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Approximate Solutions of Time Fractional Diffusion Wave Models

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Abstract: In this paper, a wavelet based collocation method is formulated for an approximate solution of (1 + 1)- and (1 + 2)-dimensional time fractional diffusion wave equations. The main objective of this study is to combine the finite difference method with Haar wavelets. One and two dimensional Haar wavelets are used for the discretization of a spatial operator while time fractional derivative is approximated using second order finite difference and quadrature rule. The scheme has an excellent feature that converts a time fractional partial differential equation to a system of algebraic equations which can be solved easily. The suggested technique is applied to solve some test problems. The obtained results have been compared with existing results in the literature. Also, the accuracy of the scheme has been checked by computing L_2 and L_{∞} error norms. Computations validate that the proposed method produces good results, which are comparable with exact solutions and those presented before.

Keywords: fractional differential equations; two-dimensional wavelets; finite differences

1. Introduction

The theory of fractional calculus is an ancient topic that has many applications. However, practical work in this direction has been recently started (see References [1–3]). Most of the physical phenomena in chemistry, physics, engineering and other fields of science can be modeled using parameters of fractional calculus [4,5], means fractional derivative and integral operators. Amongst these are electrolyte polarization [6], viscoelastic systems [7], dielectric polarization [8] and so forth. Fractional models in different circumstances lead towards more accurate behaviour than those of integer order models.

The time fractional diffusion wave equation (TFDWE) is such an important model which has extensive uses. The TFDWE is actually a wave equation [9] with a fractional time derivative which describes universal acoustic, electromagnetic and mechanical responses [10,11] with an enhanced method. Over the past few decades, extensive attention has been paid to the closed form solution of



time fractional diffusion wave equations (TFDWEs) and is still an open area of research. The closed form solution of such problems is not an easy job and needs herculean efforts. Owing to the fact several authors proposed numerical methods for the solution of fractional models, Tadjeran et al. [12] used second order accurate approximation for fractional diffusion equations. Zhuang et al. [13] applied an implicit numerical method for the anomalous sub-diffusion equation. Yuste and Acedo [14] studied fractional diffusion equations via an explicit finite difference method. Chen et al. [15] proposed the Fourier method for fractional diffusion equations. Hosseini et al. [16] solved the fractional telegraph equation with the help of radial basis functions. Zhou and Xu [17] applied the Chebyshev wavelets collocation method for the solution of time fractional diffusion wave equations. Bhrawya [18] used the spectral Tau algorithm based on the Jacobi operational matrix for the numerical solution of time fractional diffusion-wave equations. Yaseen et al. [19] solved fractional diffusion wave equations with reaction terms using finite differences and a trigonometric B-splines technique. Khader [20] and his co-author applied the finite difference method coupled with the Hermite formula for solutions of fractional diffusion wave equations. Kanwal et al. [21] implemented two-dimensional Genocchi Polynomials combined with the Ritz-Galerkin Method for solutions of fractional diffusion wave and Klein-Gordon equations. Datsko et al. [22] studied time-fractional diffusion-wave equation with mass absorption in a sphere under harmonic impact.

Recently, numerical methods using wavelets have been given more emphasis because of their simple applicability. These methods also have some other interesting properties such as the ability to detect singularities and express the function in different resolution levels, which improves the accuracy. Amongst different classes of wavelets, Haar wavelets deserve special consideration. Haar wavelets consist of piece wise constant functions. The integration of these wavelets in different times is one of the best features. Also, Haar wavelets have orthogonality and normalization properties with compact support. For more discussion on Haar wavelets one can see References [23,24].

In the present study, we propose a hybrid numerical scheme, based on Haar wavelets and finite differences, to solve (1 + 1)- and (1 + 2)-dimensional TFDWEs. The stability of the proposed method is discussed with the matrix method which is an essential part of the manuscript. The models which will be under consideration are characterized in the following types:

(1 + 1)-Dimensional Equation:

$${}^{c}D_{t}^{\delta}w(x,t) = -w_{t}(x,t) + w_{xx}(x,t) + \mathcal{A}(x,t), \quad x \in \Omega, \quad t \in [0,T], \quad 1 < \delta \le 2,$$
(1)

$$\begin{cases} w(x,0) = f(x), & w_t(x,0) = g(x) & x \in \Omega = \Omega \cup \partial \Omega, \\ w(x,t) = \alpha(t), & x \in \partial \Omega & t \in [0,T]. \end{cases}$$
(2)

(1 + 2)-Dimensional Equation:

$$^{c}D_{t}^{\delta}w(x,y,t) = \Delta w(x,y,t) + \mathcal{B}(x,y,t), \quad (x,y) \in \Phi, \quad t \in [0,T], \quad 1 < \delta \le 2,$$
 (3)

$$\begin{cases} w(x,y,0) = \chi(x,y), & w_t(x,y,0) = \kappa(x,y), & (x,y) \in \widetilde{\Phi} = \Phi \cup \partial \Phi, \\ w(x,y,t) = \chi_1(x,y,t), & (x,y) \in \partial \Phi, & t \in [0,T]. \end{cases}$$
(4)

In Equations (1)–(4), Δ is two-dimensional Laplacian; A, B, f, g, α , χ , κ , χ_1 are known functions and w is unknown function. Equations (2) and (4) are the corresponding initial and boundary conditions. The symbols, Ω and $\partial\Omega$, Φ and $\partial\Phi$ represent the domain and boundary of the domain respectively for

(1 + 1)- and (1 + 2)-dimensional problems. Also ${}^{c}D_{t}^{\delta}w$ denotes the time fractional derivative of w with respect to t in the Caputo sense which is given by

$${}^{c}D_{t}^{\delta}w = \begin{cases} \frac{1}{\Gamma(2-\delta)} \int_{0}^{t} \frac{w_{\zeta\zeta}(x,\zeta)}{(t-\zeta)^{\delta-1}} d\zeta, & 1 < \delta < 2, \\ \frac{\partial^{2}w(x,t)}{\partial t^{2}}, & \delta = 2. \end{cases}$$
(5)

2. Ground Work

In this section, some basic definitions of fractional calculus and Haar wavelets are presented, which will be required for the demonstration of our results. For a basic definition of Haar wavelets and its integrals we refer to Reference [23]. Let us consider $x \in [a, b]$ where a and b are the limits of the interval. Next, the interval is subdivided into 2*M* intervals where $M = 2^J$ and J denote the maximal level of resolution. Further, the two parameters $j = 0, \dots, J$ and $k = 0, \dots, 2^j - 1$ are introduced. These parameters show the integer decomposition of wavelet number i = m + k + 1, where $m = 2^j$. The first and ith wavelets are defined as follows:

$$\mathcal{H}_{1}(x) = \begin{cases}
1, & x \in [a, b] \\
0, & \text{otherwise.} \\
\end{cases}$$

$$\mathcal{H}_{i}(x) = \begin{cases}
1, & x \in [\xi_{1}(i), \xi_{2}(i)) \\
-1, & x \in [\xi_{2}(i), \xi_{3}(i)) \\
0, & \text{otherwise,} \\
\end{cases}$$
(6)

(7)

where

$$\xi_1(i) = a + 2k\nu\delta x, \ \xi_2(i) = a + (2k+1)\nu\delta x, \ \xi_3(i) = a + 2(k+1)\nu\delta x, \ \nu = \frac{M}{m}, \ \delta x = \frac{b-a}{2M}$$

To solve nth order time fractional PDEs the following repeated integrals are needed:

$$\mathcal{P}_{i,\beta}(x) = \int_a^x \int_a^x \cdots \int_a^x \mathcal{H}_i(z) dz^\beta = \frac{1}{(\beta - 1)!} \int_a^x (x - z)^{\beta - 1} \mathcal{H}_i(z) dz,\tag{8}$$

where

$$\beta = 1, 2, \dots n, \quad i = 1, 2, \dots 2M.$$

Keeping in view Equations (6) and (7) the close form expressions of these integrals are given by

$$\mathcal{P}_{1,\beta}(x) = \frac{(x-a)^{\beta}}{\beta!}.$$

$$\mathcal{P}_{i,\beta}(x) = \begin{cases} 0, & x < \xi_1(x) \\ \frac{1}{\beta!} [x - \xi_1(i))^{\beta} & x \in [\xi_1(i), \xi_2(i)) \\ \frac{1}{\beta!} [(x - \xi_1(i))^{\beta} - 2((x - \xi_2(i))^{\beta}) & x \in [\xi_2(i), \xi_3(i)) \\ \frac{1}{\beta!} [(x - \xi_1(i))^{\beta} - 2((x - \xi_2(i))^{\beta} + (x - \xi_3(i))^{\beta}) & x \ge \xi_3(i). \end{cases}$$
(9)

3. Description of the Method

This section is devoted to discussing the scheme for Equations (1) and (3) separately. In both cases, the fractional order time derivative has been approximated by the quadrature formula [16]

$${}^{c}\mathbb{D}_{t}^{\delta}w(x,t^{j+1}) = \frac{1}{\Gamma(2-\delta)} \int_{0}^{t^{j+1}} w^{(2)}(x,\zeta) \left(t^{j+1}-\zeta\right)^{1-\delta} d\zeta$$

$$= \frac{1}{\Gamma(2-\delta)} \sum_{k=0}^{j} \int_{t^{j}}^{t^{j+1}} \left[\frac{w^{k+1}-2w^{k}+w^{k-1}}{\tau^{2}}\right] \left(t^{j+1}-\zeta\right)^{1-\delta} d\zeta$$

$$= \frac{1}{\Gamma(2-\delta)} \sum_{k=0}^{j} \left[\frac{w^{k+1}-2w^{k}+w^{k-1}}{\tau^{2}}\right] \int_{t^{j}}^{t^{j+1}} \left[(j+1)\tau-\zeta\right]^{1-\delta} d\zeta$$

$$= \frac{1}{\Gamma(2-\delta)} \sum_{k=0}^{j} \left[\frac{w^{k+1}-2w^{k}+w^{k-1}}{\tau^{2}}\right] \frac{(j-k+1)^{2-\delta}-(j-k)^{2-\delta}}{(2-\delta)(\tau^{\delta-2})}$$

$$= \frac{\tau^{-\delta}}{\Gamma(3-\delta)} \sum_{k=0}^{j} \left[w^{j-k+1}-2w^{j-k}+w^{j-k-1}\right] \left[(k+1)^{2-\delta}-(k)^{2-\delta}\right]$$

$$= A_{\delta} \left[w^{j+1}-2w^{j}+w^{j-1}\right] + A_{\delta} \sum_{k=1}^{j} \left[w^{j-k+1}-2w^{j-k}+w^{j-k-1}\right] B(k),$$

where $A_{\delta} = \frac{\tau^{-\delta}}{\Gamma(3-\delta)}$, τ is time step size and $B(k) = (k+1)^{2-\delta} - (k)^{2-\delta}$. *Case i:*

Using Equation (11) and θ -weighted scheme ($0 \le \theta \le 1$) in Equation (1), we obtain

$$A_{\delta} \left[w^{j+1} - 2w^{j} + w^{j-1} \right] + A_{\delta} \sum_{k=1}^{j} \left[w^{j-k+1} - 2w^{j-k} + w^{j-k-1} \right] B(k) + \frac{1}{\tau} \left\{ w^{j+1} - w^{j} \right\}$$

$$= \theta w_{xx}^{j+1} + (1-\theta) w_{xx}^{j} + \mathcal{A}(x, t^{j+1}).$$
(12)

After simplification, the above equation transforms to

$$(\tau A_{\delta} + 1)w^{j+1} - \tau \theta w_{xx}^{j+1} = 2\tau A_{\delta}w^{j} - \tau A_{\delta}w^{j-1} - \tau A_{\delta}\sum_{k=1}^{j} \left[w^{j-k+1} - 2w^{j-k} + w^{j-k-1}\right]B(k) + w^{j} + \tau(1-\theta)w_{xx}^{j} + \tau \mathcal{A}(x, t^{j+1}).$$
(13)

In our analysis we take $\theta = 1/2$. Now approximating the highest order derivative by a truncated Haar wavelets series as:

$$w_{xx}^{j+1}(x) = \sum_{i=1}^{2M} a_i^{j+1} \mathcal{H}_i(x).$$
(14)

Integrating Equation (14) from 0 to x

$$w_x^{j+1}(x) = \sum_{i=1}^{2M} a_i^{j+1} \mathcal{P}_{i,1}(x) + w_x^{j+1}(0).$$
(15)

Integrating Equation (15) from 0 to 1, we get

$$w_x^{j+1}(0) = w^{j+1}(1) - w^{j+1}(0) - \sum_{i=1}^{2M} a_i^{j+1} \mathcal{P}_{i,2}(1).$$
(16)

Substituting Equation (16) in Equation (15), the resultant equation reduces to

$$w_x^{j+1}(x) = \sum_{i=1}^{2M} a_i^{j+1} \left[\mathcal{P}_{i,1}(x) - \mathcal{P}_{i,2}(1) \right] + w^{j+1}(1) - w^{j+1}(0).$$
(17)

Integration of Equation (17) from 0 to x yields

$$w^{j+1}(x) = \sum_{i=1}^{2M} a_i^{j+1} \left[\mathcal{P}_{i,2}(x) - x \mathcal{P}_{i,2}(1) \right] + x \left[w^{j+1}(1) - w^{j+1}(0) \right] + w^{j+1}(0).$$
(18)

Substituting values from Equations (14), (17) and (18) in Equation (13) and using collocation points $x_m = \frac{m-0.5}{2M}$, m = 1, 2, ... 2M, leads to the following system of algebraic equation

$$\sum_{i=1}^{2M} a_i^{j+1} \big[\left(\tau A_{\delta} + 1 \right) \{ \mathcal{P}_{i,2}(x) - x \mathcal{P}_{i,2}(1) \} - \tau \theta \mathcal{H}_i(x) \big]_{x=x_m} = \mathcal{R}(m),$$
(19)

where

$$\begin{aligned} \mathcal{R}(m) &= 2\tau A_{\delta} w^{j} - \tau A_{\delta} w^{j-1} - \tau A_{\delta} \sum_{k=1}^{j} \left[w^{j-k+1} - 2w^{j-k} + w^{j-k-1} \right] B(k) + w^{j} \\ &+ \tau (1-\theta) w^{j}_{xx} + \tau \mathcal{A}(x_{m}, t^{j+1}) - (\tau A_{\delta} + 1) \left\{ x_{m} \left(w^{j+1}(1) - w^{j+1}(0) \right) + w^{j+1}(0) \right\}. \end{aligned}$$

Equation (19) contains 2*M* equations. The unknown wavelet coefficients can be computed from this system. After determination of these unknown constants, the required solution at each time can be calculated from Equation (18).

Case ii:

Following a similar approach, as discussed earlier, Equation (3) gives

$$A_{\delta}w^{j+1} - \theta \left[w_{xx}^{j+1} + w_{yy}^{j+1} \right] = (1 - \theta) \left[w_{xx}^{j} + w_{yy}^{j} \right] + \mathcal{B}(x, y, t^{j+1}) + 2A_{\delta}w^{j} - A_{\delta}w^{j-1} - A_{\delta} \sum_{k=1}^{j} \left[w^{j-k+1} - 2w^{j-k} + w^{j-k-1} \right] B(k).$$
(20)

Now we approximate $w_{xxyy}^{j+1}(x, y)$ with a two dimensional truncated Haar wavelets series as:

$$w_{xxyy}^{j+1}(x,y) = \sum_{i=1}^{2M} \sum_{l=1}^{2M} a_{i,l}^{j+1} \mathcal{H}_i(x) \mathcal{H}_i(y),$$
(21)

where $a_{i,l}^{j+1}$ are unknowns to be determined. Integration of Equation (21) w.r.t. to *y*, between 0 and *y*, gives

$$w_{xxy}^{j+1}(x,y) = \sum_{i=1}^{2M} \sum_{l=1}^{2M} a_{i,l}^{j+1} \mathcal{H}_i(x) \mathcal{P}_{l,1}(y) + w_{xxy}^{j+1}(x,0).$$
(22)

Integrating Equation (22) w.r.t *y* from 0 to 1, the unknown term $w_{xxy}^{j+1}(x, 0)$ is given by

$$w_{xxy}^{j+1}(x,0) = w_{xx}^{j+1}(x,1) - w_{xx}^{j+1}(x,0) - \sum_{i=1}^{2M} \sum_{l=1}^{2M} a_{i,l}^{j+1} \mathcal{H}_i(x) \mathcal{P}_{l,2}(1).$$
(23)

Substituting Equation (23) in Equation (22), the obtained result is

$$w_{xxy}^{j+1}(x,y) = \sum_{i=1}^{2M} \sum_{l=1}^{2M} a_{i,l}^{j+1} \mathcal{H}_i(x) \left[\mathcal{P}_{l,1}(y) - \mathcal{P}_{l,2}(1) \right] + w_{xx}^{j+1}(x,1) - w_{xx}^{j+1}(x,0).$$
(24)

Integrating Equation (24) from 0 to *y*, we get

$$w_{xx}^{j+1}(x,y) = \sum_{i=1}^{2M} \sum_{l=1}^{2M} a_{i,l}^{j+1} \mathcal{H}_i(x) \left[\mathcal{P}_{l,2}(y) - y \mathcal{P}_{l,2}(1) \right] + y w_{xx}^{j+1}(x,1) + (1-y) w_{xx}^{j+1}(x,0).$$
(25)

Repeating the same procedure one can easily derive the subsequent expressions

$$w_{yy}^{l+1}(x,y) = \sum_{i=1}^{2M} \sum_{l=1}^{2M} a_{i,l}^{j+1} \left[\mathcal{P}_{i,2}(x) - x \mathcal{P}_{i,2}(1) \right] \mathcal{H}_l(y) + x w_{yy}^{l+1}(1,y) + (1-x) w_{yy}^{l+1}(0,y).$$
(26)

$$w_x^{j+1}(x,y) = \sum_{i=1}^{2M} \sum_{l=1}^{2M} a_{i,l}^{j+1} \left[\mathcal{P}_{i,1}(x) - \mathcal{P}_{i,2}(1) \right] \left[\mathcal{P}_{l,2}(y) - y \mathcal{P}_{l,2}(1) \right] + y w_x^{j+1}(x,1) + (1-y) w_x^{j+1}(x,0) + w^{j+1}(1,y) - w^{j+1}(0,y) - y w^{j+1}(1,1) + y w^{j+1}(0,1) + (y-1) w^{j+1}(1,0) + (1-y) w^{j+1}(0,0).$$
(27)

$$w_{y}^{l+1}(x,y) = \sum_{i=1}^{2M} \sum_{l=1}^{2M} a_{i,l}^{j+1} \left[\mathcal{P}_{i,2}(x) - x \mathcal{P}_{i,2}(1) \right] \left[\mathcal{P}_{l,1}(y) - \mathcal{P}_{l,2}(1) \right] + x w_{y}^{l+1}(1,y) + (1-x) w_{y}^{l+1}(0,y) + w^{l+1}(x,1) - w^{l+1}(x,0) - x w^{l}(1,1) + x w^{l+1}(1,0) + (x-1) w^{l+1}(0,1) + (1-x) w^{l+1}(0,0).$$
(28)

$$w^{j+1}(x,y) = \sum_{i=1}^{2M} \sum_{l=1}^{2M} a_{i,l}^{j+1} \left[\mathcal{P}_{i,2}(x) - x \mathcal{P}_{i,2}(1) \right] \left[\mathcal{P}_{l,2}(y) - y \mathcal{P}_{l,2}(1) \right] + y w^{j+1}(x,1) - y w^{j+1}(0,1) + (1-y) \left[w^{j+1}(x,0) - w^{j+1}(0,0) \right] + x w^{j+1}(1,y) - x w^{j+1}(0,y) - xy \left[w^{j+1}(1,1) - w^{j+1}(0,1) \right] + x (y-1) w^{j+1}(1,0) + x (1-y) w^{j+1}(0,0) + w^{j+1}(0,y).$$
(29)

Substitution of Equations (25), (26) and (29) in Equation (20) and using the collocation points, $x_m = \frac{m-0.5}{2M}$, $y_n = \frac{n-0.5}{2M}$, m, n = 1, 2, ..., 2M, produces the following system of equations

$$\sum_{i=1}^{2M} \sum_{l=1}^{2M} a_{i,l}^{j+1} \left[A_{\delta} \mathcal{D}(i,l,m,n) - \theta \mathcal{E}(i,l,m,n) - \theta \mathcal{F}(i,l,m,n) \right] = \mathcal{L}(m,n) + \mathcal{M}(m,n),$$
(30)

where

$$\begin{split} \mathcal{D}(i,l,m,n) &= \left[\mathcal{P}_{i,2}(x_m) - x_m \mathcal{P}_{i,2}(1)\right] \left[\mathcal{P}_{l,2}(y_n) - y_n \mathcal{P}_{l,2}(1)\right],\\ \mathcal{E}(i,l,m,n) &= \mathcal{H}_i(x_m) \left[\mathcal{P}_{l,2}(y_n) - y_n \mathcal{P}_{l,2}(1)\right],\\ \mathcal{F}(i,l,m,n) &= \left[\mathcal{P}_{i,2}(x_m) - x_m \mathcal{P}_{i,2}(1)\right] \mathcal{H}_l(y_n),\\ \mathcal{L}(m,n) &= (1-\theta) \left[w_{xx}^j + w_{yy}^j\right], + \mathcal{B}(x_m, y_n, t^{j+1}) + 2A_\delta w^j - A_\delta w^{j-1} \\ &- A_\delta \sum_{k=1}^j \left[w^{j-k+1} - 2w^{j-k} + w^{j-k-1}\right] \mathcal{B}(k),\\ \mathcal{M}(m,n) &= -A_\delta \left[y_n w_x^{j+1}(x_m, 1) - y_n w^{j+1}(0, 1) + (1-y_n) \left\{w^{j+1}(x_m, 0) - w^{j+1}(0, 0)\right\} \\ &+ x_m w^{j+1}(1, y_n) - x_m w^{j+1}(0, y_n) - x_m y_n \left\{w^{j+1}(1, 1) - w^{j+1}(0, 1)\right\} \\ &+ x_m \left(y_n - 1\right) w^{j+1}(1, 0) + x_m (1-y_n) w^{j+1}(0, 0) + w^{j+1}(0, y_n)\right] + \theta \left[y_n w_{xx}^{j+1}(x_m, 1) \\ &+ (1-y_n) w_{xx}^{j+1}(x_m, 0) + x_m w_{yy}^{j+1}(1, y_n) + (1-x_m) w_{yy}^{j+1}(0, y_n)\right]. \end{split}$$

Equation (30) represents $2M \times 2M$ equations in so many unknowns which can be solved easily. After calculation of these unknowns, an approximate solution can be obtained from Equation (29).

4. Stability Analysis

Here we present the stability analysis of the proposed scheme for (1 + 2)-dimensional problems; a similar result can be proved for (1 + 1)-dimensional problems. In matrix form Equations (25), (26) and (29) can be written as

$$w_{xx}^{j+1} = \mathcal{U}\alpha^{j+1} + \tilde{\mathcal{U}}^{j+1},\tag{31}$$

$$w_{yy}^{j+1} = \mathcal{V}\alpha^{j+1} + \tilde{\mathcal{V}}^{j+1},\tag{32}$$

$$w^{j+1} = \mathcal{Z}\alpha^{j+1} + \tilde{\mathcal{Z}}^{j+1},\tag{33}$$

where $\alpha^{j+1} = \alpha^{j+1}(i, l), \mathcal{U}, \mathcal{V}, \mathcal{Z}$ and $\tilde{\mathcal{U}}^{j+1}, \tilde{\mathcal{V}}^{j+1}, \tilde{\mathcal{Z}}^{j+1}$ are interpolation matrices of $w_{xx}^{j+1}, w_{yy}^{j+1}, w^{j+1}$ at collocation points and boundary terms, respectively. Now using Equations (31), (32) and (33) in Equation (20), we get

$$\left[A_{\delta}\mathcal{Z} - \theta(\mathcal{U} + \mathcal{V})\right]\alpha^{j+1} = \left[2A_{\delta}\mathcal{Z} + (1 - \theta)(\mathcal{U} + \mathcal{V})\right]\alpha^{j} + \mathcal{G}^{j+1},\tag{34}$$

where $\mathcal{G}^{j+1} = -A_{\delta}\tilde{\mathcal{Z}}^{j+1} + \theta(\tilde{\mathcal{U}}^{j+1} + \tilde{\mathcal{V}}^{j+1}) + 2A_{\delta}\tilde{\mathcal{Z}}^{j} + (1-\theta)(\tilde{\mathcal{U}}^{j} + \tilde{\mathcal{V}}^{j}) + \mathbf{B}^{j+1} - A_{\delta}w^{j-1} - A_{\delta}\sum_{k=1}^{j} \left[w^{j-k+1} - 2w^{j-k} + w^{j-k-1}\right]B(k).$ Now From Equation (34) one can write

$$\alpha^{j+1} = \mathcal{C}^{-1} \mathcal{T} \alpha^j + \mathcal{C}^{-1} \mathcal{G}^{j+1}, \tag{35}$$

where $C = [A_{\delta}Z - \theta(U + V)]$, $T = 2A_{\delta}Z + (1 - \theta)[U + V)$. Putting Equation (35) in Equation (33) we get

$$w^{j+1} = \mathcal{Z}\mathcal{C}^{-1}\mathcal{T}\alpha^j + \mathcal{Z}\mathcal{C}^{-1}\mathcal{G}^{j+1} + \tilde{\mathcal{Z}}^{j+1}.$$
(36)

Using Equation (33) in Equation (36) we have

$$w^{j+1} = \mathcal{Z}\mathcal{C}^{-1}\mathcal{T}\mathcal{Z}^{-1}w^j - \mathcal{Z}\mathcal{C}^{-1}\mathcal{T}\mathcal{Z}^{-1}\tilde{\mathcal{Z}}^j + \mathcal{Z}\mathcal{C}^{-1}\mathcal{G}^{j+1} + \tilde{\mathcal{Z}}^{j+1}.$$
(37)

The above equation shows a recurrence relation of a full discretization scheme which allow us refinement in time. If \tilde{w}^{j+1} is numerical solution then

$$\tilde{w}^{j+1} = \mathcal{Z}\mathcal{C}^{-1}\mathcal{T}\mathcal{Z}^{-1}\tilde{w^{j}} - \mathcal{Z}\mathcal{C}^{-1}\mathcal{T}\mathcal{C}^{-1}\tilde{\mathcal{Z}}^{j} + \mathcal{Z}\mathcal{C}^{-1}\mathcal{G}^{j+1} + \tilde{\mathcal{Z}}^{j+1}.$$
(38)

Let $e^{j+1} = w^{j+1} - \tilde{w}^{j+1}$ be the error at (j+1)th time level. Subtracting Equation (37) from Equation (38) then

$$e^{j+1} = \Lambda e^j$$
,

where $\Lambda = \mathcal{ZC}^{-1}\mathcal{TZ}^{-1}$ is the amplification matrix. According to Lax-Richtmyer criterion, the scheme will be stable if $\|\Lambda\| \leq 1$. It has been verified computationally that $\|\Lambda\| \leq 1$. For J = 1 the spectral radius is 0.01025 which lies in the stability domain.

5. Convergence Analysis

The convergence analysis of scheme (18) and (29) is similar to the following theorems, therefore the proofs are omitted.

Lemma 1 (see [24]). If $w(x) \in L^2(R)$ with $|w'(x)| \le \rho$, for all $x \in (0,1)$, $\rho > 0$ and $w(x) = \sum_{i=0}^{\infty} a_i \mathcal{H}_i(x)$ then $|a_i| \le \frac{\rho}{2^{i+1}}$.

Lemma 2 (see [25]). *If* f(x, y) *satisfies a Lipschitz condition on* $[0, 1] \times [0, 1]$ *, that is, there exists a positive* L *such that for all* $(x_1, y), (x_2, y) \in [0, 1] \times [0, 1]$ *we have* $| f(x_1, y) - f(x_2, y) | \leq L | x_1 - x_2 |$ *then*

$$a_{i,l}^2 \le \frac{L^2}{2^{4j+4}m^2}$$

Theorem 1. If w(x) and $w_{2M}(x)$ are the exact and approximate solution of Equation (1), then the error norm $|| E_I ||$ at J^{th} resolution level is

$$|| E_J || \le \frac{4\rho}{3} \left(\frac{1}{2^{J+1}}\right)^2.$$
 (39)

Proof. See [26]. \Box

Theorem 2. Assume w(x, y) and $w_{2M}(x, y)$ be the exact and approximate solution of Equation (3), then

$$|| E_J || \le \frac{L}{4\sqrt{255}} \frac{1}{2^{4J}}.$$
(40)

Proof. See [27]. □

6. Illustrative Test Problems

In this part, we chose some test problem to confirm the reliability and efficiency of the present scheme. For validation of our results L_{∞} and L_2 error norm are figured out which are defined as follows:

$$L_{2} = \sqrt{\sum_{i=1}^{2M} \left(w^{ext} - w^{app} \right)^{2}}, \quad L_{\infty} = \max_{1 \le i \le 2M} \left| w^{ext} - w^{app} \right|, \tag{41}$$

where w^{app} and w^{ext} are respectively approximate and exact solutions.

Problem 5.1

Let us take the following (1 + 1)-dimensional TFDWE with damping

$${}^{c}D_{t}^{\delta}w(x,t) = -w_{t}(x,t) + w_{xx}(x,t) + \mathcal{A}(x,t), \quad x \in [0,1], \quad t \in [0,1], \quad 1 < \delta \le 2,$$
(42)

with $A(x,t) = \frac{2x(1-x)t^{2-\delta}}{\Gamma(3-\delta)} + 2tx(1-x) + 2t^2$. Initial and boundary conditions are derived from the exact solution $w(x,t) = t^2x(1-x)$. This problem has been solved for parameters J = 4, t = 0.01, 0.1, 1, $\delta = 1.1$, 1.3, 1.5, 1.7, 1.9. The obtained error norms are shown in Table 1. From table it is obvious that results of the present scheme match well with exact solution. Also in Table 2 it has been observed that accuracy increases with increasing resolution level which shows the convergence in the spatial direction. In the same table, the results have been matched with existing results in the literature which clarify that computed solutions are in good agreement with the work of Chen et al. [28]. Table 3 shows convergence in time for fixed dx = 1/32. The convergence rate of the proposed scheme has been addressed in Table 4. the graphical solution and error plot are given in Figure 1. From this Figure it is clear that approximate solutions are matchable with exact.

Table 1. Error norms of problem 5.1 for at J = 4.

δ	$t = 0.01, \ \tau = 0.0001$		$t = 0.1, \ \tau = 0.001$		$t = 1, \ \tau = 0.01$	
-	L_{∞}	L_2	L_{∞}	L_2	L_∞	L_2
1.1	7.0694×10^{-8}	$2.9496 imes 10^{-7}$	$1.0799 imes 10^{-5}$	4.5921×10^{-5}	2.1556×10^{-3}	8.9537×10^{-3}
1.3	$3.1776 imes10^{-8}$	$1.3294 imes10^{-7}$	$7.6592 imes 10^{-6}$	$3.2979 imes 10^{-5}$	$2.1082 imes10^{-3}$	$8.7615 imes10^{-3}$
1.5	$1.1646 imes10^{-8}$	$4.8890 imes10^{-8}$	$4.8457 imes10^{-6}$	$2.1318 imes10^{-5}$	$2.0653 imes 10^{-3}$	$8.5871 imes 10^{-3}$
1.7	$5.2296 imes 10^{-9}$	$2.1899 imes 10^{-8}$	$3.3635 imes 10^{-6}$	$1.4944 imes10^{-5}$	$2.1431 imes 10^{-3}$	$8.8989 imes 10^{-3}$
1.9	$2.2087 imes 10^{-9}$	$9.2078 imes 10^{-9}$	$2.1386 imes 10^{-6}$	$9.3949 imes 10^{-5}$	2.4382×10^{-3}	1.0094×10^{-2}

Table 2. Comparison of maximum error of problem 5.1 with previous work at t = 1 and $\delta = 1.7$.

[28]			Present Method		
h	τ	Error	dx	τ	Error
0.05	0.05	$4.4333 imes 10^{-3}$	1/4	0.05	8.2306×10^{-4}
0.025	0.0125	$7.7368 imes10^{-4}$	1/8		$5.4184 imes10^{-4}$
0.0125	0.00625	$3.1827 imes10^{-4}$	1/16		4.9195×10^{-4}

Table 3. Error norms of problem 5.1 for different values of τ and δ .

	$\delta = 1.5$		$\delta=1.7$		
τ	L_{∞}	L_2	L_{∞}	L ₂	
1/4	5.4216×10^{-2}	2.2515×10^{-1}	5.4689×10^{-2}	$2.2708 imes 10^{-1}$	
1/8	$2.7891 imes 10^{-2}$	$1.1571 imes 10^{-1}$	$2.8861 imes 10^{-2}$	$1.1961 imes 10^{-1}$	
1/16	$1.3645 imes 10^{-2}$	$5.6647 imes 10^{-2}$	$1.4343 imes 10^{-2}$	$5.9443 imes 10^{-2}$	
1/32	$6.6683 imes 10^{-3}$	$2.7699 imes 10^{-2}$	7.0022×10^{-3}	$2.9034 imes 10^{-2}$	
1/64	3.2674×10^{-3}	$1.3580 imes 10^{-2}$	3.4061×10^{-3}	1.4135×10^{-2}	

J	τ	Error	Rate
1	1/10	2.6194×10^{-3}	-
2	1/20	$1.3086 imes 10^{-3}$	1.0012
3	1/40	$5.5920 imes10^{-4}$	1.2265
4	1/80	$2.2921 imes10^{-4}$	1.2866
5	1/160	$9.3261 imes 10^{-5}$	1.2973
6	1/320	$3.7377 imes 10^{-5}$	1.3191

Table 4. Convergence rate of maximum error of problem 5.1 at t = 1 and $\delta = 1.7$.



Figure 1. Graphical behaviour of problem 5.1 when t = 1, $\delta = 1.5$.

Problem 5.2:

Consider the following TFDWE with damping

$${}^{c}D_{t}^{\delta}w(x,t) = -w_{t}(x,t) + w_{xx}(x,t) + \mathcal{A}(x,t), \quad x \in [0,1], \quad t \in [0,1], \quad 1 < \delta \le 2,$$
(43)

coupled with initial and boundary conditions

$$\begin{cases} w(x,0) = 0, & w_t(x,0) = 0 \quad x \in (0,1) \\ w(0,t) = t^3, & w(1,t) = et^3, \quad t \in [0,1]. \end{cases}$$
(44)

The exact solution and source term are given by $w(x,t) = e^x t^3$ and $\mathcal{A}(x,t) = \frac{6t^{3-\delta}e^x}{\Gamma(4-\delta)} + 3t^2e^x - t^3e^x$. In Table 5 the obtained error norms are shown for parameters $t = 0.01, 0.1, \delta = 1.1, 1.3, 1.5, 1.7, 1.9, J = 4$. Table 5 shows that exact and approximate solutions agree with each other. The solution profile and absolute error are displayed Figure 2. From the Figure, the coincidence of both solutions are visible.

Table 5. Error norms of problem 5.2 at J = 4.

δ	t = 0.01, -	au = 0.0001	$t = 0.1, \ \tau = 0.001$		
	L_∞	L_2	L_∞	L_2	
1.1	1.7079×10^{-7}	$6.8446 imes10^{-7}$	1.2504×10^{-4}	$5.3397 imes10^{-4}$	
1.3	$6.5331 imes10^{-7}$	$2.5683 imes 10^{-6}$	$4.4278 imes10^{-4}$	$1.8777 imes 10^{-3}$	
1.5	$1.2494 imes10^{-6}$	$4.7989 imes10^{-6}$	$8.7071 imes10^{-4}$	$3.6354 imes10^{-3}$	
1.7	$1.3386 imes10^{-6}$	$5.0827 imes10^{-6}$	$1.0489 imes10^{-3}$	$4.2541 imes10^{-3}$	
1.9	$5.6739 imes 10^{-7}$	$2.1936 imes 10^{-6}$	$5.1085 imes10^{-4}$	2.0046×10^{-3}	





Figure 2. Graphical behaviour of problem 5.2 at t = 0.3, $\delta = 1.1$.

Problem 5.3:

Now we consider (1+2)-dimensional TFDWE [29]

$${}^{c}D_{t}^{\delta}w(x,y,t) = \Delta w(x,y,t) + \mathcal{B}(x,y,t), \quad (x,y) \in [0,1] \times [0,1], \quad t \in [0,1], \quad 1 < \delta \le 2,$$
(45)

with exact solution $w(x, y, t) = sin(\pi x)sin(\pi y)t^{\delta+3}$, and source term

$$\mathcal{B}(x,y,t) = \sin(\pi x)\sin(\pi y)\left[\frac{\Gamma(\delta+3)t^2}{2} - 2t^{\delta+2}\right].$$

We solved this problem for resolution level J = 4 and the obtained results are recorded in Table 6 for different values of time and τ . From Table 6 it is clear that the proposed scheme works well for the solution of two dimensional problems. Table 7 shows the comparison of the computed results with the previous work of Zhang [29]. One can see that our results are matchable with existing results. The same table presents convergence in time for (1 + 2)-dimensional problems. The graphical solution and absolute error of the problem are shown in Figure 3. It is obvious from Figure 3 that the exact and approximate solutions have strong agreement.

δ	$t = 0.1, \ \tau = 0.001$		$t = 0.2, \ \tau = 0.01$		$t = 0.5, \ \tau = 0.05$	
	L_∞	L_2	L_∞	L_2	L_{∞}	L_2
1.5	1.6049×10^{-4}	$8.0439 imes10^{-5}$	$4.3534 imes10^{-4}$	$2.1819 imes10^{-4}$	2.5390×10^{-2}	1.2725×10^{-2}
1.7	$1.1635 imes10^{-4}$	$5.8320 imes 10^{-5}$	$5.9673 imes10^{-4}$	$2.9908 imes10^{-4}$	$7.5824 imes10^{-3}$	$3.8003 imes 10^{-3}$
1.9	$3.0965 imes 10^{-5}$	$1.5519 imes 10^{-5}$	$3.9390 imes 10^{-4}$	$1.9742 imes 10^{-4}$	4.8842×10^{-3}	$2.4480 imes 10^{-3}$

Table 6. Comparison of problem 5.4 at t = 1 and δ with previous results.

		L_{∞}		
δ	τ	Present	[29]	
1.25	1/10 1/20 1/40	$\begin{array}{c} 8.1748 \times 10^{-3} \\ 6.5092 \times 10^{-3} \\ 5.7150 \times 10^{-3} \end{array}$	$\begin{array}{c} 8.1577 \times 10^{-2} \\ 3.4379 \times 10^{-2} \\ 1.4484 \times 10^{-2} \end{array}$	
1.5	1/10 1/20 1/40	$\begin{array}{l} 6.7087 \times 10^{-3} \\ 4.8922 \times 10^{-3} \\ 4.1390 \times 10^{-3} \end{array}$	$\begin{array}{c} 2.9942 \times 10^{-2} \\ 1.0749 \times 10^{-2} \\ 3.8291 \times 10^{-3} \end{array}$	
1.75	1/10 1/20 1/40	$\begin{array}{c} 6.7087 \times 10^{-3} \\ 4.8922 \times 10^{-3} \\ 4.1390 \times 10^{-3} \end{array}$	$\begin{array}{c} 8.4482 \times 10^{-3} \\ 2.5877 \times 10^{-3} \\ 7.8500 \times 10^{-4} \end{array}$	

Table 7. Error norms of problem 5.3 for different values of τ and δ .



Figure 3. Graphical behaviour of problem 5.3 when t = 0.5, $\delta = 1.9$.

Problem 5.4:

Consider the following TFDWE with reaction term [19]

$${}^{c}D_{t}^{\delta}w(x,t) + w(x,t) = w_{xx}(x,t) + \mathcal{A}(x,t), \quad x \in [0,1], \quad t \in [0,1], \quad 1 < \delta \le 2,$$
(46)

coupled with initial and boundary conditions

$$\begin{cases} w(x,0) = 0, & w_t(x,0) = 0 & x \in (0,1) \\ w(0,t) = o, & w(1,t) = 0, & t \in [0,1], \end{cases}$$
(47)

where the forcing terms are $A(x, t) = \frac{2t^{2-\delta}x(1-x)}{\Gamma(3-\delta)} + t^2x(1-x) - 2t^2$. This problem has been solved with the help of the proposed scheme. In Table 8 we presented the solutions at different points. Also the obtained results have been compared with the work presented in Reference [19]. It is clear from table that our results are more accurate. From the table it is also obvious that the exact and numerical solutions are in good agreement. Exact verses numerical solutions are plotted in Figure 4. Graphical solutions also indicate that the proposed scheme works in the case where the reaction term exists.

 $\delta = 1.1$ $\delta = 1.3$ $\delta = 1.5$ $\delta = 1.9$ (x,t) L_{∞} [19] L_{∞} L_{∞} [19] L_{∞} L_{∞} [19] L_{∞} [19] L_{∞} L_{∞} 2.1684×10^{-19} 1.0842×10^{-19} $3.2526\times\overline{10^{-19}}$ 9.5133×10^{-9} 6.6004×10^{-9} 4.4920×10^{-9} 5.8546×10^{-18} (0.1, 0.1) 1.9326×10^{-1} 1.0530×10^{-7} 9.5409×10^{-18} 2.1684×10^{-17} $7.9127 imes 10^{-8}$ 3.4694×10^{-18} $5.7844 imes10^{-8}$ 1.8735×10^{-16} 2.8903×10^{-8} (0.2, 0.2) $4.1633 imes 10^{-17}$ 9.6665×10^{-7} 4.1633×10^{-17} 3.3461×10^{-7} $1.0755 \times 10^{-16} \quad 2.5678 \times 10^{-7}$ 1.3634×10^{-15} $1.4105 imes 10^{-7}$ (0.3, 0.3) $1.3877 imes 10^{-17}$ 4.5033×10^{-15} 1.0813×10^{-6} 1.8735×10^{-16} 9.1574×10^{-7} $7.0776 imes 10^{-16}$ (0.4, 0.4) 7.3594×10^{-7} 4.3402×10^{-7} 1.6516×10^{-6} 1.3877×10^{-16} 2.2190×10^{-6} $3.9551 imes 10^{-16}$ 2.0261×10^{-15} 1.4231×10^{-14} 1.6516×10^{-6} 1.0367×10^{-6} (0.5, 0.5)

Table 8. Absolute error at different points of example 5.4 at $\tau = 0.001$.



Figure 4. Graphical behaviour of problem 5.4 at $\delta = 1.1$, t = 1.

Problem 5.5:

Now we consider the following equation

$${}^{c}D_{t}^{\delta}w(x,y,t) = a_{1}\Delta w(x,y,t) - b_{1}\sin(w(x,y,t)), \quad (x,y) \in \Phi, \quad t \in [0,T], \quad 1 < \delta \le 2,$$
(48)

where a_1 and b_1 are constants and the initial and boundary conditions are

$$\begin{cases} w(x,y,0) = \arctan\left(\exp(\frac{1}{2} - \sqrt{15x^2 + 15y^2})\right), & w_t(x,y,0) = 0, \quad (x,y) \in \widetilde{\Phi} = \Phi \cup \partial \Phi, \\ w(x,y,t) = 0, & (x,y) \in \partial \Phi, \quad t \in [0,T]. \end{cases}$$
(49)

Here, we examine the behaviour of circular ring soliton numerically. Due to pulsating behaviour, such waves are also known as pulsons. We choose different values of parameters a_1 , b_1 to present surface plots to study the time evolution of the circular ring soliton. We observe the effect of a_1 and b_1 on solutions. In Figure 5, numerical solutions for different values of a_1 and b_1 have been plotted. Figure 6 shows the numerical solution for $a_1 = 0.05$ while varying b_1 . In Figure 7 the results are plotted for $b_1 = 10$, in which the wave peak value at the centre becomes lower as a_1 increases. This reveals that the solitary wave moves in a stable way up to a large time under finite initial condition.



Figure 5. Graphical behaviour of problem 5.5 at $\delta = 1.9$, $a_1 = 0.1$, $b_1 = 10$.



Figure 6. Graphical behaviour of problem 5.5 at $\delta = 1.9$, $a_1 = 0.05$.



Figure 7. Graphical behaviour of problem 5.5 at $\delta = 1.9$, $b_1 = 10$.

7. Conclusions

In this paper, we proposed a hybrid method based on finite difference and Haar wavelets approximations. The scheme is applied for the numerical solution of (1 + 1)- and (1 + 2)-dimensional time fraction partial differential equations. The accuracy and applicability of the scheme is validated through some test problems. The tabulated data and graphical solution show that the scheme works very well for time fractional problems.

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