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Convergence of Two Splitting Projection Algorithms in Hilbert Spaces

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Abstract: The aim of this present paper is to study zero points of the sum of two maximally monotone mappings and fixed points of a non-expansive mapping. Two splitting projection algorithms are introduced and investigated for treating the zero and fixed point problems. Possible computational errors are taken into account. Two convergence theorems are obtained and applications are also considered in Hilbert spaces

Keywords: extragradient method; maximally monotone operator; proximal point algorithm; variational inequality; zero point

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1. Introduction—Preliminaries

Let H be an infinite-dimensional real Hilbert space. Its inner product is denoted by $\langle x, y \rangle$. The induced norm is denoted by $\|x\| = \sqrt{\langle x, x \rangle}$ for $x, y \in H$. Let C be a convex and closed set in H and let $A : C \rightarrow H$ be a single-valued mapping. We recall the following definitions.

A is said to be a monotone mapping iff

$$\langle x' - x, Ax' - Ax \rangle \geq 0, \quad \forall x', x \in C.$$

A is said to be a strongly monotone mapping iff there exists a positive real constant L such that

$$\langle x' - x, Ax' - Ax \rangle \geq L\|x' - x\|^2, \quad \forall x', x \in C.$$

A is said to be an inverse-strongly monotone mapping iff there exists a positive real constant L such that

$$\langle x' - x, Ax' - Ax \rangle \geq L\|Ax - Ax'\|^2, \quad \forall x', x \in C.$$

A is said to be L -Lipschitz continuous iff there exists a positive real constant L such that

$$\|Ax' - Ax\| \leq L\|x - x'\|, \quad \forall x', x \in C.$$

A is said to be sequentially weakly continuous iff, for any vector sequence $\{x_n\}$ in C , $\{x_n\}$ converges weakly to x , which implies that Ax_n converges weakly to Ax .

Consider that the following monotone variational inequality, which associates with mapping A and set C consists of finding an $x \in C$ such that

$$\langle Ax, x^* - x \rangle \geq 0, \quad \forall x^* \in C. \quad (1)$$

From now on, we borrow $VI(C, A)$ to present the set of solutions (1). Recently, spotlight has been shed on projection-based iterative methods, which are efficient to deal with solutions of variational inequality (1). With the aid of resolvent mapping $Proj_C^H(Id - rA)$, where $Proj_C^H$ is the metric projection from H onto C , r is some positive real number and Id denotes the identity on H , one knows that x is a solution to inequality (1) iff x is a fixed point of $Proj_C^H(Id - rA)$. When dealing with the resolvent mapping, one is required to metric projections at every iteration. In the case that C is a linear variety or a closed ball or polytope, the computation of $Proj_C^H$ is not hard to implement. If C is a bounded set, then the existence of solutions of the variational inequality is guaranteed by Browder [1]. If A is monotone and L -Lipschitz continuous, Korpelevich [2] introduced the following so-called extragradient method:

$$\begin{cases} x_0 \in C, \\ y_n = Proj_C^H(x_n - rAx_n), \\ x_{n+1} = Proj_C^H(x_n - rAy_n), \quad \forall n \geq 0, \end{cases}$$

where C is assumed to be a convex and closed set in a finite-dimensional Euclidean space R^n and r is positive real number in $(0, 1/L)$. He proved that sequence $\{x_n\}$ converges to a point in $VI(C, A)$, for more materials; see [2] and the cited references therein. We remark here that the extragradient method is an Ishikawa-like iterative method, which is efficient for solving fixed-point problems of pseudocontractive mappings whose complementary mappings are monotone, that is, A is monotone if and only if $I - A$ is pseudocontractive.

Next, we turn our attention to set-valued mappings. Let $B : H \rightrightarrows H$ be a set-valued mapping. We borrow $Graph(B) := \{(y, x) \in H \times H : x \in Ay\}$ to denote the graph of mapping B and $B^{-1}(0)$ to denote the zero set of mapping B . One says that B is monotone iff $\langle y - y', x - x' \rangle \geq 0$, for all $(y, x), (y', x') \in Graph(B)$. One says that B is maximal iff there exists no proper monotone extensions of the graph of B on $H \times H$, that is, the graph of B is not a subset of any other monotone operator graphs. For a maximally monotone operator B , one can define the single-valued resolvent mapping $J_r = (rB + Id)^{-1} : H \rightarrow Dom(B)$, where $Dom(B)$ stands for the domain of B , Id stands for the identity mapping and r is a real number. In the case in which B is the subdifferential of proper, lower semicontinuous and convex functions, then its resolvent mapping is called the proximity mapping. One knows $B^{-1}(0) = Fix(J_r)$, where $Fix(J_r)$ stands for the fixed-point set of J_r and J_r is firmly non-expansive, that is, $\langle J_r x' - J_r x, x' - x \rangle \geq \|J_r x - J_r x'\|^2$.

The class of maximally monotone mapping is under the spotlight of researchers working on the fields of optimization and functional analysis. Let f be a proper convex and closed function $f : H \rightarrow (-\infty, \infty]$. One known example of maximally mapping is ∂f , the subdifferential of f . It is defined as follows:

$$\partial f(x) := \{x^* \in H : \langle x' - x, x^* \rangle + f(x) \leq f(x'), \forall x' \in H\}, \quad \forall x \in H.$$

Rockafellar [3] asserted that ∂f is a maximally monotone operator. One can verify that $f(v) = \min_{x \in H} f(x)$ iff $0 \in \partial f(v)$. Next, we give one more example for maximally monotone mappings: $N_C + M$, where M is a continuous single-valued maximally monotone mapping, and N_C is the mapping of the normal cone:

$$N_C(x) := \{x^* \in H : \langle x^*, x' - x \rangle \leq 0, \forall x' \in C\}$$

for $x \in C$ and is empty otherwise. Then, $0 \in N_C(x) + Mx$ iff $x \in C$ is a solution to the following monotone variational inequalities: $\langle x' - x, Mx \rangle \geq 0$ for all $x' \in C$.

One of the fundamental and efficient solution methods for investigating the inclusion problem $0 \in Tx$, where T is a maximally monotone mapping is the known proximal point algorithm (PPA), which was studied by Martinet [4,5] and Rockafellar [6,7]. The PPA has been extensively studied [8–11] and is known to yield special cases' decomposition methods such as the method of

partial inverses [12], the FB splitting method, and the ADMM [13,14]. The following forward-backward splitting method $(I - r_n A)x_n \in (I + r_n B)x_{n+1}$, $n = 0, 1, \dots$, where $r_n > 0$, was proposed by Lions and Mercier [15], and Passty [16] for $T = A + B$, where A and B are two maximally monotone mappings. Furthermore, if $B = N_C$, then this method is reduced to the gradient-projection iterative method [17]. Recently, a number of researchers, who work on the fields of monotone operators, studied the splitting algorithm; see [18–22] and the references therein.

Let $S : C \rightarrow C$ be a single-valued mapping. In this paper, we use $Fix(S)$ to stand for the fixed-point set of mapping S . Recall that S is said to be non-expansive if $\|x' - x\| \geq \|Sx' - Sx\|$, $\forall x', x \in C$. If C is a bounded set, then the set of fixed points of mapping S is non-empty; see [23]. In the real world, a number of problems and modelings have reformulations that require finding fixed points of non-expansive mapping (zeros of monotone mappings). One knows that Mann-like iteration is weakly convergent for non-expansive mapping only. Recently, a number of researchers concentrated on various Mann-like iterations so that strongly convergent theorems can be obtained without additional compact restrictions on mappings; see [24–27].

For most real mathematical modelings, one often has more than one constraint. For such modelings, solutions to a problem which are simultaneously solutions to two or more problems (or desired solutions lie on the solution set of other problems); see [28–33] and the references therein.

In this paper, we, based on Tseng’s ideas, are concerned with the problem of finding a common solution of fixed-point problems of a non-expansive mapping and zero-point problems of a sum of two monotone operators based on two splitting algorithms, which take into account possible computational errors. Convergence theorems of the algorithms are obtained. Applications of the algorithms are also discussed.

In order to prove the main results of this paper, the following tools are essential.

An infinite-dimensional space X is said to satisfy Opial’s condition [34] if, for any $\{x_n\} \subset X$ with $x_n \rightarrow x$, the following inequality holds:

$$\liminf_{n \rightarrow \infty} \|x_n - y\| > \liminf_{n \rightarrow \infty} \|x_n - x\|$$

for $y \in X$ with $y \neq x$. It is well known that the above inequality is equivalent to

$$\limsup_{n \rightarrow \infty} \|x_n - y\| > \limsup_{n \rightarrow \infty} \|x_n - x\|$$

for $y \in X$ with $y \neq x$. It is well known that l^p , where $p > 1$ and L^2 satisfy the Opial’s condition.

The following lemma is trivial, so the proof is omitted.

Lemma 1. *Let $\{a_n\}$ be a real positive sequence with $a_{n+1} \leq a_n + b_n$, $\forall n \geq n_0$, where $\{b_n\}$ is a real positive sequence with $\sum_{n=1}^{\infty} b_n < \infty$ and n_0 is some nonnegative integer. Then, the limit $\lim_{n \rightarrow \infty} a_n$ exists.*

Lemma 2. *Reference [35] Let H be a Hilbert space and let $\{t_n\}$ be a real sequence with the restriction $0 < p \leq t_n \leq q < 1$ for all $n \geq 1$. Let $\{x_n\}$ and $\{y_n\}$ be two vector sequences in H with $\lim_{n \rightarrow \infty} \|y_n + t_n x_n - t_n y_n\| = r$, $\limsup_{n \rightarrow \infty} \|x_n\| \leq r$ and $\limsup_{n \rightarrow \infty} \|y_n\| \leq r$, where r is some positive real number. Then, $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$.*

Lemma 3. *Reference [34] Let C be a convex and closed set in an infinite-dimension Hilbert space H and let S be a non-expansive mapping with a non-empty fixed-point set on set C . Let $\{x_n\}$ be a vector sequence on C . If $\{x_n\}$ converges weakly to x and $\lim_{n \rightarrow \infty} \|(S - I)x_n\| = 0$, then $x \in Fix(S)$.*

2. Main Results

Theorem 1. *Let C be a convex and closed set in a Hilbert space H . Let S be a non-expansive self mapping on C , whose fixed-point set is non-empty. Let $A : C \rightarrow H$ be a monotone and both L -Lipschitz continuous*

and sequentially weakly continuous mapping. Let B be a maximally monotone mapping on H . Assume that $Dom(B)$ lies in C and $Fix(S) \cap (A + B)^{-1}(0)$ is not empty. Let $\{\alpha_n\}$ be a real number sequence in $[a, b]$ for some $a, b \in (0, 1)$, and let $\{r_n\}$ be a real number sequence in $[c, d]$ for some $c, d \in (0, 1/\sqrt{2}L)$. Let $\{x_n\}$ be a vector sequence defined and generated in the following iterative process:

$$\begin{cases} x_0 \in C, \text{ arbitrarily chosen,} \\ y_n = J_{r_n}(x_n - r_nAx_n - e_n), \\ x_{n+1} = \alpha_nx_n + (1 - \alpha_n)SProj_C^H(y_n - r_nAy_n + r_nAx_n + e_n), \quad \forall n \geq 0, \end{cases}$$

where $\{e_n\}$ is an error sequence in H with $\sum_{n=0}^\infty \|e_n\| < \infty$. Then, $\{x_n\}$ converges to a point $\bar{x} \in Fix(S) \cap (A + B)^{-1}(0)$ weakly.

Proof. Set $z_n = y_n - r_nAy_n + r_nAx_n + e_n$. Fixing $x^* \in (A + B)^{-1}(0) \cap Fix(T)$, we find that

$$\begin{aligned} \|x_n - x^*\|^2 &= \|x_n - y_n + y_n - z_n + z_n - x^*\|^2 \\ &= \|y_n - x_n\|^2 + \|y_n - z_n\|^2 + \|z_n - x^*\|^2 + 2\langle y_n - x_n, x^* - y_n \rangle \\ &\quad + 2\langle y_n - z_n, z_n - x^* \rangle \\ &= \|x_n - y_n\|^2 - \|y_n - z_n\|^2 + \|z_n - x^*\|^2 + 2\langle x_n - z_n, y_n - x^* \rangle \\ &\geq \|x_n - y_n\|^2 - \|r_nAy_n - r_nAx_n - e_n\|^2 + \|z_n - x^*\|^2. \end{aligned}$$

Using the Lipschitz continuity of mapping A , one finds that

$$\begin{aligned} \|z_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 + \|r_nAy_n - r_nAx_n - e_n\|^2 \\ &\leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 + 2r_n^2\|Ay_n - Ax_n\|^2 + 2\|e_n\|^2 \\ &\leq \|x_n - x^*\|^2 - (1 - 2r_n^2L^2)\|x_n - y_n\|^2 + 2\|e_n\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \alpha_n\|x_n - x^*\|^2 + (1 - \alpha_n)\|SProj_C^H z_n - x^*\|^2 \\ &\leq \alpha_n\|x_n - x^*\|^2 + (1 - \alpha_n)\|z_n - x^*\|^2 \\ &\leq \|x_n - x^*\|^2 - (1 - \alpha_n)(1 - 2r_n^2L^2)\|x_n - y_n\|^2 + 2\|e_n\|^2. \end{aligned} \tag{2}$$

In light of Lemma 1, one finds that the following limit $\lim_{n \rightarrow \infty} \|x_n - x^*\|$ exists; in particular, the vector sequence $\{x_n\}$ is bounded. By using (2), one gets that

$$(1 - \alpha_n)(1 - 2r_n^2L^2)\|x_n - y_n\|^2 \leq \|x^* - x_n\|^2 - \|x^* - x_{n+1}\|^2 + 2\|e_n\|^2.$$

Thanks to the condition on $\{\alpha_n\}$, $\{r_n\}$ and $\{e_n\}$, one obtains

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0. \tag{3}$$

Notice the fact that vector sequence $\{x_n\}$ is bounded. There is a vector sequence $\{x_{n_i}\}$, which is a subsequence of original sequence $\{x_n\}$ converging to $\bar{x} \in C$ weakly. In light of (3), we find that the subsequence $\{y_{n_i}\}$ of $\{y_n\}$ also converges to \bar{x} weakly.

Now, one is in a position to claim that \bar{x} lies in $(A + B)^{-1}(0)$. Notice that

$$\frac{x_n - y_n - e_n}{r_n} - Ax_n \in By_n.$$

Suppose $\mu \in Bv$. By using the continuity of mapping B , one reaches

$$\left\langle \frac{x_n - y_n - e_n}{r_n} - Ax_n - \mu, y_n - v \right\rangle \geq 0.$$

Taking into account the fact that A is a sequentially weakly continuous account, one arrives at $\langle -A\bar{x} - \mu, \bar{x} - v \rangle \geq 0$. This guarantees $-A\bar{x} \in B\bar{x}$, that is, $\bar{x} \in (A + B)^{-1}(0)$.

On the other hand, we have that $\bar{x} \in \text{Fix}(S)$. Indeed, set $\lim_{n \rightarrow \infty} \|x_n - x^*\| = d$. It follows from (2) that $\|S\text{Proj}_C^H z_n - x^*\| \leq \|x_n - x^*\| + \sqrt{2}\|e_n\|$. This shows that $\limsup_{n \rightarrow \infty} \|S\text{Proj}_C^H z_n - x^*\| \leq d$. It follows from Lemma 2 that

$$\lim_{n \rightarrow \infty} \|S\text{Proj}_C^H z_n - x_n\| = 0. \tag{4}$$

Since A is L -Lipschitz continuous, we find that

$$\begin{aligned} \|Sy_n - y_n\| &\leq \|Sy_n - S\text{Proj}_C^H z_n\| + \|S\text{Proj}_C^H z_n - x_n\| + \|x_n - y_n\| \\ &\leq \|y_n - \text{Proj}_C^H z_n\| + \|S\text{Proj}_C^H z_n - x_n\| + \|x_n - y_n\| \\ &\leq (1 + r_n L)\|y_n - x_n\| + \|e_n\| + \|S\text{Proj}_C^H z_n - x_n\|. \end{aligned}$$

Combining (3) with (4), we obtain that $\lim_{n \rightarrow \infty} \|Sy_n - y_n\| = 0$. Thanks to Lemma 3, one concludes that $\bar{x} \in \text{Fix}(S)$.

Next, one claims that vector sequence $\{x_n\}$ converges to \bar{x} weakly. If not, one finds that there exists some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and this subsequence $\{x_{n_i}\}$ converges to $\hat{x} \in C$ weakly, and $\hat{x} \neq \bar{x}$. Similarly, one has $\hat{x} \in (A + B)^{-1}(0)$. From the fact that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists, $\forall p \in (A + B)^{-1}(0)$, one may suppose that $\lim_{n \rightarrow \infty} \|x_n - \bar{x}\| = d$, where d is a nonnegative number. By using the Opial's inequality, one arrives at

$$\begin{aligned} d &= \liminf_{i \rightarrow \infty} \|x_{n_i} - \bar{x}\| < \liminf_{i \rightarrow \infty} \|x_{n_i} - \hat{x}\| \\ &= \liminf_{j \rightarrow \infty} \|x_{n_j} - \hat{x}\| < \liminf_{j \rightarrow \infty} \|x_{n_j} - \bar{x}\| = d. \end{aligned}$$

One reaches a contradiction. Hence, $\bar{x} = \hat{x}$. \square

The following result is not hard to derive from Theorem 1.

Corollary 1. *Let C be a convex and closed set in a Hilbert space H . Let $A : C \rightarrow H$ be a monotone and both L -Lipschitz continuous and sequentially weakly continuous mapping. Let B be a maximally monotone mapping on H . Suppose that $\text{Dom}(B) \subset C$ and $(A + B)^{-1}(0)$ is not empty. Let $\{\alpha_n\}$ be a real number sequence in $[a, b]$ for some $a, b \in (0, 1)$ and let $\{r_n\}$ be a real number sequence in $[c, d]$ for some $c, d \in (0, 1/\sqrt{2}L)$. Let $\{x_n\}$ be a vector sequence defined and generated in the following process:*

$$\begin{cases} x_0 \in C, \text{ arbitrarily chosen,} \\ y_n = J_{r_n}(x_n - r_n Ax_n - e_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \text{Proj}_C^H(y_n + r_n Ax_n - r_n Ay_n + e_n), \quad \forall n \geq 0, \end{cases}$$

where $\{e_n\}$ is an error sequence in H such that $\sum_{n=0}^{\infty} \|e_n\| < \infty$. Then, $\{x_n\}$ converges to a point $\bar{x} \in (A + B)^{-1}(0)$ weakly.

Next, one is ready to present the other convergence theorem.

Theorem 2. *Let C be a convex and closed set in a Hilbert space H . Let S be a non-expansive self mapping on C , whose fixed-point set is non-empty. Let $A : C \rightarrow H$ be a monotone and both L -Lipschitz continuous and sequentially weakly continuous mapping. Let B be a maximally monotone mapping on H . Assume that*

$Dom(B)$ lies in C and $Fix(S) \cap (A + B)^{-1}(0)$ is not empty. Let $\{\alpha_n\}$ be a real number sequence in $[0, a]$ for some $a \in [0, 1)$ and let $\{r_n\}$ be a real number sequence in $[b, c]$ for some $b, c \in (0, 1/\sqrt{2}L)$. Let $\{x_n\}$ be a vector sequence defined and generated in the following iterative process:

$$\begin{cases} x_0 \in H \text{ arbitrarily chosen,} \\ C_0 = C, \\ z_n = J_{r_n}(x_n - r_nAx_n - e_n), \\ y_n = \alpha_nx_n + (1 - \alpha_n)SProj_C^H(z_n - r_nAz_n + r_nAx_n + e_n), \\ C_{n+1} = \{w \in C_n : \|y_n - w\|^2 \leq \|x_n - w\|^2 - (1 - \alpha_n)(1 - 2r_n^2L^2)\|x_n - z_n\|^2 + 2\|e_n\|^2\}, \\ x_{n+1} = Proj_{C_{n+1}}^H x_0, \quad n \geq 0, \end{cases}$$

where $\{e_n\}$ is an error sequence with the restriction $\lim_{n \rightarrow \infty} \|e_n\| = 0$. Then, $\{x_n\}$ converges strongly to $Proj_{Fix(S) \cap (A+B)^{-1}(0)}^H x_0$.

Proof. First, we show that the set C_n is closed and convex. It is clear that C_n is closed. We only show the convexness of C_n . From assumption, we see that C_0 is convex. We suppose that C_m is a convex set for some $m \geq 0$. Next, one claims that C_{m+1} is also a convex set. Since

$$\|w - y_n\|^2 \leq \|w - x_n\|^2 - (1 - \alpha_n)(1 - 2r_n^2L^2)\|x_n - z_n\|^2 + 2\|e_n\|^2$$

is equivalent to

$$\|y_n\|^2 - \|x_n\|^2 + (1 - \alpha_n)(1 - 2r_n^2L^2)\|x_n - z_n\|^2 - 2\|e_n\|^2 + 2\langle x_n - y_n, w \rangle \leq 0,$$

we easily find that C_{m+1} is a convex set. This claims that set C_n is convex and closed. Next, we show that $Fix(S) \cap (A + B)^{-1}(0) \subset C_n$. From the assumption, we see that $Fix(S) \cap (A + B)^{-1}(0) \subset C_0$. Suppose that $Fix(S) \cap (A + B)^{-1}(0) \subset C_m$ for some $m \geq 0$. Next, we show that $Fix(S) \cap (A + B)^{-1}(0) \subset C_{m+1}$ for the same m . Set $v_n = z_n - r_nAz_n + r_nAx_n + e_n$. For any $w \in Fix(S) \cap (A + B)^{-1}(0) \subset C_m$, we find that

$$\begin{aligned} \|x_n - w\|^2 &= \|x_n - z_n + z_n - v_n + v_n - w\|^2 \\ &= \|z_n - x_n\|^2 + \|v_n - z_n\|^2 + \|v_n - w\|^2 + 2\langle x_n - z_n, z_n - w \rangle \\ &\quad + 2\langle z_n - v_n, v_n - w \rangle \\ &\geq \|x_n - z_n\|^2 - \|r_nAz_n - r_nAx_n - e_n\|^2 + \|v_n - w\|^2. \end{aligned}$$

Thanks to the fact that A is a Lipschitz continuous mapping, one asserts that

$$\begin{aligned} \|v_n - w\|^2 &\leq \|x_n - w\|^2 - \|z_n - x_n\|^2 + \|r_nAz_n - r_nAx_n - e_n\|^2 \\ &\leq \|x_n - w\|^2 - \|z_n - x_n\|^2 + 2r_n^2\|Az_n - Ax_n\|^2 + 2\|e_n\|^2 \\ &\leq \|x_n - w\|^2 - (1 - 2r_n^2L^2)\|z_n - x_n\|^2 + 2\|e_n\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} \|y_n - w\|^2 &\leq (1 - \alpha_n)\|SProj_C v_n - w\|^2 + \alpha_n\|x_n - w\|^2 \\ &\leq (1 - \alpha_n)\|v_n - w\|^2 + \alpha_n\|x_n - w\|^2 \\ &\leq 2\|e_n\|^2 - (1 - \alpha_n)(1 - 2r_n^2L^2)\|x_n - z_n\|^2 + \|x_n - w\|^2. \end{aligned}$$

This implies that $w \in C_{n+1}$. This proves that $Fix(S) \cap (A + B)^{-1}(0) \subset C_n$. Since $x_n = Proj_{C_n}^H x_0$

and $x_{n+1} = Proj_{C_{n+1}}^H x_0 \in C_{n+1}$, which is a subset of C_n , we find that

$$\begin{aligned} 0 &\leq \langle x_n - x_0, x_{n+1} - x_n \rangle \\ &\leq -\|x_n - x_0\|^2 + \|x_n - x_0\| \|x_{n+1} - x_0\|. \end{aligned}$$

This implies that $\|x_n - x_0\| \leq \|x_{n+1} - x_0\|$. For any $w \in Fix(S) \cap (A + B)^{-1}(0) \subset C_n$, we find from $x_n = Proj_{C_n}^H x_0$ that $\|x_0 - x_n\| \leq \|x_0 - w\|$, in particular, $\|x_0 - x_n\| \leq \|x_0 - Proj_{Fix(S) \cap (A+B)^{-1}(0)}^H x_0\|$.

This claims that vector sequence $\{x_n\}$ is bounded and limit $\lim_{n \rightarrow \infty} \|x_0 - x_n\|$ exists. Note that

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 + \|x_n - x_0\|^2 \\ &= \|x_0 - x_{n+1}\|^2 - \|x_n - x_0\|^2 + 2\langle x_0 - x_n, x_{n+1} - x_n \rangle \\ &\leq \|x_0 - x_{n+1}\|^2 - \|x_n - x_0\|^2. \end{aligned}$$

Letting $n \rightarrow \infty$, one obtains $\|x_n - x_{n+1}\| \rightarrow 0$. In light of $x_{n+1} = Proj_{C_{n+1}}^H x_0 \in C_{n+1}$, we see that

$$\|y_n - x_{n+1}\|^2 \leq 2\|e_n\|^2 - (1 - \alpha_n)(1 - 2r_n^2 L^2)\|x_n - z_n\|^2 + \|x_n - x_{n+1}\|^2.$$

It follows that $\lim_{n \rightarrow \infty} \|y_n - x_{n+1}\| = 0$. This proves that $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$. Using the restrictions imposed on the sequence $\{\alpha_n\}$, $\{r_n\}$ and $\{e_n\}$, we also find that $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$. By using the fact that $\{x_n\}$ is a bounded sequence, there exists a sequence $\{x_{n_i}\}$, which is a subsequence of $\{x_n\}$, converging to $\bar{x} \in C$ weakly. One also obtains that the sequence $\{z_{n_i}\}$ also converges to \bar{x} weakly. Note that $\frac{x_n - z_n - e_n}{r_n} - Ax_n \in Bz_n$. Next, we suppose μ is a point in Bv . The monotonicity of B yields that $\langle \frac{x_n - z_n - e_n}{r_n} - Ax_n - \mu, z_n - v \rangle \geq 0$. Since A is sequentially weakly continuous mapping, we obtain that $\langle \bar{x} - v, -A\bar{x} - \mu \rangle \geq 0$. These yield that $-A\bar{x} \in B\bar{x}$. Hence, one obtains $\bar{x} \in (A + B)^{-1}(0)$.

One now is in a position to claim that $\bar{x} \in Fix(S)$. Since $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$, which in turn implies that $\lim_{n \rightarrow \infty} \|SProj_C^H v_n - x_n\| = 0$. Since A is Lipschitz continuous, one has

$$\begin{aligned} \|Sx_n - x_n\| &\leq \|Sx_n - Sz_n\| + \|Sz_n - SProj_C^H v_n\| + \|SProj_C^H v_n - x_n\| \\ &\leq \|x_n - z_n\| + \|z_n - Proj_C^H v_n\| + \|SProj_C^H v_n - x_n\| \\ &\leq (1 + r_n L)\|z_n - x_n\| + \|e_n\| + \|SProj_C^H v_n - x_n\|. \end{aligned}$$

This proves that $\lim_{n \rightarrow \infty} \|Sx_n - x_n\| = 0$. In light of Lemma 3, one finds $\bar{x} \in Fix(S)$. Put $\tilde{x} = Proj_{(A+B)^{-1}(0)}^H x_0$. Since $x_n = Proj_{C_n}^H x_0$ and $\tilde{x} \in C_n$, we find that $\|x_0 - x_n\| \leq \|x_0 - \tilde{x}\|$. Note that

$$\|x_0 - \tilde{x}\| \leq \|x_0 - \bar{x}\| \leq \liminf_{i \rightarrow \infty} \|x_0 - x_{n_i}\| \leq \limsup_{i \rightarrow \infty} \|x_0 - x_{n_i}\| \leq \|x_0 - \tilde{x}\|.$$

It follows that

$$\|x_0 - \bar{x}\| = \lim_{i \rightarrow \infty} \|x_0 - x_{n_i}\| = \|x_0 - \tilde{x}\|,$$

from which one gets $x_{n_i} \rightarrow \bar{x} = \tilde{x}$. From the arbitrariness of $\{x_{n_i}\}$, one has $x_n \rightarrow \tilde{x}$. \square

The following results can be derived immediately from Theorem 2.

Corollary 2. *Let C be a convex and closed set in a Hilbert space H . Let $A : C \rightarrow H$ be a monotone and both L -Lipschitz continuous and sequentially weakly continuous mapping. Let B be a maximally monotone mapping on H . Assume that $Dom(B)$ lies in C and $(A + B)^{-1}(0)$ is not empty. Let $\{\alpha_n\}$ be a real number sequence in*

$[0, a]$ for some $a \in [0, 1)$ and let $\{r_n\}$ be a real number sequence in $[b, c]$ for some $b, c \in (0, 1/\sqrt{2}L)$. Let $\{x_n\}$ be a vector sequence defined and generated in the following iterative process:

$$\begin{cases} x_0 \in H \text{ chosen arbitrarily,} \\ C_0 = C, \\ z_n = J_{r_n}(x_n - r_nAx_n - e_n), \\ y_n = \alpha_nx_n + (1 - \alpha_n)Proj_C^H(z_n - r_nAz_n + r_nAx_n + e_n), \\ C_{n+1} = \{w \in C_n : \|y_n - w\|^2 \leq \|x_n - w\|^2 - (1 - \alpha_n)(1 - 2r_n^2L^2)\|x_n - z_n\|^2 + 2\|e_n\|^2\}, \\ x_{n+1} = Proj_{C_{n+1}}^Hx_0, \quad n \geq 0, \end{cases}$$

where $J_{r_n} = (I + r_nB)^{-1}$ and $\{e_n\}$ is an error sequence in H such that $\lim_{n \rightarrow \infty} \|e_n\| = 0$. Then, $\{x_n\}$ converges to $Proj_{(A+B)^{-1}(0)}x_0$ strongly.

Corollary 3. Let C be a convex and closed set in a Hilbert space H . Let S be a non-expansive self mapping on C , whose fixed-point set is non-empty. Let $\{\alpha_n\}$ be a real number sequence in $[0, a]$ for some $a \in [0, 1)$. Let $\{x_n\}$ be a vector sequence defined and generated in the following iterative process:

$$\begin{cases} x_0 \in H \text{ chosen arbitrarily,} \\ C_0 = C, \\ y_n = (1 - \alpha_n)Sx_n + \alpha_nx_n, \\ C_{n+1} = \{w \in C_n : \|y_n - w\|^2 \leq \|x_n - w\|^2 - (1 - \alpha_n)(1 - 2r_n^2L^2)\|x_n - z_n\|^2\}, \\ x_{n+1} = Proj_{C_{n+1}}^Hx_0, \quad n \geq 0. \end{cases}$$

Then, $\{x_n\}$ converges strongly to $Proj_{Fix(S)}^Hx_0$.

3. Applications

This section gives some results on solutions of variational inequalities, minimizers of convex functions, and solutions of equilibrium problems.

Let H be a real Hilbert space and let C be a convex and closed set in H . Let i_C be a function defined by

$$i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

One knows that indicator function i_C is proper convex and lower semicontinuous, and its subdifferential ∂i_C is maximally monotone. Define the resolvent mapping of subdifferential operator ∂i_C by $J_r := (I + r\partial i_C)^{-1}$. Letting $x = J_r y$, one finds

$$\begin{aligned} y \in r\partial i_Cx + x &\iff y \in rN_Cx + x \\ &\iff \langle y - x, z - x \rangle \leq 0, \forall z \in C \\ &\iff x = Proj_C^Hy, \end{aligned}$$

where $N_Cx := \{z \in H | \langle z, v - x \rangle, \forall v \in C\}$. If $B = \partial i_C$ in Theorem 1 and Theorem 2, then the following results can be derived immediately.

Theorem 3. Let C be a convex and closed set in a Hilbert space H . Let S be a non-expansive self mapping on C , whose fixed-point set is non-empty. Let $A : C \rightarrow H$ be a monotone and both L -Lipschitz continuous and sequentially weakly continuous mapping. Assume that $Fix(S) \cap VI(C, A)$ is not empty. Let $\{\alpha_n\}$ be

a real number sequence in $[a, b]$ for some $a, b \in (0, 1)$ and let $\{r_n\}$ be a real number sequence in $[c, d]$ for some $c, d \in (0, 1/\sqrt{2}L)$. Let $\{x_n\}$ be a vector sequence defined and generated in the following iterative process:

$$\begin{cases} x_0 \in C, \text{ chosen arbitrarily,} \\ y_n = \text{Proj}_C^H(x_n - r_n Ax_n - e_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S\text{Proj}_C^H(y_n - r_n Ay_n + r_n Ax_n + e_n), \quad \forall n \geq 0, \end{cases}$$

where $\{e_n\}$ is an error sequence in H such that $\sum_{n=0}^\infty \|e_n\| < \infty$. Then, $\{x_n\}$ converges to a point $\bar{x} \in \text{Fix}(S) \cap \text{VI}(C, A)$ weakly.

Theorem 4. Let C be a convex and closed set in a Hilbert space H . Let S be a non-expansive self mapping on C , whose fixed-point set is non-empty. Let $A : C \rightarrow H$ be a monotone and both L -Lipschitz continuous and sequentially weakly continuous mapping. Assume that $\text{Fix}(S) \cap \text{VI}(C, A)$ is not empty. Let $\{\alpha_n\}$ be a real number sequence in $[0, a]$ for some $a \in [0, 1)$ and let $\{r_n\}$ be a real number sequence in $[c, d]$ for some $c, d \in (0, 1/\sqrt{2}L)$. Let $\{x_n\}$ be a vector sequence defined and generated in the following iterative process:

$$\begin{cases} x_0 \in H \text{ chosen arbitrarily,} \\ C_0 = C, \\ z_n = \text{Proj}_{C_n}(x_n - r_n Ax_n - e_n), \\ y_n = \alpha_n x_n + (1 - \alpha_n) S\text{Proj}_{C_n}^H(z_n - r_n Az_n + r_n Ax_n + e_n), \\ C_{n+1} = \{w \in C_n : \|y_n - w\|^2 \leq \|x_n - w\|^2 - (1 - \alpha_n)(1 - 2r_n^2 L^2)\|x_n - z_n\|^2 + 2\|e_n\|^2\}, \\ x_{n+1} = \text{Proj}_{C_{n+1}}^H x_0, \quad n \geq 0, \end{cases}$$

where $\{e_n\}$ is an error sequence in H such that $\lim_{n \rightarrow \infty} \|e_n\| = 0$. Then, $\{x_n\}$ converges to $\text{Proj}_{\text{Fix}(S) \cap \text{VI}(C, A)}^H x_0$ strongly.

Next, we consider minimizers of a proper convex and lower semicontinuous function.

Let $f : H \rightarrow (-\infty, \infty]$ be a proper lower semicontinuous convex function. One can define subdifferential mapping ∂f by $\partial f(x) = \{x^* \in H : f(x) + \langle y - x, x^* \rangle \leq f(y), \forall y \in H\}$, $\forall x \in H$. Rockafellar [3] proved that subdifferential mappings are maximally monotone and $0 \in \partial f(v)$ if and only if $f(v) = \min_{x \in H} f(x)$.

Theorem 5. Let C be a convex and closed set in a Hilbert space H . Let $f : H \rightarrow (-\infty, +\infty]$ be a proper convex lower semicontinuous function such that $(\partial f)^{-1}(0)$ is not empty. Let $\{\alpha_n\}$ be a real number sequence in $[a, b]$ for some $a, b \in (0, 1)$ and let $\{r_n\}$ be a real number sequence in $[c, d]$ for some $c, d \in (0, 1/\sqrt{2}L)$. Let $\{x_n\}$ be a vector sequence defined and generated in the following iterative process:

$$\begin{cases} x_0 \in C, \text{ chosen arbitrarily,} \\ y_n = \arg \min_{z \in H} \{f(z) + \frac{\|z - x_n + e_n\|^2}{2r_n}\}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S\text{Proj}_C^H(y_n + e_n), \quad \forall n \geq 0, \end{cases}$$

where $\{e_n\}$ is an error sequence in H such that $\sum_{n=0}^\infty \|e_n\| < \infty$. Then, $\{x_n\}$ converges to a point $\bar{x} \in (\partial f)^{-1}(0)$ weakly.

Proof. From the assumption that $f : H \rightarrow (-\infty, \infty]$ is proper, convex, and lower semicontinuous, one obtains that subdifferential ∂f is maximally monotone. Setting $A = 0$ and $y_n = J_{r_n}(x_n - e_n)$, one sees that

$$y_n = \arg \min_{z \in H} \{f(z) + \frac{\|z - x_n + e_n\|^2}{2r_n}\}$$

is equivalent to

$$0 \in \partial f(y_n) + \frac{1}{r_n}(y_n - x_n + e_n).$$

It follows that

$$x_n - e_n \in y_n + r_n \partial f(y_n).$$

□

By using Theorem 2.1, we draw the desired conclusion immediately.

Theorem 6. Let C be a convex and closed set in a Hilbert space H . Let $f : H \rightarrow (-\infty, +\infty]$ be a proper convex lower semicontinuous function such that $(\partial f)^{-1}(0)$ is not empty. Let $\{\alpha_n\}$ be a real number sequence in $[0, a]$ for some $a \in [0, 1)$ and let $\{r_n\}$ be a real number sequence in $[b, c]$ for some $b, c \in (0, 1/\sqrt{2}L)$. Let $\{x_n\}$ be a vector sequence defined and generated in the following iterative process:

$$\begin{cases} x_0 \in H \text{ chosen arbitrarily,} \\ C_0 = C, \\ z_n = \arg \min_{z \in H} \{f(z) + \frac{\|z - x_n + e_n\|^2}{2r_n}\}, \\ y_n = \alpha_n x_n + (1 - \alpha_n) SProj_C(z_n + e_n), \\ C_{n+1} = \{w \in C_n : \|y_n - w\|^2 \leq \|x_n - w\|^2 - (1 - \alpha_n)(1 - 2r_n^2 L^2) \|x_n - z_n\|^2 + 2\|e_n\|^2\}, \\ x_{n+1} = Proj_{C_{n+1}}^H x_0, \quad n \geq 0, \end{cases}$$

where $\{e_n\}$ is an error sequence in H such that $\lim_{n \rightarrow \infty} \|e_n\| = 0$. Then, $\{x_n\}$ converges to $Proj_{(\partial f)^{-1}(0)}^H x_0$ strongly.

Proof. From the assumption that $f : H \rightarrow (-\infty, \infty]$ is proper, convex and lower semicontinuous, one sees that subdifferential ∂f is maximally monotone. Setting $A = 0$ and $z_n = J_{r_n}(x_n - e_n)$, one sees that

$$z_n = \arg \min_{z \in H} \{f(z) + \frac{\|z - x_n + e_n\|^2}{2r_n}\}$$

is equivalent to

$$0 \in \partial f(z_n) + \frac{1}{r_n}(z_n - x_n + e_n).$$

It follows that

$$x_n - e_n \in z_n + r_n \partial f(z_n).$$

□

By using Theorem 2, we draw the desired conclusion immediately.

Finally, we consider an equilibrium problem, which is also known as Ky Fan inequality [36], in the sense of Blum and Oettli [37].

We employ \mathbb{R} to denote the set of real numbers. Let F be a bifunction mapping $C \times C$ to \mathbb{R} . The equilibrium problem consists of

$$\text{Finding } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \tag{5}$$

Hereafter, $EP(F)$ means the solution set of problem (5).

In order to study solutions of equilibrium problem (5), the following routine restrictions on F are needed:

- (R1) for each $x \in C, y \mapsto F(x, y)$ is convex and lower semi-continuous;
- (R2) for each $x, y, z \in C, \limsup_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

- (R3) for each $x \in C, F(x, x) = 0$;
- (R4) for each $x, y \in C, F(x, y) + F(y, x) \leq 0$.

The following lemma is on a resolvent mapping associated with F , introduced in [38].

Lemma 4. Let $F : C \times C \rightarrow \mathbb{R}$ be a bifunction with restriction (R1)–(R4). Let $r > 0$ and $x \in H$. Then, there exists a vector $z \in C$ such that $\langle y - z, z - x \rangle + rF(z, y) \geq 0, \forall y \in C$. Define a mapping $T_r : H \rightarrow C$ by

$$T_r x = \left\{ z \in C : \langle y - z, z - x \rangle + rF(z, y) \geq 0, \forall y \in C \right\} \tag{6}$$

for each $x \in H$ and each $r > 0$. Then, (1) $Fix(T_r) = EP(F)$ is convex and closed; (2) T_r is single-valued firmly non-expansive.

Lemma 5. [39]. Let F be a bifunction with restrictions (R1)–(R4), and let A_F be a mapping on H defined by

$$A_F x = \begin{cases} \emptyset, & x \notin C, \\ \{z \in H : F(x, y) + \langle x - y, z \rangle \geq 0, \forall y \in C\}, & x \in C. \end{cases} \tag{7}$$

Then, A_F is a maximally monotone mapping such that $D(A_F) \subset C, EP(F) = A_F^{-1}(0)$, and $T_r x = (I + rA_F)^{-1}x, \forall x \in H, r > 0$, where T_r is defined as in (6).

Thanks to Lemmas 4 and 5, one finds from Theorem 1 and Theorem 2 the following results on equilibrium problem (5) immediately.

Theorem 7. Let C be a convex and closed set in a Hilbert space H . Let S be a non-expansive self mapping on C , whose fixed-point set is non-empty. Let $F : C \times C \rightarrow \mathbb{R}$ be a bifunction with restrictions (R1)–(R4). Assume that $Fix(S) \cap EP(F)$ is not empty. Let $\{\alpha_n\}$ be a real number sequence in $[a, b]$ for some $a, b \in (0, 1)$ and let $\{r_n\}$ be a real number sequence such that $r_n \geq c$, where c is some positive real number. Let $\{x_n\}$ be a vector sequence defined and generated in the following iterative process: $x_0 \in C, x_{n+1} = \alpha_n x_n + (1 - \alpha_n)S(I + rA_F)^{-1}x_n, \forall n \geq 0$, where A_F is defined by (7). Then, $\{x_n\}$ converges to a point $\bar{x} \in Fix(S) \cap EP(F)$ weakly.

Theorem 8. Let C be a convex and closed set in a Hilbert space H . Let S be a non-expansive self mapping on C , whose fixed-point set is non-empty. Let $F : C \times C \rightarrow \mathbb{R}$ be a bifunction with restrictions (R1)–(R4). Assume that $Fix(S) \cap EP(F)$ is not empty. Let $\{\alpha_n\}$ be a real number sequence in $[0, a]$ for some $a \in [0, 1)$ and let $\{r_n\}$ be a real number sequence such that $r_n \geq c$, where c is some positive real number. Let $\{x_n\}$ be a vector sequence defined and generated in the following iterative process:

$$\begin{cases} x_0 \in H \text{ chosen arbitrarily,} \\ C_0 = C, \\ y_n = \alpha_n x_n + (1 - \alpha_n)S(I + rA_F)^{-1}x_n, \\ C_{n+1} = \{w \in C_n : \|y_n - w\|^2 \leq \|x_n - w\|^2 - (1 - \alpha_n)(1 - 2r_n^2 L^2)\|x_n - z_n\|^2\}, \\ x_{n+1} = Proj_{C_{n+1}}^H x_0, \quad n \geq 0, \end{cases}$$

where A_F is defined by (7). Then, $\{x_n\}$ converges to $Proj_{Fix(S) \cap EP(F)}^H x_0$ strongly.

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