Article

On the Non-Hypercyclicity of Normal Operators, Their Exponentials, and Symmetric Operators

Marat V. Markin and Edward S. Sichel *

Department of Mathematics, California State University, Fresno, 5245 N. Backer Avenue, M/S PB 108, Fresno, CA 93740-8001, USA; mmarkin@csufresno.edu

* Correspondence: edsichel@mail.fresnostate.edu

Received: 23 August 2019; Accepted: 25 September 2019; Published: 27 September 2019

Abstract: We give a simple, straightforward proof of the non-hypercyclicity of an arbitrary (bounded or not) normal operator A in a complex Hilbert space as well as of the collection $\{e^{tA}\}_{t \geq 0}$ of its exponentials, which, under a certain condition on the spectrum of A , coincides with the C_0 -semigroup generated by it. We also establish non-hypercyclicity for symmetric operators.

Keywords: hypercyclicity; scalar type spectral operator; normal operator; C_0 -semigroup

MSC: Primary 47A16; 47B15; Secondary 47D06; 47D60; 34G10

1. Introduction

In [1], furnished is a straightforward proof of the non-hypercyclicity of an arbitrary (bounded or not) scalar type spectral operator A in a complex Banach space as well as of the collection $\{e^{tA}\}_{t \geq 0}$ of its exponentials (see, e.g., [2]), the important particular case of a normal operator A in a complex Hilbert space (see, e.g., [3,4]) following immediately.

Without the need to resort to the machinery of dual space, we provide a shorter, simpler, and more transparent direct proof for the normal operator case, in particular, generalizing the known result [5] [Corollary 5.31] for bounded normal operators, and further establish non-hypercyclicity for symmetric operators (see, e.g., [6]).

Definition 1 (Hypercyclicity). *Let*

$$A : X \supseteq D(A) \rightarrow X$$

($D(\cdot)$ is the domain of an operator) *be a (bounded or unbounded) linear operator in a (real or complex) Banach space $(X, \|\cdot\|)$. A vector*

$$f \in C^\infty(A) := \bigcap_{n=0}^{\infty} D(A^n)$$

($A^0 := I$, I is the identity operator on X) *is called hypercyclic if its orbit*

$$\text{orb}(f, A) := \{A^n f\}_{n \in \mathbb{Z}_+}$$

under A ($\mathbb{Z}_+ := \{0, 1, 2, \dots\}$ is the set of nonnegative integers) is dense in X .

Linear operators possessing hypercyclic vectors are said to be hypercyclic.

More generally, a collection $\{T(t)\}_{t \in J}$ (J is a nonempty indexing set) of linear operators in X is called hypercyclic if it possesses hypercyclic vectors, i.e., such vectors $f \in \bigcap_{t \in J} D(T(t))$, whose orbit

$$\{T(t)f\}_{t \in J}$$

is dense in X .

Cf. [5,7–12].

Remark 1.

- Clearly, hypercyclicity for a linear operator can only be discussed in a separable Banach space setting. Generally, for a collection of operators, this need not be the case.
- For a hypercyclic linear operator A , dense in $(X, \|\cdot\|)$ is the subspace $C^\infty(A)$ (cf., e.g., [1]), which, in particular, implies that any hypercyclic linear operator is densely defined (i.e., $\overline{D(A)} = X$).
- Bounded normal operators on a complex Hilbert space are known to be non-hypercyclic [5] [Corollary 5.31].

2. Preliminaries

Here, we briefly outline certain preliminaries essential for the subsequent discourse (for more, see, e.g., [13–15]).

Henceforth, unless specified otherwise, A is a *normal operator* in a complex Hilbert space $(X, (\cdot, \cdot), \|\cdot\|)$ with strongly σ -additive spectral measure (the resolution of the identity) $E_A(\cdot)$ assigning to Borel sets of the complex plane $\mathbb{C}$ orthogonal projection operators on X and having the operator's spectrum $\sigma(A)$ as its support [3,4].

Associated with a normal operator A is the Borel operational calculus assigning to any Borel measurable function $F : \sigma(A) \rightarrow \mathbb{C}$ a normal operator

$$F(A) := \int_{\sigma(A)} F(\lambda) dE_A(\lambda),$$

with

$$f \in D(F(A)) \Leftrightarrow \int_{\sigma(A)} |F(\lambda)|^2 d(E_A(\lambda)f, f) < \infty,$$

where $(E_A(\cdot)f, f)$ is a Borel measure, in which case [3,4]

$$\|F(A)f\|^2 = \int_{\sigma(A)} |F(\lambda)|^2 d(E_A(\lambda)f, f) \quad (1)$$

In particular,

$$A^n = \int_{\sigma(A)} \lambda^n dE_A(\lambda), \quad n \in \mathbb{Z}_+, \quad \text{and} \quad e^{tA} := \int_{\sigma(A)} e^{t\lambda} dE_A(\lambda), \quad t \in \mathbb{R}.$$

Provided

$$\sigma(A) \subseteq \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq \omega\}$$

with some $\omega \in \mathbb{R}$, the collection of exponentials $\{e^{tA}\}_{t \geq 0}$ is the C_0 -semigroup generated by A [4,16].

Remark 2.

- By [13] [Theorem 1], the orbits

$$y(t) = e^{tA}f, \quad t \geq 0, \quad f \in \bigcap_{t \geq 0} D(e^{tA}), \quad (2)$$

describe all weak/mild solutions of the abstract evolution equation

$$y'(t) = Ay(t), \quad t \geq 0, \quad (3)$$

(see [17], cf. also [16] [Ch. II, Definition 6.3]).

- The subspaces

$$C^\infty(A) \quad \text{and} \quad \bigcap_{t \geq 0} D(e^{tA})$$

of all possible initial values for the corresponding orbits are dense in X since they contain the subspace

$$\bigcup_{\alpha > 0} E_A(\Delta_\alpha)X, \text{ where } \Delta_\alpha := \{\lambda \in \mathbb{C} \mid |\lambda| \leq \alpha\}, \alpha > 0,$$

which is dense in X and coincides with the class $\mathcal{E}^{\{0\}}(A)$ of the entire vectors of A of exponential type (see, e.g., [18,19], cf. also [20]).

3. Normal Operators and Their Exponentials

We are to prove [1] [Corollary 4.1] directly generalizing in part [5] [Corollary 5.31].

Theorem 1 ([1] (Corollary 4.1)). *An arbitrary normal, in particular self-adjoint, operator A in a nonzero complex Hilbert space $(X, (\cdot, \cdot), \|\cdot\|)$ with spectral measure $E_A(\cdot)$ is not hypercyclic and neither is the collection $\{e^{tA}\}_{t \geq 0}$ of its exponentials, which, provided the spectrum of A is located in a left half-plane*

$$\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq \omega\},$$

with some $\omega \in \mathbb{R}$, is the C_0 -semigroup generated by A .

Proof. Let $f \in C^\infty(A) \setminus \{0\}$ be arbitrary.

There are two possibilities: either

$$E_A(\{\lambda \in \sigma(A) \mid |\lambda| > 1\})f \neq 0$$

or

$$E_A(\{\lambda \in \sigma(A) \mid |\lambda| > 1\})f = 0.$$

In the first case, for any $n \in \mathbb{Z}_+$,

$$\begin{aligned} \|A^n f\|^2 & \qquad \qquad \qquad \text{by (1);} \\ &= \int_{\sigma(A)} |\lambda|^{2n} d(E_A(\lambda)f, f) \geq \int_{\{\lambda \in \sigma(A) \mid |\lambda| > 1\}} |\lambda|^{2n} d(E_A(\lambda)f, f) \\ &\geq \int_{\{\lambda \in \sigma(A) \mid |\lambda| > 1\}} 1 d(E_A(\lambda)f, f) = (E_A(\{\lambda \in \sigma(A) \mid |\lambda| > 1\})f, f) \\ &= \|E_A(\{\lambda \in \sigma(A) \mid |\lambda| > 1\})f\|^2 > 0, \end{aligned}$$

which implies that the orbit $\operatorname{orb}(f, A)$ of f under A cannot approximate the zero vector, and hence, is not dense in X .

In the second case, since

$$f = E_A(\{\lambda \in \sigma(A) \mid |\lambda| > 1\})f + E_A(\{\lambda \in \sigma(A) \mid |\lambda| \leq 1\})f,$$

we infer that

$$f = E_A(\{\lambda \in \sigma(A) \mid |\lambda| \leq 1\})f \neq 0$$

and hence, for any $n \in \mathbb{Z}_+$,

$$\begin{aligned} \|A^n f\|^2 &= \|A^n E_A(\{\lambda \in \sigma(A) \mid |\lambda| \leq 1\}) f\|^2 \\ &\quad \text{by (1) and the properties of the operational calculus;} \\ &= \int_{\{\lambda \in \sigma(A) \mid |\lambda| \leq 1\}} |\lambda|^{2n} d(E_A(\lambda) f, f) \leq \int_{\{\lambda \in \sigma(A) \mid |\lambda| \leq 1\}} 1 d(E_A(\lambda) f, f) \\ &= (E_A(\{\lambda \in \sigma(A) \mid |\lambda| \leq 1\}) f, f) = \|E_A(\{\lambda \in \sigma(A) \mid |\lambda| \leq 1\}) f\|^2 = \|f\|^2, \end{aligned}$$

which also implies that the orbit $\text{orb}(f, A)$ of f under A , being bounded, is not dense in X and completes the proof for the operator case.

Now, let us consider the case of the exponential collection $\{e^{tA}\}_{t \geq 0}$ assuming that $f \in \bigcap_{t \geq 0} D(e^{tA}) \setminus \{0\}$ is arbitrary.

There are two possibilities: either

$$E_A(\{\lambda \in \sigma(A) \mid \text{Re } \lambda > 0\}) f \neq 0$$

or

$$E_A(\{\lambda \in \sigma(A) \mid \text{Re } \lambda > 0\}) f = 0.$$

In the first case, for any $t \geq 0$,

$$\begin{aligned} \|e^{tA} f\|^2 &\quad \text{by (1);} \\ &= \int_{\sigma(A)} |e^{t\lambda}|^2 d(E_A(\lambda) f, f) = \int_{\sigma(A)} e^{2t \text{Re } \lambda} d(E_A(\lambda) f, f) \\ &\geq \int_{\{\lambda \in \sigma(A) \mid \text{Re } \lambda > 0\}} e^{2t \text{Re } \lambda} d(E_A(\lambda) f, f) \geq \int_{\{\lambda \in \sigma(A) \mid \text{Re } \lambda > 0\}} 1 d(E_A(\lambda) f, f) \\ &= (E_A(\{\lambda \in \sigma(A) \mid \text{Re } \lambda > 0\}) f, f) = \|E_A(\{\lambda \in \sigma(A) \mid \text{Re } \lambda > 0\}) f\|^2 > 0, \end{aligned}$$

which implies that the orbit $\{e^{tA} f\}_{t \geq 0}$ of f cannot approximate the zero vector, and hence, is not dense in X .

In the second case, since

$$f = E_A(\{\lambda \in \sigma(A) \mid \text{Re } \lambda > 0\}) f + E_A(\{\lambda \in \sigma(A) \mid \text{Re } \lambda \leq 0\}) f,$$

we infer that

$$f = E_A(\{\lambda \in \sigma(A) \mid \text{Re } \lambda \leq 0\}) f \neq 0$$

and hence, for any $t \geq 0$,

$$\begin{aligned} \|e^{tA} f\|^2 &= \|e^{tA} E_A(\{\lambda \in \sigma(A) \mid \text{Re } \lambda \leq 0\}) f\|^2 \\ &\quad \text{by (1) and the properties of the operational calculus;} \\ &= \int_{\{\lambda \in \sigma(A) \mid \text{Re } \lambda \leq 0\}} |e^{t\lambda}|^2 d(E_A(\lambda) f, f) = \int_{\{\lambda \in \sigma(A) \mid \text{Re } \lambda \leq 0\}} e^{2t \text{Re } \lambda} d(E_A(\lambda) f, f) \\ &\leq \int_{\{\lambda \in \sigma(A) \mid \text{Re } \lambda \leq 0\}} 1 d(E_A(\lambda) f, f) = (E_A(\{\lambda \in \sigma(A) \mid \text{Re } \lambda \leq 0\}) f, f) \\ &= \|E_A(\{\lambda \in \sigma(A) \mid \text{Re } \lambda \leq 0\}) f\|^2 = \|f\|^2, \end{aligned}$$

which also implies that the orbit $\{e^{tA}f\}_{t \geq 0}$ of f , being bounded, is not dense on X and completes the proof of the exponential case and the entire statement. $\square$

4. Symmetric Operators

The following generalizes in part [5] [Lemma 2.53 (a)] to the case of a densely defined unbounded linear operator in a Hilbert space.

Lemma 1. *Let A be a hypercyclic linear operator in a nonzero Hilbert space $(X, (\cdot, \cdot), \|\cdot\|)$ over the scalar field $\mathbb{F}$ of real or complex numbers (i.e., $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$). Then*

1. *the adjoint operator A^* has no eigenvalues, or equivalently, for any $\lambda \in \mathbb{F}$, the range of the operator $A - \lambda I$ is dense in X , i.e.,*

$$\overline{R(A - \lambda I)} = X$$

($R(\cdot)$ is the range of an operator);

2. *provided the space X is complex (i.e., $\mathbb{F} = \mathbb{C}$) and the operator A is closed, the residual spectrum of A is empty, i.e.,*

$$\sigma_r(A) = \emptyset.$$

Proof.

1. Let $f \in X$ be a hypercyclic vector for A .

We proceed by contradiction, assuming that the adjoint operator A^* , which exists since A is densely defined (see Remark 1), has an eigenvalue $\lambda \in \mathbb{F}$, and hence,

$$\exists g \in X \setminus \{0\} : A^*g = \lambda g,$$

which, in particular, implies that $g \in C^\infty(A^*) := \bigcap_{n=0}^\infty D((A^*)^n)$ and

$$\forall n \in \mathbb{N} : (A^*)^n g = \lambda^n g.$$

In view of the above, we have inductively:

$$\forall n \in \mathbb{N} : (A^n f, g) = (A^{n-1} f, A^* g) = (f, (A^*)^n g) = (f, \lambda^n g) = \bar{\lambda}^n (f, g),$$

the conjugation being superfluous when the space is real.

Since $g \neq 0$, by the Riesz representation theorem (see, e.g., [21,22]), the hypercyclicity of f implies that the set

$$\{(A^n f, g)\}_{n \in \mathbb{N}}$$

is dense in $\mathbb{F}$, which contradicts the fact that the same set

$$\{\bar{\lambda}^n (f, g)\}_{n \in \mathbb{N}}$$

is clearly not.

Thus, the adjoint operator A^* has no eigenvalues.

The rest of the statement of part (1) immediately follows from the orthogonal sum decomposition

$$X = \ker(A^* - \bar{\lambda}I) \oplus \overline{R(A - \lambda I)}, \quad \lambda \in \mathbb{F},$$

the conjugation being superfluous when the space is real, (see, e.g., [21]).

2. Suppose that the space X is complex (i.e., $\mathbb{F} = \mathbb{C}$) and the operator A is closed. Recalling that

$$\sigma_r(A) = \left\{ \lambda \in \mathbb{C} \mid A - \lambda I \text{ is one-to-one and } \overline{R(A - \lambda I)} \neq X \right\}$$

(see, e.g., [21,23]), we infer from part (1) that

$$\sigma_r(A) = \emptyset.$$

□

We immediately arrive at the following

Proposition 1 (Non-Hypercyclicity Test). *Any densely defined closed linear operator A in a nonzero complex Hilbert space X with a nonempty residual spectrum (i.e., $\sigma_r(A) \neq \emptyset$) is not hypercyclic.*

Now, we are ready to prove the subsequent.

Theorem 2. *An arbitrary symmetric operator A in a complex Hilbert space X is not hypercyclic.*

Proof. Since

$$A \subseteq A^*,$$

without loss of generality, we can regard the symmetric operator A to be closed (see, e.g., [24]).

If both deficiency indices of the operator A are equal to zero, A is self-adjoint ($A = A^*$) (see, e.g., [6]), and hence, by Theorem 1, is not hypercyclic.

If at least one of the deficiency indices of the operator A is nonzero, then

$$\sigma_r(A) \neq \emptyset$$

(see, e.g., [6,22]), and hence, by Proposition 1, A is not hypercyclic. □

5. Some Examples

Example 1.

1. In the complex Hilbert space $L_2(\mathbb{R})$, the self-adjoint differential operator $A := i \frac{d}{dx}$ (i is the imaginary unit) with the domain

$$D(A) := W_2^1(\mathbb{R}) := \{f \in L_2(\mathbb{R}) \mid f(\cdot) \in AC(\mathbb{R}), f' \in L_2(\mathbb{R})\}$$

($AC(\cdot)$ is the set of absolutely continuous functions on an interval) is non-hypercyclic by Theorem 1 (cf. [1] [Corollary 5.1]).

2. In the complex Hilbert space $L_2(0, \infty)$, the symmetric differential operator $A := i \frac{d}{dx}$ with the domain

$$D(A) := \{f \in L_2(0, \infty) \mid f(\cdot) \in AC[0, \infty), f' \in L_2(0, \infty), f(0) = 0\}$$

and deficiency indices $(0, 1)$ is non-hypercyclic by Theorem 2.

3. In the complex Hilbert space $L_2(0, 2\pi)$, the symmetric differential operator $A := i \frac{d}{dx}$ with the domain

$$D(A) := \{f \in L_2(0, 2\pi) \mid f(\cdot) \in AC[0, 2\pi], f' \in L_2(0, 2\pi), f(0) = f(2\pi) = 0\}$$

and deficiency indices $(1, 1)$ is non-hypercyclic by Theorem 2.

Cf. [6] (Sections 49 and 80).

Author Contributions: Conceptualization, M.V.M.; methodology, M.V.M.; validation, E.S.S.; formal analysis, E.S.S.; investigation, M.V.M., E.S.S.; writing—original draft preparation, M.V.M., E.S.S.; writing—review and editing, M.V.M., E.S.S.; supervision, M.V.M.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

- Markin, M.V. On the non-hypercyclicity of scalar type spectral operators and collections of their exponentials. *arXiv* **2018**, arXiv:1807.07423.
- Dunford, N.; Schwartz, J.T.; Bade, W.G.; Bartle, R.G. *Linear Operators. Part III: Spectral Operators*; Interscience Publishers: New York, NY, USA, 1971.
- Dunford, N.; Schwartz, J.T.; Bade, W.G.; Bartle, R.G. *Linear Operators. Part II: Spectral Theory. Self Adjoint Operators in Hilbert Space*; Interscience Publishers: New York, NY, USA, 1963.
- Plesner, A.I. *Spectral Theory of Linear Operators*; Nauka: Moscow, Russia, 1965. (In Russian)
- Grosse-Erdmann, K.; Manguillot, A.P. *Linear Chaos*; Universitext; Springer: London, UK, 2011.
- Akhiezer, N.I.; Glazman, I.M. *Theory of Linear Operators in Hilbert Space*; Dover Publications, Inc.: New York, NY, USA, 1993.
- Bès, J.; Chan, K.C.; Seubert, S.M. Chaotic unbounded differentiation operators. *Integral Equations Oper. Theory* **2001**, *40*, 257–267. [[CrossRef](#)]
- deLaubenfels, R.; Emamirad, H.; Grosse-Erdmann, K. Chaos for semigroups of unbounded operators. *Math. Nachr.* **2003**, *261/262*, 47–59. [[CrossRef](#)]
- Guirao, A.J.; Montesinos, V.; Zizler, V. *Open Problems in the Geometry and Analysis of Banach Spaces*; Springer International Publishing: Cham, Switzerland, 2016.
- Markin, M.V. On the chaoticity and spectral structure of Rolewicz-type unbounded operators. *arXiv* **2018**, arXiv:1811.06640.
- Markin, M.V. On general construct of chaotic unbounded linear operators in Banach spaces with Schauder bases. *arXiv* **2018**, arXiv:1812.02294.
- Rolewicz, S. On orbits of elements. *Studia Math.* **1969**, *32*, 17–22. [[CrossRef](#)]
- Markin, M.V. On the strong smoothness of weak solutions of an abstract evolution equation. I. Differentiability. *Appl. Anal.* **1999**, *73*, 573–606 [[CrossRef](#)]
- Markin, M.V. On the strong smoothness of weak solutions of an abstract evolution equation. II. Gevrey ultradifferentiability. *IBID* **2001**, *78*, 97–137 [[CrossRef](#)]
- Markin, M.V. On the strong smoothness of weak solutions of an abstract evolution equation. III. Gevrey ultradifferentiability of orders less than one. *IBID* **2001**, *78*, 139–152. [[CrossRef](#)]
- Engel, K.; Nagel, R. *One-Parameter Semigroups for Linear Evolution Equations*; Graduate Texts in Mathematics; Springer: New York, NY, USA, 2000; Volume 194.
- Ball, J.M. Strongly continuous semigroups, weak solutions, and the variation of constants formula. *Proc. Am. Math. Soc.* **1977**, *63*, 370–373. [[CrossRef](#)]
- Gorbachuk, V.I.; Knyazyuk, A.V. Boundary values of solutions of operator-differential equations. *Russ. Math. Surv.* **1989**, *44*, 67–111. [[CrossRef](#)]
- Radyno, Y.V. The space of vectors of exponential type. *Dokl. Akad. Nauk BSSR* **1983**, *27*, 791–793.
- Markin, M.V. On the Carleman ultradifferentiable vectors of a scalar type spectral operator. *Methods Funct. Anal. Topol.* **2015**, *21*, 361–369.
- Markin, M.V. *Elementary Operator Theory*; De Gruyter Graduate; Walter de Gruyter GmbH: Berlin, Germany; Boston, MA, USA. (to appear).
- Markin, M.V. *Elementary Functional Analysis*; De Gruyter Graduate; Walter de Gruyter GmbH: Berlin, Germany; Boston, MA, USA, 2018.

23. Markin, M.V. On certain spectral features inherent to scalar type spectral operators. *Methods Funct. Anal. Topol.* **2017**, *23*, 60–65.
24. Dunford, N.; Schwartz, J.T.; Bade, W.G.; Bartle, R.G. *Linear Operators. Part I: General Theory*; Interscience Publishers: New York, NY, USA, 1958.

© 2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).