mathematics

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Line and Subdivision Graphs Determined by $\mathbb{T}_4$-Gain Graphs

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Received: 16 August 2019; Accepted: 26 September 2019; Published: 5 October 2019

Abstract: Let $\mathbb{T}_4 = \{\pm 1, \pm i\}$ be the subgroup of fourth roots of unity inside $\mathbb{T}$, the multiplicative group of complex units. For a $\mathbb{T}_4$-gain graph $\Phi = (\Gamma, \mathbb{T}_4, \varphi)$, we introduce gain functions on its line graph $\mathcal{L}(\Gamma)$ and on its subdivision graph $\mathcal{S}(\Gamma)$. The corresponding gain graphs $\mathcal{L}(\Phi)$ and $\mathcal{S}(\Phi)$ are defined up to switching equivalence and generalize the analogous constructions for signed graphs. We discuss some spectral properties of these graphs and in particular we establish the relationship between the Laplacian characteristic polynomial of a gain graph Φ, and the adjacency characteristic polynomials of $\mathcal{L}(\Phi)$ and $\mathcal{S}(\Phi)$. A suitably defined incidence matrix for $\mathbb{T}_4$-gain graphs plays an important role in this context.

Keywords: complex unit gain graph; line graph; subdivision graph; oriented gain graph; voltage graph

1. Introduction

Let Γ be a simple graph with the vertex set $V(\Gamma) = \{v_1, v_2, \ldots, v_n\}$ and the set of oriented edges $\vec{E}(\Gamma)$ that contains two copies of each edge of Γ with opposite directions. We write e_{ij} for the oriented edge from v_i to v_j. Given any group $\mathfrak{G}$, a ($\mathfrak{G}$-)*gain graph* is a triple $\Phi = (\Gamma, \mathfrak{G}, \varphi)$ consisting of an *underlying graph* Γ, the *gain group* $\mathfrak{G}$ and a map $\varphi : \vec{E}(\Gamma) \to \mathfrak{G}$ such that $\varphi(e_{ij}) = \varphi(e_{ji})^{-1}$ called the *gain function*.

Gain graphs (also known in the literature as *voltage graphs*) are studied in many, not necessarily pure mathematics, research areas (for more details, see [1] and the annotated bibliography [2]). During the last decade, there has been a growing interest for the study of matrices and eigenvalues associated to gain graphs. For instance, in [3], N. Reff studied some spectral properties of the adjacency and the Laplacian matrix of $\mathbb{T}$-graphs, where $\mathbb{T}$ denotes the *circle group*, i.e., the multiplicative group of all complex numbers with norm 1. Such gain graphs are also known as *complex unit gain graphs*. In [4], the same author introduced a notion of orientation for gain graphs in order to provide a suitable setting to build up line graphs of gain graphs. This setting works reasonably well when $\mathfrak{G}$ is abelian. More recently, in [5], the third and the fourth authors of the present paper began to explore the spectral properties of $\mathbb{T}_4$-gain graphs, where $\mathbb{T}_4$ denotes the group of the fourth roots of unity $\{\pm 1, \pm i\}$, showing in particular how the least Laplacian eigenvalue of a $\mathbb{T}_4$-gain graph is related to its frustration index and number. Other spectral results concerning $\mathbb{T}_4$ are obtained in [6–8], where gain graphs are called weighted directed graphs (see also their list of references).

In this paper, we also investigate $\mathbb{T}_4$-gain graphs. The interest towards $\mathbb{T}_4$-gain graphs is due to the fact that every spectral result concerning $\mathbb{T}_4$-gain graphs applies as well to $\mathbb{T}_2$-gain graphs, which are well-known as *signed graphs*. In fact, the latter ones can be seen as $\mathbb{T}_4$-gain graph such that $\varphi(\vec{E}(\Gamma)) \subseteq \{\pm 1\}$. Moreover, $\mathbb{T}_4$ is the minimal complex unit gain context allowing to retrieve the spectral theory of digraphs and mixed graphs as developed, for instance, in [9]. In other words, digraphs and mixed graphs can be also seen as $\mathbb{T}_4$-gain graphs $\Phi = (\Gamma, \varphi)$, such that $\varphi(\vec{E}(\Gamma)) \subseteq \{1, \pm i\}$.

The rest of the paper is organized as follows. In Section 2, we recall some background theory on gain graphs, including notions of balancedness and switching equivalence. In Section 3, we revisit the N. Reff's notion of line graph associated to $\mathfrak{G}$-gain graph emphasizing his results in the case $\mathfrak{G} = \mathbb{T}_4$. Finally, in Section 4, we introduce subdivision graphs determined by $\mathbb{T}_4$-gain graphs. To best of our knowledge, no attempts in the same direction have been done in the literature. Our constructions are consistent with those carried out for signed graphs in [10] (Section 2) (see also [11] (Section 2)).

2. Preliminaries

From now on, a $\mathbb{T}_4$-gain graph is simply denoted by $\Phi = (\Gamma, \varphi)$. We write $(\Gamma, 1)$ for the $\mathbb{T}_4$-gain graph with all neutral edges. The *all-negative* $\mathbb{T}_4$-gain graph $(\Gamma, -1)$ is a gain graph (Γ, φ) such that φ maps all oriented edges onto $\{-1\} \subset \mathbb{T}_4$. We say that a $\mathbb{T}_4$-gain graph $\Phi = (\Gamma, \varphi)$ is of order n and size m if its underlying graph Γ has n vertices and m edges.

Moreover, we adopt the following notation

$$V(\Gamma) = \{ v_1, \dots, v_n \} \qquad \text{and} \qquad \mathrm{E}(\Gamma) = \{ e_1, \dots, e_m \}$$

for the set of vertices and the set of (unoriented) edges of Γ, respectively.

Let $M_{m,n}(\mathbb{C})$ be the set of m times n complex matrices. For a matrix $A = (a_{ij}) \in M_{m,n}(\mathbb{C})$, we denote by $A^* = (a^*_{ij}) \in M_{n,m}(\mathbb{C})$ its *conjugate* (or *Hermitian*) *transpose*; i.e., $a^*_{ij} = \bar{a}_{ji}$.

The *adjacency matrix* $A(\Phi) = (a_{ij}) \in M_{n,n}(\mathbb{C})$ of a $\mathbb{T}_4$-gain graph $\Phi = (\Gamma, \varphi)$ is defined by

$$a_{ij} = \begin{cases} \varphi(e_{ij}) & \text{if } v_i \text{ is adjacent to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

If e_{ij} is an arc from v_i to v_j, then $a_{ij} = \varphi(e_{ij}) = \varphi(e_{ji})^{-1} = \overline{\varphi(e_{ji})} = \bar{a}_{ji}$. Consequently, $A(\Phi)$ is Hermitian and its eigenvalues are real. The *Laplacian matrix* $L(\Phi)$ is defined as $D(\Gamma) - A(\Phi)$, where $D(\Gamma)$ stands for the diagonal matrix of vertex degrees of Γ. Therefore, $L(\Phi)$ is also Hermitian. As shown in [3] by N. Reff, the matrix $L(\Phi)$ is positive semidefinite, and all its eigenvalues are nonnegative. The multiset of eigenvalues of $A(\Phi)$ (respectively, of $L(\Phi)$) is called the adjacency (respectively, the Laplacian) spectrum of Φ and is denoted by $\mathrm{Spec}(A(\Phi))$ (respectively, $\mathrm{Spec}(L(\Phi))$). A *switching function* of a given gain graph Φ is any map $\zeta : V(\Gamma) \to \mathbb{T}_4$. In other words, the switching the $\mathbb{T}_4$-gain graph $\Phi = (\Gamma, \varphi)$ means replacing φ by φ^ζ, where $\varphi^\zeta(e_{ij}) = \zeta(v_i)^{-1}\varphi(e_{ij})\zeta(v_j)$ and obtaining in this way the new $\mathbb{T}_4$-gain graph $\Phi^\zeta = (\Gamma, \varphi^\zeta)$. We say that $\Phi_1 = (\Gamma, \varphi_1)$ and $\Phi_2 = (\Gamma, \varphi_2)$ (and their corresponding gain functions) are *switching equivalent* if there exists a switching function ζ such that $\Phi_2 = \Phi_1^\zeta$. By writing $\Phi_1 \sim \Phi_2$ or $\varphi_1 \sim \varphi_2$, we mean that Φ_1 and Φ_2 are switching equivalent.

To each switching function ζ, we associate a diagonal matrix $D(\zeta) = \mathrm{diag}(\zeta(v_1), \dots, \zeta(v_n))$ also known as *switching matrix*. Note that

$$A(\Phi_2) = D(\zeta)^* A(\Phi_1) D(\zeta) \qquad \text{and} \qquad L(\Phi_2) = D(\zeta)^* L(\Phi_1) D(\zeta).$$

Hence, given any pair (Φ_1, Φ_2) of switching equivalent $\mathbb{T}_4$-gain graphs, we get the following equality between their spectra:

$$\mathrm{Spec}(A(\Phi_1)) = \mathrm{Spec}(A(\Phi_2)) \qquad \text{and} \qquad \mathrm{Spec}(L(\Phi_1)) = \mathrm{Spec}(L(\Phi_2)).$$

One of the key notions in the theory of gain graphs (and of the more general theory of biased graphs as well) is balancedness (see [1]). An oriented edge $e_{i_h i_k} \in \vec{E}(\Gamma)$ is said to be *neutral* for $\Phi = (\Gamma, \varphi)$ if $\varphi(e_{i_h i_k}) = 1$. Similarly, the walk $W = e_{i_1 i_2} e_{i_2 i_3} \cdots e_{i_{l-1} i_l}$ is said to be *neutral* if its *gain*

$$\varphi(W) := \varphi(e_{i_1 i_2})\varphi(e_{i_2 i_3}) \cdots \varphi(e_{i_{l-1} i_l})$$

is equal to 1. An edge set $S \subseteq E$ is said to be *balanced* if every directed cycle $\vec{C}$ with edges in S is neutral. A subgraph is *balanced* if its edge set is balanced (see [3] and [5] for further details).

The following proposition gives necessary and sufficient conditions for a $\mathbb{T}_4$-gain graph to be balanced. It also holds in the more general context of complex unit gain graphs (see [5] for a proof).

Proposition 1. *Let $\Phi = (\Gamma, \varphi)$ be a $\mathbb{T}_4$-gain graph. Then, the following are equivalent:*

1. *Φ is balanced.*
2. *$\Phi \sim (\Gamma, 1)$.*
3. *There exists a function $\theta : V(\Gamma) \to \mathbb{T}_4$ such that*

$$\theta(v_i)^{-1}\theta(v_j) = \varphi(e_{ij}) \qquad \text{for all } e_{ij} \in \vec{E}(\Gamma).$$

Although the following characterization of balanced $\mathbb{T}_4$-gain graph Φ is not used in our paper, we recall by sake of completeness that a connected $\mathbb{T}_4$-gain graph Φ is balanced if and only if its least Laplacian eigenvalue $\lambda_n(\Phi)$ is 0. This follows by [3] (Lemma 2.1 (2)) or [12] (Theorem 2.8).

The next proposition restates the result of [4] (Lemma 2.2) in the case of $\mathbb{T}_4$-gain graphs.

Proposition 2. *Let $\Phi_1 = (\Gamma, \varphi_1)$ and $\Phi_2 = (\Gamma, \varphi_2)$ be $\mathbb{T}_4$-gain graphs with the same underlying graph Γ. If for every cycle C in Γ there exists a directed cycle $\vec{C}_v$ with base vertex v such that $\varphi_1(\vec{C}_v) = \varphi_2(\vec{C}_v)$, then there exists a switching function ζ such that $\Phi_2 = \Phi_1^{\zeta}$.*

By Proposition 2, it follows that a gain graph Φ is balanced if and only if all its directed cycles are neutral. Moreover, if in Φ, there exists a directed cycle with an imaginary gain, then Φ cannot be switching equivalent to a signed graph.

To depict $\mathbb{T}_4$-gain graphs in Figures 1–5, each continuous (respectively, dashed) thick undirected line represents two opposite oriented edges with gain 1 (respectively, -1), whereas the arrows detect the oriented edges uv's such that $\varphi(uv) = i$. The other possible choice for the arrow direction not employed here—namely using an arrow from v to u to denote the oriented edge uv such that $\varphi(uv) = i$—would lead to an alternative and fully satisfactory way to "read" the imaginary gains from the drawings.

3. Line Graphs Associated to $\mathbb{T}_4$-Gain Graphs

Recall that $\mathbb{T}$ stands for the multiplicative group of all complex numbers with norm 1. In other words, $\mathbb{T} = \{z \in \mathbb{C} : z\bar{z} = 1\}$ is a subgroup of the multiplicative group $\mathbb{C}^\times$ of all nonzero complex numbers. Clearly, $\mathbb{T}_4 = \{\pm 1, \pm i\}$ is a subgroup of $\mathbb{T}$.

We start with an elementary algebraic lemma. For its relevance in the sequel, we also provide its proof.

Lemma 1. *Let (a, b) and (c, d) be two pairs in $\mathbb{T} \times \mathbb{T}$ such that*

$$\bar{b}a = \bar{d}c. \tag{1}$$

Then, there exists $\rho \in \mathbb{T}$ such that $(c, d) = \rho(a, b)$.

Proof of Lemma 1. By multiplying both sides of (1) by bd, we get $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$. Therefore,

$$(c, d) = \rho\,(a, b),$$

for some $\rho \in \mathbb{C}^{\times}$. Together with (1), this implies $|\rho|^2 \bar{b}a = \bar{b}a$, i.e., $\rho \in \mathbb{T}$. □

Let $\Phi = (\Gamma, \varphi)$ be a $\mathbb{T}_4$-gain graph. We say that the $n \times m$ matrix $\mathrm{H}(\Phi) = (\eta_{ve})$ with entries in $\mathbb{T}_4 \cup \{0\}$ is an *incidence matrix* of Φ if

$$\eta_{v_i e_h} = \begin{cases} -\eta_{v_j e_h}\varphi(e_{ij}) & \text{if the endpoints of } e_h \text{ are precisely } v_i \text{ and } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

In the case when e_h joins v_i and v_j, we also require that $\eta_{v_i e_h} \in \mathbb{T}_4$. We say "an" incidence matrix, because by this definition $\mathrm{H}(\Phi)$ is unique only if Γ is empty, i.e., if it is of size 0. If each column is multiplied by any element in $\mathbb{T}_4$ the result will still be an incidence matrix. The next proposition shows that all the other possible incident matrices can be obtained from a fixed $\mathrm{H}(\Phi)$ in that way.

Proposition 3. *Let $\mathrm{H}(\Phi) = (\eta_{ve})$ and $\mathrm{H}(\Phi)' = (\eta'_{ve})$ be two incidence matrices both related to the $\mathbb{T}_4$-gain graph $\Phi = (\Gamma, \varphi)$. There exists an $m \times m$ diagonal matrix S with entries in $\mathbb{T}_4 \cup \{0\}$ such that $\mathrm{H}(\Phi)' = \mathrm{H}(\Phi)S$ and $S^*S = I$.*

Proof of Proposition 3. Let v_i and v_j be the endpoints of a fixed edge $e_h \in E(\Gamma)$. Clearly, the only non-zero elements in the hth columns of $\mathrm{H}(\Phi)$ and $\mathrm{H}(\Phi)'$ are $\eta_{v_i e_h}, \eta_{v_j e_h}, \eta'_{v_i e_h}$ and $\eta'_{v_j e_h}$, where

$$\overline{\eta}_{v_j e_h}\eta_{v_i e_h} = -\varphi(e_{ij}) = \overline{\eta}'_{v_j e_h}\eta'_{v_i e_h}.$$

By Lemma 1, there exists a $\rho_h \in \mathbb{T}_4$ such that $(\eta'_{v_i e_h}, \eta'_{v_j e_h}) = \rho_h(\eta_{v_i e_h}, \eta_{v_j e_h})$. For $S = \mathrm{diag}(\rho_1, \ldots, \rho_m)$, it can be easily verified that $\mathrm{H}(\Phi)' = \mathrm{H}(\Phi)S$ and $S^*S = I$. □

In particular, by Proposition 3, for a fixed edge $e_h \in E(\Gamma)$ with endpoints v_i and v_j, we have four different possibilities for the corresponding column in the incidence matrix:

$$(\eta_{v_i e_h}, \eta_{v_j e_h}) = \begin{cases} (1, -\overline{\varphi(e_{ij})}); \\ (-1, \overline{\varphi(e_{ij})}); \\ (i, -i \cdot \overline{\varphi(e_{ij})}); \\ (-i, i \cdot \overline{\varphi(e_{ij})}). \end{cases} \tag{2}$$

In other words, every $\mathbb{T}_4$-gain graph $\Phi = (\Gamma, \varphi)$ admits 4^m different incidence matrices related to it.

Proposition 4. *Let $\mathrm{H}(\Phi) = (\eta_{ve})$ be an incidence matrix related to the $\mathbb{T}_4$-gain graph $\Phi = (\Gamma, \varphi)$. Then*

$$\mathrm{H}(\Phi)\mathrm{H}(\Phi)^* = D(\Gamma) - A(\Gamma) = L(\Phi). \tag{3}$$

Proof of Proposition 4. Let $\mathrm{H}(\Phi)\mathrm{H}(\Phi)^* = (c_{ij})$. By definition,

$$c_{ii} = \sum_{e_h \in E(\Gamma)} |\eta_{v_i e_h}|^2 = d_\Gamma(v_i). \tag{4}$$

In fact, $|\eta_{v_i e_h}| = 1$ whenever $\eta_{v_i e_h} \neq 0$, and there are precisely $d_\Gamma(v_i)$ summands of this type in (4). If $i \neq j$, then

$$c_{ij} = \sum_{e_h \in E(\Gamma)} \eta_{v_i e_h} \overline{\eta}_{v_j e_h} = -\varphi(e_{ij}),$$

where $\varphi(e_{ij})$ is 0 whenever v_i and v_j are not adjacent. This completes the proof. □

To have a lighter notation, in what follows, we denote by H a specific incidence matrix related to the $\mathbb{T}_4$-gain graph $\Phi = (\Gamma, \varphi)$. We next explain how H determines a $\mathbb{T}_4$-gain structure on the line graph $\mathcal{L}(\Gamma)$. It is well-known that $V(\mathcal{L}(\Gamma)) = E(\Gamma)$ and $ef \in E(\mathcal{L}(\Gamma))$, whenever e and f share an endpoint. We denote by $\mathcal{L}_{\mathrm{H}}(\Phi)$ the $\mathbb{T}_4$-gain graph $(\mathcal{L}(\Gamma), \varphi_{\mathrm{H}}^{\mathcal{L}})$, where

$$\varphi_{\mathrm{H}}^{\mathcal{L}} : ef \in \vec{E}(\mathcal{L}(\Gamma)) \longmapsto \overline{\eta}_{we}\eta_{wf} \in \mathbb{T}_4, \tag{5}$$

where w is the endpoint shared by the edges e and f. It is easy to verify that $\varphi_{\mathrm{H}}^{\mathcal{L}}$ is a gain function. In fact,

$$\varphi_{\mathrm{H}}^{\mathcal{L}}(fe) = \overline{\varphi_{\mathrm{H}}^{\mathcal{L}}(ef)}.$$

The proof of our Theorem 1 reads the same line as the one of [4] (Theorem 5.1).

Theorem 1. *Let* H *be one of the incidence matrices related to the* $\mathbb{T}_4$*-gain graph* $\Phi = (\Gamma, \varphi)$*. Then,*

$$\mathrm{H}(\Phi)^*\mathrm{H}(\Phi) = 2I_m + A(\mathcal{L}_H(\Phi)). \tag{6}$$

Proof of Theorem 1. First, note that $\mathrm{H}(\Phi)^*\mathrm{H}(\Phi)$ is an $m \times m$ matrix. Consider next the dot product of row $\mathbf{r}_h$ of $\mathrm{H}(\Phi)^*$ with column $\mathbf{c}_k$ of $\mathrm{H}(\Phi)$. We differ two cases:

- $h = k$ (same edge). Let u and w be the endpoints of e_h. Then, η_{ue_h} and η_{we_h} are the only non-zero entries in column $\mathbf{c}_h$. Therefore,

$$\begin{aligned} \mathbf{r}_h \cdot \mathbf{c}_h = \mathbf{c}_h^* \cdot \mathbf{c}_h &= \bar{\eta}_{ue_h}\eta_{ue_h} + \bar{\eta}_{we_h}\eta_{we_h} \\ &= |\eta_{ue_h}|^2 + |\eta_{we_h}|^2 \\ &= 2. \end{aligned}$$

- $h \neq k$. By definition,

$$\mathbf{r}_h \cdot \mathbf{c}_k = \mathbf{c}_h^* \cdot \mathbf{c}_k = \sum_{v_i \in V(\Gamma)} \bar{\eta}_{v_i e_h}\eta_{v_i e_k}.$$

  In the last sum there is at most one non-zero summand, which actually exists if and only if e_h and e_k are adjacent in the line graph, i.e., when e_h and e_k share a common endpoint, say w. Hence, supposing $e_h e_k \in \vec{E}(\mathcal{L}(\Gamma))$,

$$\mathbf{r}_h \cdot \mathbf{c}_k = \bar{\eta}_{we_h}\eta_{we_k}.$$

Now, the statement follows from Equation (5). □

Let $\mathfrak{G}$ be an abelian group. In [4], N. Reff already introduced a line graph associated to the gain graph $(\Gamma, \mathfrak{G}, \varphi)$. Its gains not only depend on the chosen incidence matrix, but also on the pick of a *weak involution* in $\mathfrak{G}$, i.e., on an element $\mathfrak{s} \in \mathfrak{G}$ such that $\mathfrak{s}^2 = 1_{\mathfrak{G}}$. Our definition of $\mathcal{L}_{\mathrm{H}}(\Phi)$ is consistent with N. Reff's for $\mathfrak{s} = 1_{\mathfrak{G}}$ and $\mathfrak{G} = \mathbb{T}_4$. In the case when $\varphi(\vec{E}(\Gamma)) \subseteq \{-1, 1\}$, i.e., when the $\mathbb{T}_4$-gain graph Φ is actually a signed graph, with a gain function $\varphi_{\mathrm{H}}^{\mathcal{L}}$ defined as in Equation (5), we retrieve the same signature on $\mathcal{L}(\Phi)$ as assigned in [10] (Section 2) and [11] (Section 2). It is also possible to define the adjacency matrix of a line graph as $A(\mathcal{L}(\Phi)) = 2I_m - H^*H$. In that case the signature of the line graph is consistent to the one defined by T. Zaslavsky.

Proposition 5. *Let* H *and* H' *be two incidence matrices both associated to the same* $\mathbb{T}_4$*-gain graph* $\Phi = (\Gamma, \varphi)$*. Then,* $\mathcal{L}_{\mathrm{H}}(\Phi)$ *and* $\mathcal{L}_{\mathrm{H}'}(\Phi)$ *share the same adjacency spectrum. Moreover, every gain graph which is switching equivalent to* $\mathcal{L}_{\mathrm{H}}(\Phi)$ *is a line graph associated to* Φ*.*

Proof of Proposition 5. By Proposition 3, there exists a diagonal matrix S with entries in $\mathbb{T}_4 \cup \{0\}$ such that $\mathrm{H}' = \mathrm{H}S$ and $S^*S = I$. Next by Equation (6):

$$\begin{aligned} A(\mathcal{L}_{\mathrm{H}'}(\Phi)) &= (\mathrm{H}')^*\mathrm{H}' - 2I_m \\ &= S^*\mathrm{H}^*\mathrm{H}S - 2S^*I_mS \\ &= S^*(H^*H - 2I_m)S \\ &= S^*A(\mathcal{L}_{\mathrm{H}}(\Phi))S. \end{aligned}$$

This proves the first assertion.

Let now Ψ be a $\mathbb{T}_4$-gain graph, which is switching equivalent to $\mathcal{L}_{\mathrm{H}}(\Phi)$. Then, there exists a map $\zeta_{\mathcal{L}_{\mathrm{H}}} : V(\mathcal{L}_{\mathrm{H}}(\Phi)) = E(\Gamma) \to \mathbb{T}_4$ such that $\mathcal{L}_{\mathrm{H}}(\Phi)^{\zeta_{\mathcal{L}_{\mathrm{H}}}} = \Psi$. For $D(\zeta_{\mathcal{L}_{\mathrm{H}}}) = \mathrm{diag}(\zeta_{\mathcal{L}_{\mathrm{H}}}(e_1), \ldots, \zeta_{\mathcal{L}_{\mathrm{H}}}(e_m))$, it easily follows $\Psi = A(\mathcal{L}_{\mathrm{H}'}(\Phi))$, where $\mathrm{H}' = \mathrm{H}D(\zeta_{\mathcal{L}_{\mathrm{H}}})$. □

By Proposition 5 line graphs associated to $\mathbb{T}_4$-gain graphs fully represent a class of switching equivalent gain graphs, similarly as the corresponding construction in the smaller context of signed graphs (see [10]).

Preservation of switching equivalence classes is often recognized as a minimum requirement to judge positively new constructions involving signed or gain graphs. Next, proposition shows that the introduced notion of line graph associated to a $\mathbb{T}_4$-gain graph is appropriate in this sense.

Proposition 6. *Line graphs of switching equivalent* $\mathbb{T}_4$*-gain graphs* $\Phi_1 = (\Gamma, \varphi_1)$ *and* $\Phi_2 = (\Gamma, \varphi_2)$ *are switching equivalent.*

Proof of Proposition 6. Let $\mathrm{H}_2 = (\eta^{(2)}_{ve})$ be a fixed incident matrix for Φ_2. Since $\Phi_1 \sim \Phi_2$, there exists a map $\zeta : V(\Gamma) \to \mathbb{T}_4$ such that $\varphi_2 = \varphi_1^{\zeta}$.

Suppose that v_i and v_j are the endpoint of the edge $e \in E(\Gamma)$. By definition, we obtain

$$\eta^{(2)}_{v_ie}\bar{\eta}^{(2)}_{v_je} = -\varphi_2(e_{ij}) = -\overline{\zeta(v_i)}\varphi_1(e_{ij})\zeta(v_j),$$

and consequently

$$\left(\zeta(v_i)\eta^{(2)}_{v_ie}\right)\left(\overline{\zeta(v_j)\eta^{(2)}_{v_je}}\right) = -\varphi_1(e_{ij}).$$

It turns out that, for the switching matrix $D(\zeta) = \mathrm{diag}(\zeta(v_1), \ldots, \zeta(v_n))$, the matrix $\mathrm{H}_1 = D(\zeta)\mathrm{H}_2$ is an incidence matrix of Φ_1, and $\varphi^{\mathcal{L}}_{\mathrm{H}_1(\Phi_1)} = \varphi^{\mathcal{L}}_{\mathrm{H}_2(\Phi_2)}$. In fact, if w is the endpoint shared by two edges e and f of Γ,

$$\varphi^{\mathcal{L}}_{\mathrm{H}_1(\Phi_1)}(ef) = \bar{\eta}^{(1)}_{we}\eta^{(1)}_{wf} = \bar{\eta}^{(2)}_{we}\bar{\zeta}(w)\zeta(w)\eta^{(2)}_{wf} = \bar{\eta}^{(2)}_{we}\eta^{(2)}_{wf} = \varphi^{\mathcal{L}}_{\mathrm{H}_2(\Phi_2)}(ef).$$

Hence, $\mathcal{L}_{\mathrm{H}_1}(\Phi_1) = \mathcal{L}_{\mathrm{H}_2}(\Phi_2)$. Now, the proof follows by Proposition 5. □

The line graph of a balanced $\mathbb{T}_4$-gain graph (Γ, φ) does not have necessarily to be balanced, i.e., it may happen that $\mathcal{L}(\Phi)$ is balanced while Φ is not, as shown in the following example.

Example 1. *Let* $\Phi = (C_3, \varphi)$ *be the* $\mathbb{T}_4$*-gain graph depicted on the left in Figure 1. Since* $\varphi(e_{12}) = 1$, $\varphi(e_{23}) = i$ *and* $\varphi(e_{31}) = i$, *then* $\varphi(e_{12})\varphi(e_{23})\varphi(e_{31}) \neq 1$, *thus the gain graph* Φ *is unbalanced. For the incident matrix of* Φ

$$\mathrm{H} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -i & 0 \\ 0 & 1 & -i \end{pmatrix}$$

a direct calculation shows that

$$A(\mathcal{L}_{\mathrm{H}}(\Phi)) = \mathrm{H}^*\mathrm{H} - 2I_3 = \begin{pmatrix} 0 & -i & -1 \\ i & 0 & -i \\ -1 & i & 0 \end{pmatrix}. \tag{7}$$

In addition,

$$\varphi^{\mathcal{L}}_{\mathrm{H}}(e_1e_2)\, \varphi^{\mathcal{L}}_{\mathrm{H}}(e_2e_3)\, \varphi^{\mathcal{L}}_{\mathrm{H}}(e_3e_1) = (-i)\cdot(-i)\cdot(-1) = 1$$

which shows that the graph $\mathcal{L}_{\mathrm{H}}(\Phi)$ *is balanced.*

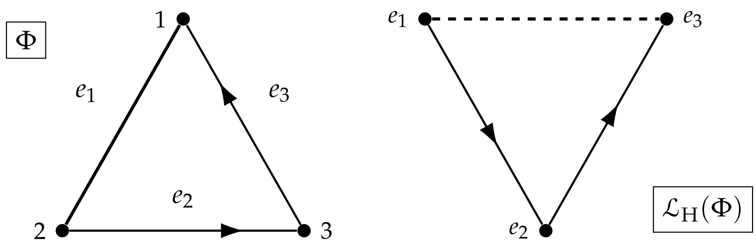

Figure 1. The unbalanced graph $\Phi = (C_3, \varphi)$ and its balanced line graph $\mathcal{L}_{\mathrm{H}}(\Phi)$.

The following theorem gives a characterization of $\mathbb{T}_4$-gain graphs whose associated line graphs are balanced.

Theorem 2. *Let* $\Phi = (\Gamma, \varphi)$ *be a* $\mathbb{T}_4$*-gain graph. Its associated line graphs are balanced if and only if each even directed cycle in* Γ *is neutral, and the gain of every odd directed cycle in* Γ *is* -1, *i.e., if and only if* Φ *is switching equivalent to* $(\Gamma, -1)$.

Proof of Theorem 2. By Proposition 1 all directed cycles of a balanced gain graph are neutral. Let H be any incidence matrix of Φ. Observe that $\mathcal{L}_{\mathrm{H}}(\Phi)$ has three types of cycles:

(i) cycles arising from cycles of Γ;
(ii) cycles arising from induced stars in Φ (forming cliques); and
(iii) cycles obtained by combining the cycles of Types (i) and (ii).

A directed cycle originating from a directed cycle $\vec{C}_k$ of Γ has gain $(-1)^k\varphi(\vec{C}_k)$. Since every induced star $(K_{1,r}, \varphi|_{K_{1,r}})$ is switching equivalent to $(K_{1,r}, -1)$, then the induced cliques on $\mathcal{L}_{\mathrm{H}}(\Phi)$ are all balanced. Finally, for the cycles of Type (iii), the theory of biased graphs (see [1]) tells that combining positive cycles leads to positive cycles. Hence, the cycles of Type (iii) are positive if and only if so are the cycles of Type (i). The statement now follows easily by Propositions 2 and 6. □

Corollary 1. *If a* $\mathbb{T}_4$*-gain graph* $\Phi = (\Gamma, \varphi)$ *and its associated line graphs are all balanced then* Γ *is bipartite.*

Proof of Corollary 1. By Theorem 2, if Φ and $\mathcal{L}_{\mathrm{H}}(\Phi)$ are both balanced, then Γ does not contain any odd cycle. Hence, the graph Γ is bipartite. □

4. Subdivision Graphs Associated to $\mathbb{T}_4$-Gain Graphs

For any graph Γ, the subdivision graph $\mathcal{S}(\Gamma)$ is obtained from Γ by replacing each of its edges by a path of length 2, or, equivalently, by inserting an additional vertex into each edge e of Γ. Adopting an abuse of notation (which has become classical in this context), we denote by e the additional vertex inserted on the homonymous edge. For the set $V(\mathcal{S}(\Gamma))$, we choose the ordering $\{v_1, \ldots, v_n, e_1, \ldots, e_m\}$.

Any incident matrix on Φ induces a gain structure on $\mathcal{S}(\Gamma)$, $\varphi^{\mathcal{S}}_{\mathrm{H}} : \vec{E}(\mathcal{S}(\Gamma)) \to \mathbb{T}_4$ defined in the following way:

$$\varphi^{\mathcal{S}}_{\mathrm{H}}(ve) = \overline{\varphi^{\mathcal{S}}_{\mathrm{H}}(ev)} = \eta_{ve}$$

for any $v \in V(\Gamma)$ and for any $e \in E(\Gamma)$.

According to the chosen vertex ordering the adjacency matrix of the gain graph $\mathcal{S}_{\mathrm{H}}(\Phi) = (\mathcal{S}(\Gamma), \varphi^{\mathcal{S}}_{\mathrm{H}})$ is

$$A(\mathcal{S}_{\mathrm{H}}(\Phi)) = \begin{pmatrix} O_n & \mathrm{H} \\ \mathrm{H}^* & O_m \end{pmatrix}. \tag{8}$$

By Proposition 3, or equivalently by Equation (2), for every edge $e_h \in E(\Gamma)$, we have four different choices for gains of the corresponding pair of 'new' edges in the gain subdivision graph. Figures 2–4 analyze several possibilities.

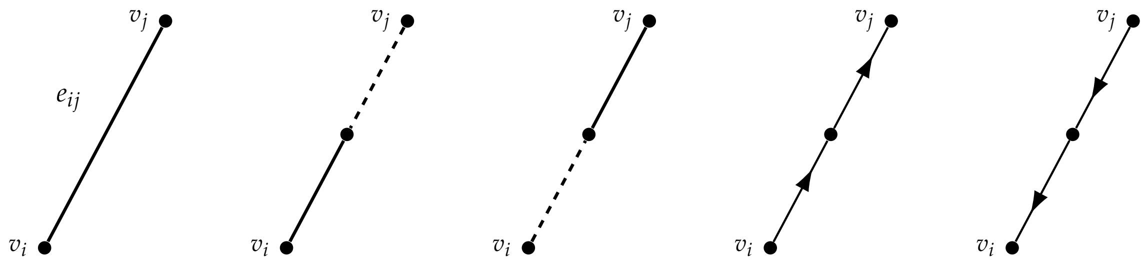

Figure 2. Four possibilities for gains on the pair of edges in the subdivision graph corresponding to $e_{ij} \in \vec{E}(\Phi)$ when $\varphi(e_{ij}) = 1$.

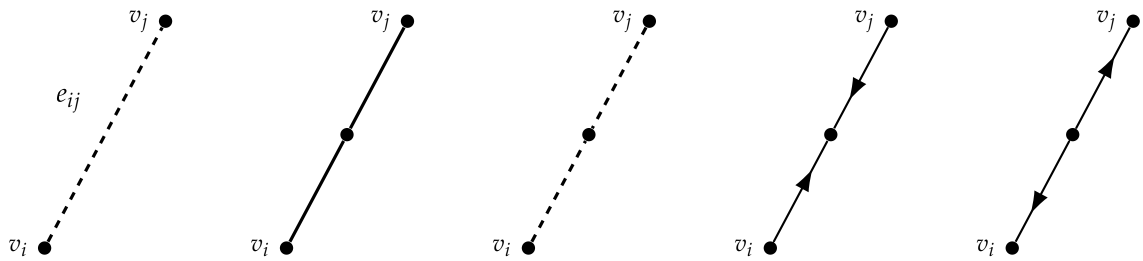

Figure 3. Four possibilities for gains on the pair of edges in the subdivision graph corresponding to $e_{ij} \in \vec{E}(\Phi)$ when $\varphi(e_{ij}) = -1$.

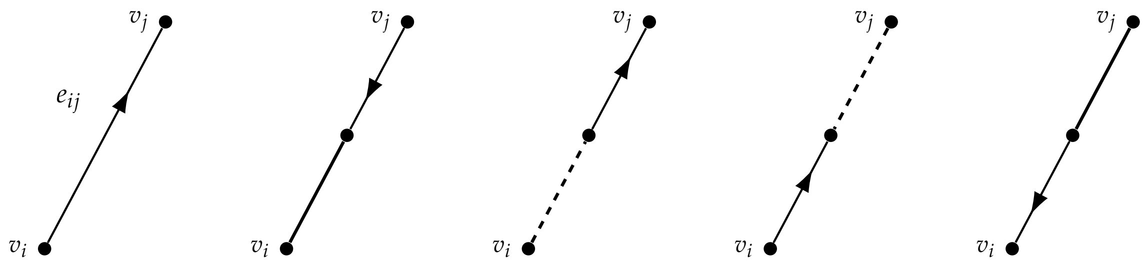

Figure 4. Four possibilities for gains on the pair of edges in the subdivision graph corresponding to $e_{ij} \in \vec{E}(\Phi)$ when $\varphi(e_{ij}) = i$.

It is reasonable to ask what the relation between two different subdivision graphs of the same $\mathbb{T}_4$-gain graph Φ is. The following proposition provides an answer to this question.

Proposition 7. *Let $\Phi = (\Gamma, \varphi)$ be a $\mathbb{T}_4$-gain graph, and let H and H' be two of its incidence matrices. Then, $\mathcal{S}_{\mathrm{H}}(\Phi)$ and $\mathcal{S}_{\mathrm{H}'}(\Phi)$ are switching equivalent; $A(\mathcal{S}_{\mathrm{H}}(\Phi))$ and $A(\mathcal{S}_{\mathrm{H}'}(\Phi))$ are similar and share the same adjacency spectrum.*

Proof of Proposition 7. Proposition 3 guarantees that $\mathrm{H}' = \mathrm{H}S$ for a suitably chosen diagonal matrix $S = \mathrm{diag}(s_1, \dots, s_m)$ with diagonal entries in $\mathbb{T}_4$. Taking into account Equation (8), it is not hard to verify that

$$A(\mathcal{S}_{\mathrm{H}'}(\Phi)) = (I_n \oplus S)^* A(\mathcal{S}_{\mathrm{H}}(\Phi))(I_n \oplus S),$$

where the symbol $\oplus$ denote the block diagonal sum of two matrices.

The switching function $\zeta_{\mathrm{H}}^{\mathrm{H}'}$ such that $\mathcal{S}_{\mathrm{H}'}(\Phi) = \mathcal{S}_{\mathrm{H}}(\Phi)^{\zeta_{\mathrm{H}}^{\mathrm{H}'}}$ is defined as follows:

$$\zeta_{\mathrm{H}}^{\mathrm{H}'}(v_h) = 1, \qquad h = 1, \dots, n, \qquad \text{and} \qquad \zeta_{\mathrm{H}}^{\mathrm{H}'}(e_k) = s_k, \quad k = 1, \dots, m. \quad \square$$

Proposition 7 implies that a subdivision graph of a $\mathbb{T}_4$-gain graph Φ is balanced if and only if any other subdivision graph of Φ is balanced.

Proposition 8. *Subdivision graphs of two switching equivalent $\mathbb{T}_4$-gain graphs $\Phi_1 = (\Gamma, \varphi_1)$ and $\Phi_2 = (\Gamma, \varphi_2)$ are switching equivalent.*

Proof of Proposition 8. We argue as in the proof of Proposition 6. Let $\zeta : V(\Gamma) \to \mathbb{T}_4$ be the map such that $\varphi_2 = \varphi_1^{\zeta}$. If $\mathrm{H}_2 = (\eta_{ve}^{(2)})$ is a fixed incident matrix of Φ_2, then $\mathrm{H}_1 := D(\zeta)\mathrm{H}_2$ is an incidence matrix of Φ_1, where $D(\zeta)$ is the state matrix.

Consider the map $Z : V(\mathcal{S}_{\mathrm{H}_1}(\Phi_1)) \to \mathbb{T}_4$ defined as follows:

$$Z(v_h) = \zeta(v_h), \qquad h = 1, \dots, n \qquad \text{and} \qquad Z(e_k) = 1, \qquad k = 1, \dots, m.$$

It turns out that $\mathcal{S}_{\mathrm{H}_2(\Phi_2)}(\Phi_2) = \mathcal{S}_{\mathrm{H}_1(\Phi_1)}(\Phi_1)^{Z}$. In fact,

$$\varphi^{\mathcal{S}}_{\mathrm{H}_2(\Phi_2)}(ve) = \eta_{ve}^{(2)} = Z(v)^{-1}\eta_{ve}^{(1)} = Z(v)^{-1}\eta_{ve}^{(1)} \cdot Z(e) = Z(v)^{-1}\varphi^{\mathcal{S}}_{\mathrm{H}_1(\Phi_1)}(ve)Z(e).$$

$\square$

We now investigate which conditions on Φ ensure the balancedness of its subdivision graphs. Figure 5 gives an example of an unbalanced $\mathbb{T}_4$-gain graph having balanced subdivision graphs.

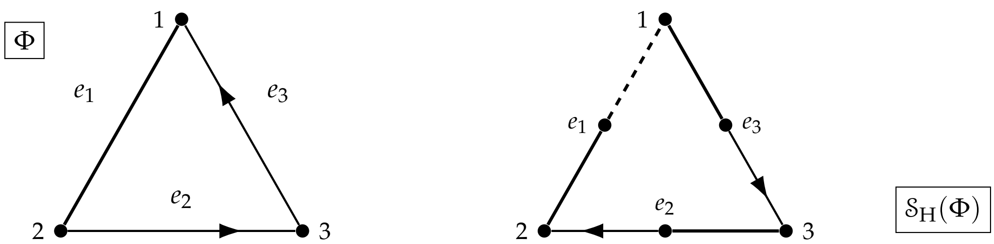

Figure 5. The graph $\Phi = (C_3, \varphi)$ and one of its subdivision graphs for an incidence matrix H of Φ.

Lemma 2. *Let $\Phi = (\Gamma, \varphi)$ be a $\mathbb{T}_4$-gain graph, and let $\vec{C} = e_{i_1 i_2} e_{i_2 i_3} \cdots e_{i_k i_1}$ be a directed cycle in Γ. Then, for any incident matrix H of Φ, we have*

$$\varphi^{\mathcal{S}}_{\mathrm{H}}(\mathcal{S}(\vec{C})) = (-1)^k \varphi(\vec{C}). \tag{9}$$

Proof of Lemma 2. The directed cycle $\mathcal{S}(\vec{C})$ is obtained by considering consecutively the following elements in $\vec{E}(\mathcal{S}(\Gamma))$:

$$v_{i_1}e_{i_1i_2}, \quad e_{i_1i_2}v_{i_2}, \quad \dots \quad ,v_{i_k}e_{i_ki_1}, \quad e_{i_ki_1}v_{i_1}.$$

Hence,

$$\begin{aligned}
\varphi_{\mathrm{H}}^{\mathcal{S}}(\mathcal{S}(\vec{C})) &= \varphi_{\mathrm{H}}^{\mathcal{S}}(v_{i_1}e_{i_1i_2}) \cdot \varphi_{\mathrm{H}}^{\mathcal{S}}(e_{i_1i_2}v_{i_2}) \cdot \ \cdots \ \cdot \varphi_{\mathrm{H}}^{\mathcal{S}}(v_{i_k}e_{i_ki_1})\varphi_{\mathrm{H}}^{\mathcal{S}}(e_{i_ki_1}v_{i_1}) \\
&= \varphi_{\mathrm{H}}^{\mathcal{S}}(v_{i_1}e_{i_1i_2}) \cdot \overline{\varphi_{\mathrm{H}}^{\mathcal{S}}(v_{i_2}e_{i_1i_2})} \cdot \ \cdots \ \cdot \varphi_{\mathrm{H}}^{\mathcal{S}}(v_{i_k}e_{i_ki_1})\overline{\varphi_{\mathrm{H}}^{\mathcal{S}}(v_{i_1}e_{i_ki_1})} \\
&= \eta_{v_{i_1}e_{i_1i_2}}\bar{\eta}_{v_{i_2}e_{i_1i_2}} \cdots \ \eta_{v_{i_k}e_{i_ki_1}}\bar{\eta}_{v_{i_1}e_{i_ki_1}} \\
&= (-\varphi(e_{i_1i_2}))\dots(-\varphi(e_{i_ki_1})) \\
&= (-1)^k\varphi(\vec{C}).
\end{aligned}$$

□

Theorem 3. *Let $\Phi = (\Gamma, \varphi)$ a $\mathbb{T}_4$-gain graph. Its subdivision graphs are balanced if and only if each even directed cycle of Γ is neutral, and the gain of every odd directed cycle in Γ is -1, i.e., if and only if Φ is switching equivalent to $(\Gamma, -1)$.*

Proof of Theorem 3. Let Φ_1 and Φ_2 be $\mathbb{T}_4$-gain graphs with the same underlying graph Γ. By Proposition 2, Φ_1 and Φ_2 are switching equivalent if and only if, for every directed cycle $\vec{C}$ in Γ, $\varphi_1(\vec{C}) = \varphi_2(\vec{C})$. Lemma 2 provides a necessary and sufficient condition to obtain all neutral directed cycle in the subdivision graphs coming from Φ: every even directed cycle of Γ should be neutral, and the gain of every odd directed cycle in Γ should be -1. Clearly $(\Gamma, -1)$ satisfies this condition. Since Proposition 8 holds, the proof is completed. □

Corollary 2. *If Γ contains a directed cycle having an imaginary gain, then the $\mathbb{T}_4$-gain graph $\Phi = (\Gamma, \varphi)$ and its subdivision graphs are all unbalanced.*

By Theorems 2 and 3, we may conclude that the structural conditions on Φ to have balanced associated line graphs, or balanced associated subdivision graphs are the same.

Our final result concerns the mutual interrelationships between the Laplacian polynomial of a $\mathbb{T}_4$-gain graph Φ and the adjacency characteristic polynomial of its line graphs and its subdivision graphs. Propositions 5 and 7 allow us to drop the incident matrix out of notations in the statements.

For a $\mathbb{T}_4$-gain graph Φ, let $\varphi(\Phi; x)$ and $\psi(\Phi; x)$ be the adjacency characteristic polynomial and the Laplacian characteristic polynomial of Φ, respectively.

Theorem 4. *Let Γ be a graph of order n and size m, and Φ a $\mathbb{T}_4$-gain graph having Γ as underlying graph. Then,*

1. $\varphi(\mathcal{L}(\Phi), x) = (x+2)^{m-n}\psi(\Phi, x+2);$
2. $\varphi(\mathcal{S}(\Phi), x) = x^{m-n}\psi(\Phi, x^2).$

Proof of Theorem 4. To prove (1), we use Equations (3) and (6), and the fact that $\mathrm{H}^*\mathrm{H}$ and $\mathrm{H}\mathrm{H}^*$ share the same non-zero eigenvalues.

The argument to prove (2) is essentially the same as the one used in the proof of [10] (Theorem 2.2). We use the Schur's formula for computing the determinant of a 2×2 block matrix, namely:

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D||A - BD^{-1}C|.$$

Now,

$$\begin{aligned} \varphi(\mathcal{S}(\Phi), x) &= \begin{vmatrix} xI_n & -\mathrm{H} \\ -\mathrm{H}^* & xI_m \end{vmatrix} = x^m |(xI_n) - \mathrm{H}(xI_m)^{-1}\mathrm{H}^*| \\ &= x^m |x^{-1}\left(x^2 I_n - \mathrm{HH}^*\right)| = x^{m-n}|x^2 I_n - L(\Phi)| \\ &= x^{m-n}\psi(\Phi, x^2). \quad \square \end{aligned}$$

Example 2. *Let Φ be the $\mathbb{T}_4$ gain graph depicted on the left of Figures 1 and 5. It is immediately seen that*

$$L(\Phi) = \begin{pmatrix} 2 & -1 & i \\ -1 & 2 & -i \\ -i & i & 2 \end{pmatrix}.$$

A direct computation shows that $\mathrm{Spec}(L(\Phi)) = \{1^{(2)}, 4\}$. *We use the incidence matrix* H *from Example 1 and (8) to calculate*

$$\mathrm{Spec}(A(\mathcal{L}(\Phi))) = \{(-1)^{(2)}, 2\} \qquad \text{and} \qquad \mathrm{Spec}(A(\mathcal{S}(\Phi))) = \{-2, (-1)^{(2)}, 1^{(2)}, 2\},$$

which turns out to be precisely as what expected by Theorem 4.

Author Contributions: All authors contributed equally in the preparation of this manuscript.

Funding: This research was funded by Kuwait University, Research Grant No. SM03/18.

Acknowledgments: The authors would like to thank the anonymous referees for their careful reading and their kind appreciation. One of them pointed out that our results possibly encapsulate fruitful applications to the spectral theory of distance matrices. This suggested line of research goes beyond the framework of our paper.

Conflicts of Interest: The authors declare no conflict of interest.

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