m}(y(t_m)) + \hat{F}_m, \Delta y'|_{t=t_m} = N_{2m}(y(t_m)) + \hat{F}_m, \Delta y''|_{t=t_m} = O_{2m}(y(t_m)) + \hat{F}_m, \\ x(0) = x'(0) = 0, {}^cD^\varepsilon x(\Omega) = x''(1), \\ y(0) = y'(0) = 0, {}^cD^\rho y(\Phi) = y''(1). \end{cases} \quad (33)$$

Since  $(x, y)$  is a solution of the inequality (28) and  $t \in J$ ; hence, by Remark 1, we obtain

$$x(t) = \begin{cases} -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s) [h(s, {}^c D^\alpha x(s), {}^c D^b y(s)) + F_h(s)] ds + e^* t^2, & t \in [0, t_1], \\ -\frac{1}{\Gamma(\alpha)} \int_{t_m}^t (t-s)^{\alpha-1} [h(s, {}^c D^\alpha x(s), {}^c D^b y(s)) + F_h(s)] ds \\ -\frac{1}{\Gamma(\alpha)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-1} [h(s, {}^c D^\alpha x(s), {}^c D^b y(s)) + F_h(s)] ds \\ -\frac{1}{\Gamma(\alpha-1)} \sum_{j=1}^m (t - t_j) \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-2} [h(s, {}^c D^\alpha x(s), {}^c D^b y(s)) + F_h(s)] ds \\ -\frac{1}{2\Gamma(\alpha-2)} \sum_{j=1}^m (t - t_j)^2 \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} [h(s, {}^c D^\alpha x(s), {}^c D^b y(s)) + F_h(s)] ds \\ + \sum_{j=1}^m [M_{1j}(x(t_j)) + F_m] + \sum_{j=1}^m (t - t_j) [N_{1j}(x(t_j)) + F_m] + \sum_{j=1}^m \frac{(t - t_j)^2}{2} \\ \times [O_{1j}(x(t_j)) + F_m] + e^* t^2, & t \in (t_m, t_{m+1}], \quad 1 \leq m \leq n. \end{cases}$$

$$y(t) = \begin{cases} -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [w(s, {}^c D^\alpha x(s), {}^c D^b y(s)) + F_w(s)] ds + c^* t^2, & t \in [0, t_1], \\ -\frac{1}{\Gamma(\beta)} \int_{t_m}^t (t-s)^{\beta-1} [w(s, {}^c D^\alpha x(s), {}^c D^b y(s)) + F_w(s)] ds \\ -\frac{1}{\Gamma(\beta)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta-1} w(s, {}^c D^\alpha x(s), {}^c D^b y(s)) ds \\ -\frac{1}{\Gamma(\beta-1)} \sum_{j=1}^m (t - t_j) \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta-2} w(s, {}^c D^\alpha x(s), {}^c D^b y(s)) ds \\ -\frac{1}{2\Gamma(\beta-2)} \sum_{j=1}^m (t - t_j)^2 \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta-3} [w(s, {}^c D^\alpha x(s), {}^c D^b y(s)) + F_w(s)] ds \\ + \sum_{j=1}^m [M_{2j}(y(t_j)) + F_m] + \sum_{j=1}^m (t - t_j) [M_{2j}(y(t_j)) + F_m] + \sum_{j=1}^m \frac{(t - t_j)^2}{2} \\ \times [O_{2j}(y(t_j)) + F_m] + c^* t^2, & t \in (t_m, t_{m+1}], \quad 1 \leq m \leq n, \end{cases}$$

where

$$e^* = Z_\alpha \left[ -\frac{1}{\Gamma(\alpha-\varepsilon)} \int_{t_j}^\Omega (\Omega - s)^{\alpha-\varepsilon-1} [h(s, {}^c D^\alpha x(s), {}^c D^b y(s)) + F_h(s)] ds \right.$$

$$-\frac{\Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-2} [h(s, {}^c D^\alpha x(s), {}^c D^b y(s)) + F_h(s)] ds$$

$$-\frac{\Omega^{2-\varepsilon}}{\Gamma(\alpha-2)\Gamma(3-\varepsilon)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} [h(s, {}^c D^\alpha x(s), {}^c D^b y(s)) + F_h(s)] ds$$

$$+\frac{\Omega^{1-\varepsilon}}{\Gamma(\alpha-2)\Gamma(2-\varepsilon)} \sum_{j=1}^m t_j \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} [h(s, {}^c D^\alpha x(s), {}^c D^b y(s)) + F_h(s)] ds$$

$$+\frac{\Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m [N_{1j}(x(t_j)) + F_m] + \frac{\Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m [O_{1j}(x(t_j)) + F_m] - \frac{\Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m t_j$$

$$\times [O_{1j}(x(t_j)) + F_m] + \frac{1}{\Gamma(\alpha-2)} \int_{t_m}^1 (1-s)^{\alpha-3} [h(s, {}^c D^\alpha x(s), {}^c D^b y(s)) + F_h(s)] ds$$

$$\left. + \frac{1}{\Gamma(\alpha-2)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} [h(s, {}^c D^\alpha x(s), {}^c D^b y(s)) + F_h(s)] ds - \sum_{j=1}^m [O_{1j}(x(t_j)) + F_m] \right],$$

$$c^* = Z_\beta \left[ -\frac{1}{\Gamma(\beta-\rho)} \int_{t_j}^\Phi (\Phi - s)^{\beta-\rho-1} [w(s, {}^c D^\alpha x(s), {}^c D^b y(s)) + F_w(s)] ds \right.$$

$$-\frac{\Phi^{1-\rho}}{\Gamma(\beta-1)\Gamma(2-\rho)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta-2} [w(s, {}^c D^\alpha x(s), {}^c D^b y(s)) + F_w(s)] ds$$

$$-\frac{\Phi^{2-\rho}}{\Gamma(\beta-2)\Gamma(3-\rho)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta-3} [w(s, {}^c D^\alpha x(s), {}^c D^b y(s)) + F_w(s)] ds$$

$$\left. + \frac{\Phi^{1-\rho}}{\Gamma(\beta-2)\Gamma(2-\rho)} \sum_{j=1}^m t_j \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta-3} [w(s, {}^c D^\alpha x(s), {}^c D^b y(s)) + F_w(s)] ds \right]$$

$$\begin{aligned}
& + \frac{\Phi^{1-\rho}}{\Gamma(2-\rho)} \sum_{j=1}^m [N_{2j}(y(t_j)) + F_m] + \frac{\Phi^{2-\rho}}{\Gamma(3-\rho)} \sum_{j=1}^m [O_{2j}(y(t_j)) + F_m] - \frac{\Phi^{1-\rho}}{\Gamma(2-\rho)} \sum_{j=1}^m t_j \\
& \times [O_{2j}(y(t_j)) + F_m] + \frac{1}{\Gamma(\beta-2)} \int_{t_m}^1 (1-s)^{\beta-3} [w(s, {}^c D^a x(s), {}^c D^b y(s)) + F_w(s)] ds \\
& + \frac{1}{\Gamma(\beta-2)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\beta-3} [w(s, {}^c D^a x(s), {}^c D^b y(s)) + F_w(s)] ds - \sum_{j=1}^m [O_{2j}(y(t_j)) + F_m].
\end{aligned}$$

For  $t \in [0, t_1]$ , we have

$$\begin{aligned}
x(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [h(s, {}^c D^a x_1(s), {}^c D^b y_1(s)) + F_h(s)] ds \\
& + \frac{z_\alpha t^2}{\Gamma(\alpha-\varepsilon)} \int_{t_j}^\Omega (\Omega-s)^{\alpha-\varepsilon-1} [h(s, {}^c D^a x_1(s), {}^c D^b y_1(s)) + F_h(s)] ds \\
& + \frac{z_\alpha t^2 \Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-2} [h(s, {}^c D^a x_1(s), {}^c D^b y_1(s)) + F_h(s)] ds \\
& + \frac{z_\alpha t^2 \Omega^{2-\varepsilon}}{\Gamma(\alpha-2)\Gamma(3-\varepsilon)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} [h(s, {}^c D^a x_1(s), {}^c D^b y_1(s)) + F_h(s)] ds \\
& + \frac{z_\alpha t^2 \Omega^{1-\varepsilon}}{\Gamma(\alpha-2)\Gamma(2-\varepsilon)} \sum_{j=1}^m t_j \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} [h(s, {}^c D^a x_1(s), {}^c D^b y_1(s)) + F_h(s)] ds \quad (34) \\
& + \frac{z_\alpha t^2 \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m [N_{1j}(x(t_j)) + F_m] + \frac{z_\alpha t^2 \Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m [O_{1j}(x(t_j)) + F_m] \\
& + \frac{z_\alpha t^2}{\Gamma(\alpha-2)} \int_{t_m}^1 (1-s)^{\alpha-3} [h(s, {}^c D^a x_1(s), {}^c D^b y_1(s)) + F_h(s)] ds \\
& + \frac{z_\alpha t^2}{\Gamma(\alpha-2)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} [h(s, {}^c D^a x_1(s), {}^c D^b y_1(s)) + F_h(s)] ds \\
& + \frac{z_\alpha t^2 \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m t_j [O_{1j}(x(t_j)) + F_m] + z_\alpha t^2 \sum_{j=1}^m [O_{1j}(x(t_j)) + F_m].
\end{aligned}$$

For computational convenience, we use  $s(t)$  for the sum of terms which are free of  $F$ ; then, (34) becomes

$$\begin{aligned}
& |x(t) - s_1(t)| \\
\leq & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |F_h(s)| ds + \frac{z_\alpha}{\Gamma(\alpha-\varepsilon)} \int_{t_j}^\Omega (\Omega-s)^{\alpha-\varepsilon-1} |F_h(s)| ds \\
& + \frac{z_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-2} |F_h(s)| ds + \frac{z_\alpha \Omega^{2-\varepsilon}}{\Gamma(\alpha-2)\Gamma(3-\varepsilon)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} |F_h(s)| ds \\
& + \frac{z_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha-2)\Gamma(2-\varepsilon)} \sum_{j=1}^m |t_j| \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} |F_h(s)| ds + \frac{z_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m |F_m| + \frac{z_\alpha \Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m |F_m| \\
& + \frac{z_\alpha}{\Gamma(\alpha-2)} \int_{t_m}^1 (1-s)^{\alpha-3} |F_h(s)| ds + \frac{z_\alpha}{\Gamma(\alpha-2)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} |F_h(s)| ds \\
& + \frac{z_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m |t_j| |F_m| + z_\alpha \sum_{j=1}^m |F_m|.
\end{aligned}$$

By utilizing Remark 1, we get

$$\begin{aligned} & |x(t) - s_1(t)| \\ & \leq \left[ \frac{1}{\Gamma(\alpha+1)} + \frac{\mathcal{Z}_\alpha}{\Gamma(\alpha-\varepsilon+1)} + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha)\Gamma(2-\varepsilon)} + \frac{\mathcal{Z}_\alpha \Omega^{2-\varepsilon}}{\Gamma(\alpha-1)\Gamma(3-\varepsilon)} + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} + \frac{2\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \right. \\ & \quad \left. + \frac{\mathcal{Z}_\alpha \Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} + \frac{2\mathcal{Z}_\alpha}{\Gamma(\alpha-1)} + \mathcal{Z}_\alpha \right] \epsilon_\alpha. \end{aligned} \quad (35)$$

Let

$$\begin{aligned} Q_1 = & \frac{1}{\Gamma(\alpha+1)} + \frac{\mathcal{Z}_\alpha}{\Gamma(\alpha-\varepsilon+1)} + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha)\Gamma(2-\varepsilon)} + \frac{\mathcal{Z}_\alpha \Omega^{2-\varepsilon}}{\Gamma(\alpha-1)\Gamma(3-\varepsilon)} + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} + \frac{2\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \\ & + \frac{\mathcal{Z}_\alpha \Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} + \frac{2\mathcal{Z}_\alpha}{\Gamma(\alpha-1)} + \mathcal{Z}_\alpha. \end{aligned}$$

Thus, (35) becomes

$$|x(t) - s_1(t)| \leq Q_1 \epsilon_\alpha. \quad (36)$$

Let

$$|x(t) - \zeta(t)| = |x(t) - s_1(t) + s_1(t) - \zeta(t)| \leq |x(t) - s_1(t)| + |s_1(t) - \zeta(t)|. \quad (37)$$

Using (36) in (37), we have

$$\begin{aligned} & |x(t) - \zeta(t)| \\ & \leq Q_1 \epsilon_\alpha + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |h(s, {}^c D^a x(s), {}^c D^b y(s)) - h(s, {}^c D^a \zeta(s), {}^c D^b \chi(s))| ds \\ & \quad + \frac{\mathcal{Z}_\alpha t^2}{\Gamma(\alpha-\varepsilon)} \int_{t_j}^{\Omega} (\Omega-s)^{\alpha-\varepsilon-1} |h(s, {}^c D^a x(s), {}^c D^b y(s)) - h(s, {}^c D^a \zeta(s), {}^c D^b \chi(s))| ds \\ & \quad + \frac{\mathcal{Z}_\alpha t^2 \Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-2} |h(s, {}^c D^a x(s), {}^c D^b y(s)) - h(s, {}^c D^a \zeta(s), {}^c D^b \chi(s))| ds \\ & \quad + \frac{\mathcal{Z}_\alpha t^2 \Omega^{2-\varepsilon}}{\Gamma(\alpha-2)\Gamma(3-\varepsilon)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-3} |h(s, {}^c D^a x(s), {}^c D^b y(s)) - h(s, {}^c D^a \zeta(s), {}^c D^b \chi(s))| ds \\ & \quad + \frac{\mathcal{Z}_\alpha t^2 \Omega^{1-\varepsilon}}{\Gamma(\alpha-2)\Gamma(2-\varepsilon)} \sum_{j=1}^m t_j \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-3} |h(s, {}^c D^a x(s), {}^c D^b y(s)) - h(s, {}^c D^a \zeta(s), {}^c D^b \chi(s))| ds \\ & \quad + \frac{\mathcal{Z}_\alpha t^2 \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m |N_{1j}(x(t_j)) - N_{1j}(\zeta(t_j))| + \frac{\mathcal{Z}_\alpha t^2 \Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m |O_{1j}(x(t_j)) - O_{1j}(\zeta(t_j))| \\ & \quad + \frac{\mathcal{Z}_\alpha t^2}{\Gamma(\alpha-2)} \int_{t_m}^1 (1-s)^{\alpha-3} |h(s, {}^c D^a x(s), {}^c D^b y(s)) - h(s, {}^c D^a \zeta(s), {}^c D^b \chi(s))| ds \\ & \quad + \frac{\mathcal{Z}_\alpha t^2}{\Gamma(\alpha-2)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-3} |h(s, {}^c D^a x(s), {}^c D^b y(s)) - h(s, {}^c D^a \zeta(s), {}^c D^b \chi(s))| ds \\ & \quad + \frac{\mathcal{Z}_\alpha t^2 \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m |t_j| |O_{1j}(x(t_j)) - O_{1j}(\zeta(t_j))| + \mathcal{Z}_\alpha t^2 \sum_{j=1}^m |O_{1j}(x(t_j)) - O_{1j}(\zeta(t_j))|. \end{aligned}$$

Utilizing  $(\mathbf{M}_1)$  and  $(\mathbf{M}_2)$ , we get

$$\begin{aligned} &\leq Q_1 \epsilon_\alpha + \frac{1}{\Gamma(\alpha+1)} [\mu_1 \|x - \zeta\|_0 + \mu_2 \|y - \chi\|_0] + \frac{Z_\alpha}{\Gamma(\alpha-\varepsilon+1)} [\mu_1 \|x - \zeta\|_0 + \mu_2 \|y - \chi\|_0] \\ &+ \frac{Z_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha)\Gamma(2-\varepsilon)} [\mu_1 \|x - \zeta\|_0 + \mu_2 \|y - \chi\|_0] + \frac{Z_\alpha \Omega^{2-\varepsilon}}{\Gamma(\alpha-1)\Gamma(3-\varepsilon)} [\mu_1 \|x - \zeta\|_0 + \mu_2 \|y - \chi\|_0] \\ &+ \frac{Z_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} [\mu_1 \|x - \zeta\|_0 + \mu_2 \|y - \chi\|_0] + \frac{Z_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \hat{I}_{1j} \|x - \zeta\|_0 \\ &+ \frac{Z_\alpha \Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m \hat{I}_{1j} \|x - \zeta\|_0 + \frac{Z_\alpha}{\Gamma(\alpha-1)} [\mu_1 \|x - \zeta\|_0 + \mu_2 \|y - \chi\|_0] + \frac{Z_\alpha}{\Gamma(\alpha-1)} \\ &\times [\mu_1 \|x - \zeta\|_0 + \mu_2 \|y - \chi\|_0] + \frac{Z_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \hat{I}_{1j} \|x - \zeta\|_0 + Z_\alpha \sum_{j=1}^m \hat{I}_{1j} \|x - \zeta\|_0. \end{aligned} \quad (38)$$

After some calculation and rearrangement in (38), we get

$$\|x - \zeta\|_0 - \frac{\Lambda_2}{(1 - \Lambda_1)} \|y - \chi\|_0 \leq \frac{Q_1 \epsilon_\alpha}{(1 - \Lambda_1)}, \quad (39)$$

where

$$\begin{aligned} \Lambda_1 &= Z_\alpha \left[ \mu_1 \left( \frac{1}{Z_\alpha \Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-\varepsilon+1)} + \frac{\Omega^{1-\varepsilon}}{\Gamma(\alpha)\Gamma(2-\varepsilon)} + \frac{\Omega^{2-\varepsilon}}{\Gamma(\alpha-1)\Gamma(3-\varepsilon)} + \frac{\Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} \right. \right. \\ &\quad \left. \left. + \frac{2}{\Gamma(\alpha-1)} \right) + \frac{\Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \hat{I}_{1j} + \frac{\Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m \hat{I}_{1j} + \frac{\Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \hat{I}_{1j} + \sum_{j=1}^m \hat{I}_{1j} \right], \\ \Lambda_2 &= Z_\alpha \left[ \mu_2 \left( \frac{1}{Z_\alpha \Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-\varepsilon+1)} + \frac{\Omega^{1-\varepsilon}}{\Gamma(\alpha)\Gamma(2-\varepsilon)} + \frac{\Omega^{2-\varepsilon}}{\Gamma(\alpha-1)\Gamma(3-\varepsilon)} + \frac{\Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} \right. \right. \\ &\quad \left. \left. + \frac{2}{\Gamma(\alpha-1)} \right) \right]. \end{aligned}$$

In addition, for  $t \in (t_m, t_{m+1}]$ , we have

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)} \int_{t_m}^t (t-s)^{\alpha-1} [h(s, {}^c D^\alpha x(s), {}^c D^b y(s)) + F_h(s)] ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-1} [h(s, {}^c D^\alpha x(s), {}^c D^b y(s)) + F_h(s)] ds \\ &\quad + \frac{1}{\Gamma(\alpha-1)} \sum_{j=1}^m (t - t_j) \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-2} [h(s, {}^c D^\alpha x(s), {}^c D^b y(s)) + F_h(s)] ds \\ &\quad + \frac{1}{2\Gamma(\alpha-2)} \sum_{j=1}^m (t - t_j)^2 \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} [h(s, {}^c D^\alpha x(s), {}^c D^b y(s)) + F_h(s)] ds \\ &\quad + \sum_{j=1}^m [M_{1j} x(t_j) + F_m] + \sum_{j=1}^m (t - t_j) [N_{1j}(x(t_j)) + F_m] \\ &\quad + \frac{Z_\alpha t^2}{\Gamma(\alpha-\varepsilon)} \int_{t_j}^\Omega (\Omega - s)^{\alpha-\varepsilon-1} [h(s, {}^c D^\alpha x(s), {}^c D^b y(s)) + F_h(s)] ds \\ &\quad + \frac{Z_\alpha t^2 \Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-2} [h(s, {}^c D^\alpha x(s), {}^c D^b y(s)) + F_h(s)] ds \end{aligned}$$

$$\begin{aligned}
& + \frac{\mathcal{Z}_\alpha t^2 \Omega^{2-\varepsilon}}{\Gamma(\alpha-2)\Gamma(3-\varepsilon)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} [h(s, {}^c \mathbf{D}^a x(s), {}^c \mathbf{D}^b y(s)) + F_h(s)] ds \\
& + \frac{\mathcal{Z}_\alpha t^2 \Omega^{1-\varepsilon}}{\Gamma(\alpha-2)\Gamma(2-\varepsilon)} \sum_{j=1}^m t_j \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} [h(s, {}^c \mathbf{D}^a x(s), {}^c \mathbf{D}^b y(s)) + F_h(s)] ds \\
& + \frac{\mathcal{Z}_\alpha t^2 \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m [N_{1j}(x(t_j)) + F_m] + \frac{\mathcal{Z}_\alpha t^2 \Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m [O_{1j}(x(t_j)) + F_m] \\
& + \frac{\mathcal{Z}_\alpha t^2}{\Gamma(\alpha-2)} \int_{t_m}^1 (1-s)^{\alpha-3} [h(s, {}^c \mathbf{D}^a x(s), {}^c \mathbf{D}^b y(s)) + F_h(s)] ds \\
& + \frac{\mathcal{Z}_\alpha t^2}{\Gamma(\alpha-2)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} [h(s, {}^c \mathbf{D}^a x(s), {}^c \mathbf{D}^b y(s)) + F_h(s)] ds \\
& + \frac{\mathcal{Z}_\alpha t^2 \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m t_j [O_{1j}(x(t_j)) + F_m] + \mathcal{Z}_\alpha t^2 \sum_{j=1}^m [O_{1j}(x(t_j)) + F_m] \\
& + \sum_{j=1}^m \frac{(t - t_j)^2}{2} [O_{1j}(x(t_j)) + F_m].
\end{aligned}$$

For computational convenience, we use  $s_1^*(t)$  for the sum of terms which are free of  $F$ , so we have

$$\begin{aligned}
& |x(t) - s_1^*(t)| \\
& \leq \frac{1}{\Gamma(\alpha)} \int_{t_m}^t (t-s)^{\alpha-1} |F_h(s)| ds + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-1} |F_h(s)| ds \\
& + \frac{1}{\Gamma(\alpha-1)} \sum_{j=1}^m |(t - t_j)| \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-2} |F_h(s)| ds + \frac{1}{2\Gamma(\alpha-2)} \sum_{j=1}^m |(t - t_j)^2| \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} \\
& \times |F_h(s)| ds + \sum_{j=1}^m |F_m| + \sum_{j=1}^m |(t - t_j)| |F_m| + \frac{\mathcal{Z}_\alpha}{\Gamma(\alpha-\varepsilon)} \int_{t_j}^\Omega (\Omega - s)^{\alpha-\varepsilon-1} |F_h(s)| ds \\
& + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-2} |F_h(s)| ds + \frac{\mathcal{Z}_\alpha \Omega^{2-\varepsilon}}{\Gamma(\alpha-2)\Gamma(3-\varepsilon)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} |F_h(s)| ds \\
& + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha-2)\Gamma(2-\varepsilon)} \sum_{j=1}^m t_j \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} |F_h(s)| ds + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m |F_m| + \frac{\mathcal{Z}_\alpha \Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m |F_m| \\
& + \frac{\mathcal{Z}_\alpha}{\Gamma(\alpha-2)} \int_{t_m}^1 (1-s)^{\alpha-3} |F_h(s)| ds + \frac{\mathcal{Z}_\alpha}{\Gamma(\alpha-2)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} |F_h(s)| ds \\
& + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m |t_j| |F_m| + \mathcal{Z}_\alpha \sum_{j=1}^m |F_m| + \sum_{j=1}^m \frac{|(t - t_j)^2|}{2} |F_m|.
\end{aligned}$$

By utilizing Remark 1, we get

$$\begin{aligned}
& |x(t) - s_1^*(t)| \\
& \leq \left( \frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} + \frac{4\mathcal{Z}_\alpha+1}{2\Gamma(\alpha-1)} + \frac{3\mathcal{Z}_\alpha+4}{2} + \frac{\mathcal{Z}_\alpha}{\Gamma(\alpha-\varepsilon+1)} + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha)\Gamma(2-\varepsilon)} + \frac{2\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} + \frac{\mathcal{Z}_\alpha \Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \right. \\
& \quad \left. + \frac{\mathcal{Z}_\alpha \Omega^{2-\varepsilon}}{\Gamma(\alpha-1)\Gamma(3-\varepsilon)} + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} \right) \epsilon_\alpha. \tag{40}
\end{aligned}$$

For computational convenience, let

$$\begin{aligned} \mathcal{Q}_\alpha &= \frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} + \frac{4\mathcal{Z}_\alpha + 1}{2\Gamma(\alpha-1)} + \frac{3\mathcal{Z}_\alpha + 4}{2} + \frac{\mathcal{Z}_\alpha}{\Gamma(\alpha-\varepsilon+1)} + \frac{\mathcal{Z}_\alpha\Omega^{1-\varepsilon}}{\Gamma(\alpha)\Gamma(2-\varepsilon)} + \frac{2\mathcal{Z}_\alpha\Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} + \frac{\mathcal{Z}_\alpha\Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \\ &\quad + \frac{\mathcal{Z}_\alpha\Omega^{2-\varepsilon}}{\Gamma(\alpha-1)\Gamma(3-\varepsilon)} + \frac{\mathcal{Z}_\alpha\Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)}. \end{aligned}$$

Thus, (40) becomes

$$|x(t) - s_1^*(t)| \leq \mathcal{Q}_\alpha \epsilon_\alpha. \quad (41)$$

Let

$$|x(t) - \zeta(t)| = |x(t) - s_1^*(t) + s_1^*(t) - \zeta(t)| \leq |x(t) - s_1^*(t)| + |s_1^*(t) - \zeta(t)|. \quad (42)$$

Using (41) in (42), we get

$$\begin{aligned} &|x(t) - \zeta(t)| \\ &\leq \mathcal{Q}_\alpha \epsilon_\alpha + \frac{1}{\Gamma(\alpha)} \int_{t_m}^t (t-s)^{\alpha-1} |h(s, {}^c D^a x(s), {}^c D^b y(s)) - h(s, {}^c D^a \zeta(s), {}^c D^b \chi(s))| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-1} |h(s, {}^c D^a x(s), {}^c D^b y(s)) - h(s, {}^c D^a \zeta(s), {}^c D^b \chi(s))| ds \\ &\quad + \frac{1}{\Gamma(\alpha-1)} \sum_{j=1}^m |(t - t_j)| \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-2} |h(s, {}^c D^a x(s), {}^c D^b y(s)) - h(s, {}^c D^a \zeta(s), {}^c D^b \chi(s))| ds \\ &\quad + \frac{1}{2\Gamma(\alpha-2)} \sum_{j=1}^m |(t - t_j)^2| \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} |h(s, {}^c D^a x(s), {}^c D^b y(s)) - h(s, {}^c D^a \zeta(s), {}^c D^b \chi(s))| ds \\ &\quad + \sum_{j=1}^m |M_{1j}(x(t_j)) - M_{1j}(\zeta(t_j))| + \sum_{j=1}^m |(t - t_j)| |N_{1j}(x(t_j)) - N_{1j}(\zeta(t_j))| \\ &\quad + \frac{\mathcal{Z}_\alpha t^2}{\Gamma(\alpha-\varepsilon)} \int_{t_j}^\Omega (\Omega - s)^{\alpha-\varepsilon-1} |h(s, {}^c D^a x(s), {}^c D^b y(s)) - h(s, {}^c D^a \zeta(s), {}^c D^b \chi(s))| ds \\ &\quad + \frac{\mathcal{Z}_\alpha t^2 \Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-2} |h(s, {}^c D^a x(s), {}^c D^b y(s)) - h(s, {}^c D^a \zeta(s), {}^c D^b \chi(s))| ds \\ &\quad + \frac{\mathcal{Z}_\alpha t^2 \Omega^{2-\varepsilon}}{\Gamma(\alpha-2)\Gamma(3-\varepsilon)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} |h(s, {}^c D^a x(s), {}^c D^b y(s)) - h(s, {}^c D^a \zeta(s), {}^c D^b \chi(s))| ds \\ &\quad + \frac{\mathcal{Z}_\alpha t^2 \Omega^{1-\varepsilon}}{\Gamma(\alpha-2)\Gamma(2-\varepsilon)} \sum_{j=1}^m |t_j| \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} |h(s, {}^c D^a x(s), {}^c D^b y(s)) - h(s, {}^c D^a \zeta(s), {}^c D^b \chi(s))| ds \\ &\quad + \frac{\mathcal{Z}_\alpha t^2 \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m |N_{1j}(x(t_j)) - N_{1j}(\zeta(t_j))| + \frac{\mathcal{Z}_\alpha t^2 \Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m |O_{1j}(x(t_j)) - O_{1j}(\zeta(t_j))| \\ &\quad + \frac{\mathcal{Z}_\alpha t^2}{\Gamma(\alpha-2)} \int_{t_m}^1 (1-s)^{\alpha-3} |h(s, {}^c D^a x(s), {}^c D^b y(s)) - h(s, {}^c D^a \zeta(s), {}^c D^b \chi(s))| ds \\ &\quad + \frac{\mathcal{Z}_\alpha t^2}{\Gamma(\alpha-2)} \sum_{j=1}^m \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-3} |h(s, {}^c D^a x(s), {}^c D^b y(s)) - h(s, {}^c D^a \zeta(s), {}^c D^b \chi(s))| ds \end{aligned}$$

$$\begin{aligned}
& + \frac{\mathcal{Z}_\alpha t^2 \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m |t_j| |O_{1j}(x(t_j)) - O_{1j}(\zeta(t_j))| + \mathcal{Z}_\alpha t^2 \sum_{j=1}^m |O_{1j}(x(t_j)) - O_{1j}(\zeta(t_j))| \\
& + \sum_{j=1}^m \frac{|(t-t_j)^2|}{2} |O_{1j}(x(t_j)) - O_{1j}(\zeta(t_j))|.
\end{aligned}$$

Using  $(M_1)$  and  $(M_2)$ , we have

$$\begin{aligned}
& \leq \mathcal{Q}_\alpha \epsilon_\alpha + \frac{1}{\Gamma(\alpha+1)} [\mu_1 \|x - \zeta\|_0 + \mu_2 \|y - \chi\|_0] + \frac{1}{\Gamma(\alpha+1)} [\mu_1 \|x - \zeta\|_0 + \mu_2 \|y - \chi\|_0] \\
& + \frac{1}{\Gamma(\alpha)} [\mu_1 \|x - \zeta\|_0 + \mu_2 \|y - \chi\|_0] + \frac{1}{2\Gamma(\alpha-1)} [\mu_1 \|x - \zeta\|_0 + \mu_2 \|y - \chi\|_0] \\
& + \sum_{j=1}^m I_{1j} \|x - \zeta\|_0 + \sum_{j=1}^m \hat{I}_{1j} \|x - \zeta\|_0 + \frac{\mathcal{Z}_\alpha}{\Gamma(\alpha-\varepsilon+1)} [\mu_1 \|x - \zeta\|_0 + \mu_2 \|y - \chi\|_0] \\
& + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha)\Gamma(2-\varepsilon)} [\mu_1 \|x - \zeta\|_0 + \mu_2 \|y - \chi\|_0] + \frac{\mathcal{Z}_\alpha \Omega^{2-\varepsilon}}{\Gamma(\alpha-1)\Gamma(3-\varepsilon)} [\mu_1 \|x - \zeta\|_0 + \mu_2 \|y - \chi\|_0] \\
& + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} [\mu_1 \|x - \zeta\|_0 + \mu_2 \|y - \chi\|_0] + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \hat{I}_{1j} \|x - \zeta\|_0 \\
& + \frac{\mathcal{Z}_\alpha \Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m \check{I}_{1j} \|x - \zeta\|_0 + \frac{\mathcal{Z}_\alpha}{\Gamma(\alpha-1)} [\mu_1 \|x - \zeta\|_0 + \mu_2 \|y - \chi\|_0] \\
& + \frac{\mathcal{Z}_\alpha}{\Gamma(\alpha-1)} [\mu_1 \|x - \zeta\|_0 + \mu_2 \|y - \chi\|_0] + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \check{I}_{1j} \|x - \zeta\|_0 + \mathcal{Z}_\alpha \sum_{j=1}^m \check{I}_{1j} \|x - \zeta\|_0 \\
& + \frac{1}{2} \sum_{j=1}^m \check{I}_{1j} \|x - \zeta\|_0.
\end{aligned} \tag{43}$$

After some calculation and rearrangement in (43), we get

$$\|x - \zeta\|_0 - \frac{\Lambda_2^*}{(1 - \Lambda_1^*)} \|y - \chi\|_0 \leq \frac{\mathcal{Q}_\alpha \epsilon_\alpha}{(1 - \Lambda_1^*)}, \tag{44}$$

where

$$\begin{aligned}
\Lambda_1^* &= \left[ \mu_1 \left( \frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha-1)} + \frac{\mathcal{Z}_\alpha}{\Gamma(\alpha-\varepsilon+1)} + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha)\Gamma(2-\varepsilon)} + \frac{\mathcal{Z}_\alpha \Omega^{2-\varepsilon}}{\Gamma(\alpha-1)\Gamma(3-\varepsilon)} \right. \right. \\
&\quad \left. \left. + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} + \frac{2\mathcal{Z}_\alpha}{\Gamma(\alpha-1)} \right) + \sum_{j=1}^m I_{1j} + \sum_{j=1}^m \hat{I}_{1j} + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \hat{I}_{1j} + \frac{\mathcal{Z}_\alpha \Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m \check{I}_{1j} \right. \\
&\quad \left. + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \check{I}_{1j} + \mathcal{Z}_\alpha \sum_{j=1}^m \check{I}_{1j} + \frac{1}{2} \sum_{j=1}^m \check{I}_{1j} \right], \\
\Lambda_2^* &= \mu_2 \left( \frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha-1)} + \frac{\mathcal{Z}_\alpha}{\Gamma(\alpha-\varepsilon+1)} + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha)\Gamma(2-\varepsilon)} + \frac{\mathcal{Z}_\alpha \Omega^{2-\varepsilon}}{\Gamma(\alpha-1)\Gamma(3-\varepsilon)} \right. \\
&\quad \left. + \frac{\mathcal{Z}_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} + \frac{2\mathcal{Z}_\alpha}{\Gamma(\alpha-1)} \right).
\end{aligned}$$

On the similar fashion, for  $t \in [0, t_1]$ , and utilizing  $(M_1) - (M_2)$ , we can find

$$|y(t) - s_2(t)| \leq \mathcal{Q}_2 \epsilon_\beta, \tag{45}$$

where  $s_2(t)$  are those terms which are free of  $F$  and

$$\begin{aligned}
\mathcal{Q}_2 &= \frac{1}{\Gamma(\beta+1)} + \frac{\mathcal{Z}_\beta}{\Gamma(\beta-\rho+1)} + \frac{\mathcal{Z}_\beta \Phi^{1-\rho}}{\Gamma(\beta)\Gamma(2-\rho)} + \frac{\mathcal{Z}_\beta \Phi^{2-\rho}}{\Gamma(\beta-1)\Gamma(3-\rho)} + \frac{\mathcal{Z}_\beta \Phi^{1-\rho}}{\Gamma(\beta-1)\Gamma(2-\rho)} + \frac{2\mathcal{Z}_\beta \Phi^{1-\rho}}{\Gamma(2-\rho)} \\
&+ \frac{\mathcal{Z}_\beta \Phi^{2-\rho}}{\Gamma(3-\rho)} + \frac{2\mathcal{Z}_\beta}{\Gamma(\beta-1)} + \mathcal{Z}_\beta,
\end{aligned}$$

and

$$\|y - x\|_0 - \frac{\Lambda_4}{(1 - \Lambda_3)} \|x - \zeta\|_0 \leq \frac{Q_2 \epsilon_\beta}{(1 - \Lambda_3)}, \quad (46)$$

where

$$\begin{aligned} \Lambda_3 &= z_\beta \left[ \mu'_1 \left( \frac{1}{z_\beta \Gamma(\beta+1)} + \frac{1}{\Gamma(\beta-\rho+1)} + \frac{\Phi^{1-\rho}}{\Gamma(\beta)\Gamma(2-\rho)} + \frac{\Phi^{2-\rho}}{\Gamma(\beta-1)\Gamma(3-\rho)} + \frac{\Phi^{1-\rho}}{\Gamma(\beta-1)\Gamma(2-\rho)} \right. \right. \\ &\quad \left. \left. + \frac{2}{\Gamma(\beta-1)} \right) + \frac{\Phi^{1-\rho}}{\Gamma(2-\rho)} \sum_{j=1}^m \hat{I}_{2j} + \frac{\Phi^{2-\rho}}{\Gamma(3-\rho)} \sum_{j=1}^m \check{I}_{2j} + \frac{\Phi^{1-\rho}}{\Gamma(2-\rho)} \sum_{j=1}^m \check{I}_{2j} + \sum_{j=1}^m \check{I}_{2j} \right], \\ \Lambda_4 &= z_\beta \left[ \mu'_2 \left( \frac{1}{z_\beta \Gamma(\beta+1)} + \frac{1}{\Gamma(\beta-\rho+1)} + \frac{\Phi^{1-\rho}}{\Gamma(\beta)\Gamma(2-\rho)} + \frac{\Phi^{2-\rho}}{\Gamma(\beta-1)\Gamma(3-\rho)} + \frac{\Phi^{1-\rho}}{\Gamma(\beta-1)\Gamma(2-\rho)} \right. \right. \\ &\quad \left. \left. + \frac{2}{\Gamma(\beta-1)} \right) \right]. \end{aligned}$$

In addition, for  $t \in (t_m, t_{m+1}]$ ,  $1 \leq m \leq n$ , we can get

$$|y(t) - s_2^*(t)| \leq Q_\beta \epsilon_\beta, \quad (47)$$

where  $s_2^*(t)$  are those terms which are free of  $F$  and

$$\begin{aligned} Q_\beta &= \frac{2}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta)} + \frac{4z_\beta + 1}{2\Gamma(\beta-1)} + \frac{3z_\beta + 4}{2} + \frac{z_\beta}{\Gamma(\beta-\rho+1)} + \frac{z_\beta \Phi^{1-\rho}}{\Gamma(\beta)\Gamma(2-\rho)} + \frac{2z_\beta \Phi^{1-\rho}}{\Gamma(2-\rho)} + \frac{z_\beta \Phi^{2-\rho}}{\Gamma(3-\rho)} \\ &\quad + \frac{z_\beta \Phi^{2-\rho}}{\Gamma(\beta-1)\Gamma(3-\rho)} + \frac{z_\beta \Phi^{1-\rho}}{\Gamma(\beta-1)\Gamma(2-\rho)}. \end{aligned}$$

In addition,

$$\|y - x\|_0 - \frac{\Lambda_4^*}{(1 - \Lambda_3^*)} \|x - \zeta\|_0 \leq \frac{Q_\beta \epsilon_\beta}{(1 - \Lambda_3^*)}, \quad (48)$$

where

$$\begin{aligned} \Lambda_3^* &= \left[ \mu'_2 \left( \frac{2}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta)} + \frac{1}{2\Gamma(\beta-1)} + \frac{z_\beta}{\Gamma(\beta-\rho+1)} + \frac{z_\beta \Phi^{1-\rho}}{\Gamma(\beta)\Gamma(2-\rho)} + \frac{z_\beta \Phi^{2-\rho}}{\Gamma(\beta-1)\Gamma(3-\rho)} \right. \right. \\ &\quad \left. \left. + \frac{z_\beta \Phi^{1-\rho}}{\Gamma(\beta-1)\Gamma(2-\rho)} + \frac{2z_\beta}{\Gamma(\beta-1)} \right) + \sum_{j=1}^m I_{2j} + \sum_{j=1}^m \hat{I}_{2j} + \frac{z_\beta \Phi^{1-\rho}}{\Gamma(2-\rho)} \sum_{j=1}^m \hat{I}_{2j} + \frac{z_\beta \Phi^{2-\rho}}{\Gamma(3-\rho)} \sum_{j=1}^m \check{I}_{2j} \right. \\ &\quad \left. + \frac{z_\beta \Phi^{1-\rho}}{\Gamma(2-\rho)} \sum_{j=1}^m \check{I}_{2j} + z_\beta \sum_{j=1}^m \check{I}_{2j} + \frac{1}{2} \sum_{j=1}^m \check{I}_{2j} \right], \\ \Lambda_4^* &= \mu'_1 \left( \frac{2}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta)} + \frac{1}{2\Gamma(\beta-1)} + \frac{z_\beta}{\Gamma(\beta-\rho+1)} + \frac{z_\beta \Phi^{1-\rho}}{\Gamma(\beta)\Gamma(2-\rho)} + \frac{z_\beta \Phi^{2-\rho}}{\Gamma(\beta-1)\Gamma(3-\rho)} \right. \\ &\quad \left. + \frac{z_\beta \Phi^{1-\rho}}{\Gamma(\beta-1)\Gamma(2-\rho)} + \frac{2z_\beta}{\Gamma(\beta-1)} \right). \end{aligned}$$

The equivalent matrix of Equations (44) and (48) is given as:

$$\begin{bmatrix} 1 & -\frac{\Lambda_2^*}{(1 - \Lambda_1^*)} \\ -\frac{\Lambda_4^*}{(1 - \Lambda_3^*)} & 1 \end{bmatrix} \begin{bmatrix} \|x - \zeta\|_0 \\ \|y - x\|_0 \end{bmatrix} \leq \begin{bmatrix} \frac{Q_\alpha \epsilon_\alpha}{(1 - \Lambda_1^*)} \\ \frac{Q_\beta \epsilon_\beta}{(1 - \Lambda_3^*)} \end{bmatrix}.$$

Solving the above inequality, we get

$$\begin{bmatrix} \|x - \zeta\|_0 \\ \|y - \chi\|_0 \end{bmatrix} \leq \begin{bmatrix} \frac{1}{\Lambda_0} & \frac{\Lambda_2^*}{\Lambda_0(1-\Lambda_1^*)} \\ \frac{\Lambda_4^*}{\Lambda_0(1-\Lambda_3^*)} & \frac{1}{\Lambda_0} \end{bmatrix} \begin{bmatrix} \frac{Q_\alpha \epsilon_\alpha}{(1-\Lambda_1^*)} \\ \frac{Q_\beta \epsilon_\beta}{(1-\Lambda_3^*)} \end{bmatrix},$$

where

$$\Lambda_0 = 1 - \frac{\Lambda_2^* \Lambda_4^*}{(1-\Lambda_1^*)(1-\Lambda_3^*)} > 0.$$

Further simplification of the above system gives

$$\begin{aligned} \|x - \zeta\|_0 &\leq \frac{Q_\alpha \epsilon_\alpha}{\Lambda_0(1-\Lambda_1^*)} + \frac{\Lambda_2^* Q_\beta \epsilon_\beta}{\Lambda_0(1-\Lambda_1^*)(1-\Lambda_3^*)}, \\ \|y - \chi\|_0 &\leq \frac{Q_\beta \epsilon_\beta}{\Lambda_0(1-\Lambda_3^*)} + \frac{\Lambda_4^* Q_\alpha \epsilon_\alpha}{\Lambda_0(1-\Lambda_1^*)(1-\Lambda_3^*)}, \end{aligned}$$

from which we have

$$\|x - \zeta\|_0 + \|y - \chi\|_0 \leq \frac{Q_\alpha \epsilon_\alpha}{\Lambda_0(1-\Lambda_1^*)} + \frac{Q_\beta \epsilon_\beta}{\Lambda_0(1-\Lambda_3^*)} + \frac{\Lambda_4^* Q_\alpha \epsilon_\alpha}{\Lambda_0(1-\Lambda_1^*)(1-\Lambda_3^*)} + \frac{\Lambda_2^* Q_\beta \epsilon_\beta}{\Lambda_0(1-\Lambda_1^*)(1-\Lambda_3^*)}. \quad (49)$$

Let  $\max \{ \epsilon_\alpha, \epsilon_\beta \} = \epsilon$ ; then, from (49), we get

$$\|(x - \zeta, y - \chi)\| \leq \mathbf{K}_{\alpha, \beta} \epsilon,$$

which implies that

$$\|(x, y) - (\zeta, \chi)\| \leq \mathbf{K}_{\alpha, \beta} \epsilon, \quad (50)$$

where

$$\mathbf{K}_{\alpha, \beta} = \left[ \frac{Q_\alpha}{\Lambda_0(1-\Lambda_1^*)} + \frac{Q_\beta}{\Lambda_0(1-\Lambda_3^*)} + \frac{\Lambda_4^* Q_\alpha}{\Lambda_0(1-\Lambda_1^*)(1-\Lambda_3^*)} + \frac{\Lambda_2^* Q_\beta}{\Lambda_0(1-\Lambda_1^*)(1-\Lambda_3^*)} \right].$$

Hence, problem (1) is Ulam–Hyers stable. Moreover, if we set  $\Theta(\epsilon) = \mathbf{K}_{x,y} \epsilon$ ;  $\Theta(0) = 0$  in (50), then problem (1) is generalized Ulam–Hyers stable.  $\square$

**(M<sub>5</sub>)**: Let  $\Psi_\alpha, \Psi_\beta \in PC(J, \mathbb{R}^+)$  be nondecreasing functions; then, for  $t \in J$ , there are  $\lambda_\alpha, \lambda_\beta > 0$  such that

$$\begin{aligned} I^\alpha \Psi_\alpha(t) &\leq \lambda_\alpha \Psi_\alpha(t), \quad I^{\alpha-1} \Psi_\alpha(t) \leq \lambda_\alpha \Psi_\alpha(t), \\ I^{\alpha-2} \Psi_\alpha(t) &\leq \lambda_\alpha \Psi_\alpha(t), \quad I^{\alpha-\varepsilon} \Psi_\alpha(t) \leq \lambda_\alpha \Psi_\alpha(t). \end{aligned}$$

Similarly,

$$\begin{aligned} I^\beta \Psi_\beta(t) &\leq \lambda_\beta \Psi_\beta(t), \quad I^{\beta-1} \Psi_\beta(t) \leq \lambda_\beta \Psi_\beta(t), \\ I^{\beta-2} \Psi_\beta(t) &\leq \lambda_\beta \Psi_\beta(t), \quad I^{\beta-\rho} \Psi_\beta(t) \leq \lambda_\beta \Psi_\beta(t). \end{aligned}$$

**Theorem 5.** Assume that (M<sub>1</sub>) – (M<sub>2</sub>) and (M<sub>5</sub>) are satisfied; then, by Definition 6 and Definition 7, Problem (1) is Ulam–Hyers–Rassias stable with respect to  $(\Psi_{\alpha, \beta}, \varphi_{\alpha, \beta})$ , as well as generalized Ulam–Hyers–Rassias stable.

## 5. Example

To substantiate the aforementioned demonstrated theory, we supply the following problem:

$$\begin{cases} {}^cD^\alpha x(t) + h(t, {}^cD^a x(t), {}^cD^b y(t)) = 0, t \neq t_m, \\ {}^cD^\beta y(t) + w(t, {}^cD^a x(t), {}^cD^b y(t)) = 0, \\ \Delta x|_{t=t_m} = M_{1m}x(t_m), \Delta x'|_{t=t_m} = N_{1m}x(t_m), \Delta x''|_{t=t_m} = O_{1m}x(t_m), \\ \Delta y|_{t=t_m} = M_{2m}y(t_m), \Delta y'|_{t=t_m} = N_{2m}y(t_m), \Delta y''|_{t=t_m} = O_{2m}y(t_m), \\ x(t) = x(t) = 0, {}^cD^\varepsilon x(\Omega) = x''(1), y(0) = y'(0) = 0, {}^cD^\rho y(\Phi) = y''(1), \\ 0 < \varepsilon, \Omega < 1, 0 < \rho, \Phi < 1. \end{cases} \quad (51)$$

Take  $J = [0, 1]$ ,  $\alpha = \frac{5}{2}$ ,  $\beta = \sqrt{5}$ ,  $\varepsilon = \frac{1}{2}$ ,  $\Omega = \frac{1}{3}$ ,  $a = \frac{1}{2}$ ,  $b = \frac{1}{8}$ ,  $t_1 = \frac{1}{2}$ ,  $\rho = \frac{1}{3}$ ,  $\Phi = \frac{1}{5}$ ,  $h(t, x, y) = e^t + \frac{|x|+|y|}{40(1+|x|+|y|)}$ ,  $w(t, x, y) = \frac{\cos t+|x|+|y|}{40(1+|x|+|y|)}$ ,  $M_{11}(x) = M_{21}(x) = \frac{|x|}{5+|x|}$ ,  $N_{11}(x) = N_{21}(x) = \frac{|x|}{10+|x|}$ ,  $O_{11}(x) = O_{21}(x) = \frac{|x|}{15+|x|}$ . By direct computation, we have  $\mu_1 = \mu_2 = \mu'_1 = \mu'_2 = \frac{1}{40}$ ,  $I_{11} = I_{21} = \frac{1}{5}$ ,  $\hat{I}_{11} = \hat{I}_{21} = \frac{1}{10}$ ,  $\check{I}_{11} = \check{I}_{21} = \frac{1}{15}$ ,

$$\left[ (\mu_1 + \mu_2) \left( \frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} + \frac{1}{2\Gamma(\alpha-1)} + \frac{Z_\alpha}{\Gamma(\alpha-\varepsilon+1)} + \frac{Z_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha)\Gamma(2-\varepsilon)} + \frac{Z_\alpha \Omega^{2-\varepsilon}}{\Gamma(\alpha-1)\Gamma(3-\varepsilon)} \right. \right. \\ \left. \left. + \frac{Z_\alpha \Omega^{1-\varepsilon}}{\Gamma(\alpha-1)\Gamma(2-\varepsilon)} + \frac{2Z_\alpha}{\Gamma(\alpha-1)} \right) + \sum_{j=1}^m I_{1j} + \sum_{j=1}^m \hat{I}_{1j} + \frac{Z_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \hat{I}_{1j} + \frac{Z_\alpha \Omega^{2-\varepsilon}}{\Gamma(3-\varepsilon)} \sum_{j=1}^m \check{I}_{1j} \right. \\ \left. + \frac{Z_\alpha \Omega^{1-\varepsilon}}{\Gamma(2-\varepsilon)} \sum_{j=1}^m \check{I}_{1j} + Z_\alpha \sum_{j=1}^m \check{I}_{1j} + \frac{1}{2} \sum_{j=1}^m \check{I}_{1j} \right] \approx 0.680277 < 1.$$

Similarly,

$$\left[ (\mu'_1 + \mu'_2) \left( \frac{2}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta)} + \frac{1}{2\Gamma(\beta-1)} + \frac{Z_\beta}{\Gamma(\beta-\rho+1)} + \frac{Z_\beta \Phi^{1-\rho}}{\Gamma(\beta)\Gamma(2-\rho)} + \frac{Z_\beta \Phi^{2-\rho}}{\Gamma(\beta-1)\Gamma(3-\rho)} \right. \right. \\ \left. \left. + \frac{Z_\beta \Phi^{1-\rho}}{\Gamma(\beta-1)\Gamma(2-\rho)} + \frac{2Z_\beta}{\Gamma(\beta-1)} \right) + \sum_{j=1}^m I_{2j} + \sum_{j=1}^m \hat{I}_{2j} + \frac{Z_\beta \Phi^{1-\rho}}{\Gamma(2-\rho)} \sum_{j=1}^m \hat{I}_{2j} + \frac{Z_\beta \Phi^{2-\rho}}{\Gamma(3-\rho)} \sum_{j=1}^m \check{I}_{2j} \right. \\ \left. + \frac{Z_\beta \Phi^{1-\rho}}{\Gamma(2-\rho)} \sum_{j=1}^m \check{I}_{2j} + Z_\beta \sum_{j=1}^m \check{I}_{2j} + \frac{1}{2} \sum_{j=1}^m \check{I}_{2j} \right] \approx 0.427420 < 1.$$

Thus, by Theorem 2, Problem (51) has a unique solution. Furthermore, all of the assumptions are satisfied, so the Problem (51) is Ulam–Hyers, generalized Ulam–Hyers, Ulam–Hyers–Rassias and generalized Ulam–Hyers–Rassias stable.

## 6. Conclusions

In the above study, we have successfully built up existence theory for the solutions of system (1). The required analysis has been developed with the help of the Banach contraction principle and Schauder fixed point theorem. We found that the fractional order coupled system is additionally complicated and challenging as compared to the single FDEs. We also concluded that, if we increase the order or boundary conditions, then the end result turns into extra accurate. Our results are new and fascinating. Our methods can be used to study the existence of solutions for the high order or multiple-point boundary value systems of a nonlinear coupled system of FDEs. Furthermore, we have presented different kinds of Ulam–Hyers stability results for the solution of the considered system (1). In addition, we have presented our main theoretical results with the help of an example. In the

future, this concept can be extended to more applied and complicated problems of applied nature. The obtained results can be used in fields like numerical analysis and managerial sciences including business mathematics and economics, etc.

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