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Fixed Point Results for Generalized $\mathcal{F}$-Contractions in Modular b-Metric Spaces with Applications

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Abstract: The aim of this paper is to generalize the $\mathcal{F}$-contractive condition in the framework of $\alpha - \nu$-complete modular b-metric spaces. Some results in ordered modular b-metric spaces are also presented. Moreover, an illustrative example and some related applications are presented to support the obtained results.

Keywords: fixed point; modular b-metric space; $\mathcal{F}$-contraction

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1. Introduction

The concept of a b-metric space has been introduced by Bakhtin [1] and Czerwik [2,3]. So far, many interesting results about the existence of fixed points in b-metric spaces have been presented (see, e.g., [4–18].)

Definition 1 ([2]). *Let Ω be a nonempty set and $s \geq 1$ be a given real number. A function $B : \Omega \times \Omega \to [0, \infty)$ is a b-metric on Ω if, for all $\rho, \varrho, \sigma \in \Omega$, the following assertions hold:*

(b_1) $B(\rho, \varrho) = 0$ iff $\rho = \varrho$;
(b_2) $B(\rho, \varrho) = B(\varrho, \rho)$;
(b_3) $B(\rho, \sigma) \leq s[B(\rho, \varrho) + B(\varrho, \sigma)]$.

The pair (Ω, B) is called a b-metric space.

Note that a b-metric is not continuous in its two variables. On the other hand, a modular metric space is an applicable extension of classical modulars over linear spaces. The concept of a modular metric space has been introduced in [19–22]. Here, we deal with modular metric spaces as the nonlinear version of the classical one introduced by Nakano [23] on a vector space and a modular function space presented by Musielak [24] and Orlicz [25].

Let Ω be a nonempty set and let $\xi : (0, \infty) \times \Omega \times \Omega \to [0, \infty]$ be a function. For simplicity, we will write

$$\xi_\lambda(\rho, \varrho) = \xi(\lambda, \rho, \varrho),$$

for all $\lambda > 0$ and for all $\rho, \varrho \in \Omega$.

Definition 2 ([19,20]). *A function $\xi : (0,\infty) \times \Omega \times \Omega \to [0,\infty]$ is called a modular metric on Ω if, for all $\rho, \varrho, \sigma \in \Omega$ and $\lambda > 0$, the following assertions hold:*

(i) $\rho = \varrho$ *iff* $\xi_\lambda(\rho, \varrho) = 0$;
(ii) $\xi_\lambda(\rho, \varrho) = \xi_\lambda(\varrho, \rho)$;
(iii) $\xi_{\lambda+\mu}(\rho, \varrho) \leq \xi_\lambda(\rho, \sigma) + \xi_\mu(z, \varrho)$.

A modular metric ξ on Ω is called regular if the following weaker version of (i) holds:

$$\rho = \varrho \quad \text{iff} \quad \xi_\lambda(\rho, \varrho) = 0 \quad \text{for some} \quad \lambda = \omega > 0.$$

Ege and Alaca in [26] introduced the notion of modular b-metric spaces as follows.

Definition 3 ([26]). *Let Ω be a nonempty set and $s \geq 1$. A mapping $\nu : (0,\infty) \times \Omega \times \Omega \to [0,\infty]$ is called a modular b-metric, if for all $\rho, \varrho, \sigma \in \Omega$ and $\lambda > 0$, we have the following assertions:*

(i) $\nu_\lambda(\rho, \varrho) = 0$ *iff* $\rho = \varrho$;
(ii) $\nu_\lambda(\rho, \varrho) = \nu_\lambda(\varrho, \rho)$;
(iii) $\nu_{\lambda+\mu}(\rho, \varrho) \leq s[\nu_\lambda(\rho, \sigma) + \nu_\mu(z, \varrho)]$ *for all* $\lambda, \mu > 0$.

Then, we say that (Ω, ν) is a modular b-metric space.

Definition 4. *Let (Ω, ν) be a modular b-metric space. Let M be a subset of Ω and $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence in Ω. Then, we have the following statements:*

(i) *$\{\rho_n\}_{n\in\mathbb{N}}$ is called ν-convergent to ρ if there is $\lambda > 0$ (maybe it depends on $\{\rho_n\}$ and ρ), such that $\nu_\lambda(\rho_n, \rho) \to 0$, as $n \to \infty$. ρ will be called the ν-limit of (ρ_n).*
(ii) *$\{\rho_n\}_{n\in\mathbb{N}}$ is called ν-Cauchy if there is $\lambda > 0$ (maybe it depends on the sequence) such that $\nu_\lambda(\rho_m, \rho_n) \to 0$, as $m, n \to \infty$.*
(iii) *M is called ν-complete if any ν-Cauchy sequence in M is ν-convergent in M.*
(iv) *The function $f : (\Omega, \nu) \to (\Omega, \nu)$ is called ν-continuous if $\nu_\lambda(f\rho_n, f\rho) \to 0$, whenever $\nu_\lambda(\rho_n, \rho) \to 0$.*

Example 1. *Let (Ω, ξ) be an MbMS and let $p \geq 1$ be a real number. Take $\nu_\lambda(\rho, \varrho) = (\xi_\lambda(\rho, \varrho))^p$. Using the convexity of $f(t) = t^p$ for $t \geq 0$, by Jensen inequality, we have*

$$(a+b)^p \leq 2^{p-1}(a^p + b^p) \tag{1}$$

for nonnegative real numbers a, b. Thus, (Ω, ν) is an MbMS with $s = 2^{p-1}$.

Lemma 1. *Let (Ω, ν) be an MbMS. Suppose that $\{\rho_n\}$ and $\{\varrho_n\}$ ν-converge to ρ and ϱ, respectively. Then,*

$$\frac{1}{s^2}\nu_\lambda(\rho, \varrho) \leq \limsup_{n\longrightarrow\infty} \nu_{\frac{\lambda}{4}}(\rho_n, \varrho_n)$$

and

$$\limsup_{n\longrightarrow\infty} \nu_\lambda(\rho_n, \varrho_n) \leq s^2 \nu_{\frac{\lambda}{4}}(\rho, \varrho).$$

In particular, if $\rho = \varrho$, then $\lim_{n\longrightarrow\infty} \nu_\lambda(\rho_n, \varrho_n) = 0$. Moreover, for each $\sigma \in \Omega$, we have

$$\frac{1}{s}\nu_\lambda(\rho, \sigma) \leq \limsup_{n\longrightarrow\infty} \nu_{\frac{\lambda}{2}}(\rho_n, \sigma)$$

and

$$\limsup_{n\longrightarrow\infty} \nu_\lambda(\rho_n,\sigma) \le s\nu_{\frac{\lambda}{2}}(\rho,\sigma).$$

Proof. (a) Using the triangular inequality, one writes

$$\begin{aligned}\nu_\lambda(\rho,\varrho) &\le s(\nu_{\frac{\lambda}{2}}(\rho,\rho_n)+\nu_{\frac{\lambda}{2}}(\rho_n,\varrho))\\ &\le s(\nu_{\frac{\lambda}{2}}(\rho,\rho_n)+s(\nu_{\frac{\lambda}{4}}(\rho_n,\varrho_n)+\nu_{\frac{\lambda}{4}}(\varrho_n,\varrho)))\end{aligned}$$

and

$$\nu_\lambda(\rho_n,\varrho_n) \le s(\nu_{\frac{\lambda}{2}}(\rho_n,\rho)+s(\nu_{\frac{\lambda}{4}}(\rho,\varrho)+\nu_{\frac{\lambda}{4}}(\varrho,\varrho_n))).$$

As $n \to \infty$, we have

$$\nu_\lambda(\rho,\varrho) \le s^2(\limsup_{n\longrightarrow\infty} \nu_{\frac{\lambda}{4}}(\rho_n,\varrho_n)).$$

Taking the upper limit as $n \to \infty$ in the second inequality implies that

$$\limsup_{n\longrightarrow\infty} \nu_\lambda(\rho_n,\varrho_n) \le s^2(\nu_{\frac{\lambda}{4}}(\rho,\varrho)).$$

(b) Using the triangular inequality, one has

$$\nu_\lambda(\rho,\sigma) \le s(\nu_{\frac{\lambda}{2}}(\rho,\rho_n)+\nu_{\frac{\lambda}{2}}(\rho_n,\sigma))$$

and

$$\nu_\lambda(\rho_n,\sigma) \le s(\nu_{\frac{\lambda}{2}}(\rho_n,\rho)+\nu_{\frac{\lambda}{2}}(\rho,\sigma)).$$

We find that

$$\nu_\lambda(\rho,\sigma) \le s(\limsup_{n\longrightarrow\infty} \nu_{\frac{\lambda}{2}}(\rho_n,\sigma))$$

and

$$\limsup_{n\longrightarrow\infty} \nu_\lambda(\rho_n,\sigma) \le s(\nu_{\frac{\lambda}{2}}(\rho,\sigma)).$$

□

Definition 5 ([27]). *Given $\alpha:\Omega\times\Omega\to[0,\infty)$, where Ω is a nonempty set, the self-mapping T on Ω is said to be α-admissible if*

$$\rho,\varrho\in\Omega,\quad \alpha(\rho,\varrho)\ge 1 \quad\Longrightarrow\quad \alpha(T\rho,T\varrho)\ge 1.$$

Definition 6 ([28]). *Given $\alpha:\Omega\times\Omega\to[0,\infty)$, the mapping $T:\Omega\to\Omega$ is said to be triangular α-admissible if*

(i) $\rho,\varrho\in\Omega,\quad \alpha(\rho,\varrho)\ge 1 \quad\Longrightarrow\quad \alpha(T\rho,T\varrho)\ge 1;$

(ii) $\rho,\varrho,\sigma\in\Omega,\quad \begin{cases}\alpha(\rho,\sigma)\ge 1\\ \alpha(\sigma,\varrho)\ge 1\end{cases} \quad\Longrightarrow\quad \alpha(\rho,\varrho)\ge 1.$

Lemma 2 ([28]). *Let f be a triangular α-admissible mapping. Assume that there is $\rho_0 \in \Omega$ such that $\alpha(\rho_0, f\rho_0) \geq 1$. Define $\rho_n = f^n \rho_0$. Then,*

$$\alpha(\rho_m, \rho_n) \geq 1 \text{ for all } m, n \in \mathbb{N} \text{ with } m < n.$$

Definition 7. *Let (Ω, d) be a metric space and let $\alpha : \Omega \times \Omega \to [0, \infty)$ be a function. The metric space Ω is said to be α-complete if every Cauchy sequence $\{\rho_n\}$ in Ω with $\alpha(\rho_n, \rho_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, converges in Ω.*

Remark 1. *If (Ω, d) is a complete metric space, then Ω is also an α-complete metric space. The converse is not true.*

Definition 8 ([29]). *Let (Ω, d) be a metric space. Given $\alpha : \Omega \times \Omega \to [0, \infty)$, the mapping $T : \Omega \to \Omega$ is said to be α-continuous on (Ω, d), if for given $\rho \in \Omega$ and sequence $\{\rho_n\}$,*

$$\rho_n \to \rho \text{ as } n \to \infty \text{ and } \alpha(\rho_n, \rho_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N},$$

we have $T\rho_n \to T\rho$ as $n \to \infty$.

We extend Definitions 7 and 8 to modular b-metric spaces.

Definition 9. *Let (Ω, ν) be an MbMS and Υ be a self-mapping on Ω. Then, we have the following statements:*

(i) *(Ω, ν) is said to be $\alpha - \nu$-complete if every ν-Cauchy sequence $\{\rho_n\}$ in Ω with $\alpha(\rho_n, \rho_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ is ν-convergent to some ρ in Ω.*
(ii) *Υ is said to be $\alpha - \nu$-continuous on (Ω, ν), if for a ν-convergent sequence $\{\rho_n\}$ to some $\rho \in \Omega$ so that $\alpha(\rho_n, \rho_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $\{\Upsilon\rho_n\}$ is ν-convergent to $\Upsilon\rho$.*

Wardowski [30] presented a new type of contractions called $\mathcal{F}$-contractions and proved a new fixed point theorem as a generalization of the Banach contraction principle.

In this paper, we prove some fixed point results for generalized $\mathcal{F}$-contractive mappings in the setup of modular b-metric spaces. An example is presented to verify the effectiveness of our obtained results. Some applications are also presented at the end.

2. Main Results

Motivated by Wardowski [30] (see also [31]), we denote by Δ the set of all functions $\mathcal{F} : (0, \infty) \to \mathbb{R}$ such that:

$(\mathcal{F}_1)$ $\mathcal{F}$ is a continuous and strictly increasing mapping;
$(\mathcal{F}_2)$ for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n\to\infty} t_n = 0$ iff $\lim_{n\to\infty} \mathcal{F}(t_n) = -\infty$.

Example 2. *Let $t > 0$. The following functions:*

$\mathcal{F}(t) = \ln(t)$;

$\mathcal{F}(t) = 1 - \frac{1}{t^p}$ *(with $p > 0$)*;

$\mathcal{F}(t) = 1 - \frac{1}{e^t - 1}$;

$\mathcal{F}(t) = \frac{1}{e^{-t} - e^t}$

belong to Δ.

The class Δ is different from the class of functions introduced by Wardowski [30]. It suffices to $\mathcal{U}(t) = -\frac{1}{t} + t$ for $t > 0$. Note that $\lim_{\alpha \to 0^+} \alpha^k \mathcal{U}(\alpha) = -\infty$ $(0 < k < 1)$, that is, $\mathcal{U} \in \Delta$, but it is not a Wardowski mapping.

Let Θ denote the set of all functions $\vartheta : \mathbb{R} \to \mathbb{R}$ satisfying:

(ϑ_1) $\lim\limits_{n\to\infty} \vartheta^n(t) = -\infty$ for all $t > 0$;
(ϑ_2) $\vartheta(t) < t$ for all $t \geq 0$;
(ϑ_3) ϑ is an increasing continuous function.

Example 3. *The following functions: $\vartheta(t) = t - \delta$ (with $\delta > 0$) and $\vartheta = \sqrt[3]{t} - 1$ are elements in Θ.*

Definition 10. *Let (Ω, ν) be an MbMS and Υ be a self-mapping on Ω. Given $\alpha : \Omega \times \Omega \to [0, \infty)$. We say that Υ is an $\alpha - \vartheta$-$\mathcal{F}$-contraction if the following inequality:*

$$\mathcal{F}\big(s^3 \cdot \nu_\lambda(\Upsilon\rho, \Upsilon\varrho)\big) \leq \vartheta\big(\mathcal{F}(\nu_\lambda(\rho, \varrho)\big), \tag{2}$$

holds for all $\rho, \varrho \in \Omega$ with $\alpha(\rho, \varrho) \geq 1$ and $\nu_\lambda(\Upsilon\rho, \Upsilon\varrho) > 0$, where $\mathcal{F} \in \Delta$ and $\vartheta \in \Theta$.

From now on, assume that (Ω, ν) is regular.

Theorem 1. *Let $\alpha : \Omega \times \Omega \to [0, \infty)$ be a function and let (Ω, ν) be an α-ν_λ-complete MbMS. Assume that $\Upsilon : \Omega \to \Omega$ is such that*

(i) *Υ is triangular α-admissible;*
(ii) *Υ is an α-ϑ-$\mathcal{F}$-contraction;*
(iii) *there is $\rho_0 \in \Omega$ such that $\alpha(\rho_0, \Upsilon\rho_0) \geq 1$;*
(iv) *Υ is $\alpha - \nu$-continuous.*

Then, Υ has a fixed point. In addition, Υ has a unique fixed point, provided that $\alpha(\rho, \varrho) \geq 1$ for all $\rho, \varrho \in Fix(\Upsilon)$.

Proof. Let $\eta_0 \in \Omega$ satisfy $\alpha(\eta_0, \Upsilon\eta_0) \geq 1$. Define a sequence $\{\eta_n\}$ by $\eta_n = \Upsilon^n\eta_0 = \Upsilon\eta_{n-1}$. Since Υ is α-admissible, $\alpha(\eta_1, \eta_2) = \alpha(\Upsilon\eta_0, \Upsilon\eta_1) \geq 1$. Continuing this process, we have

$$\alpha(\eta_{n-1}, \eta_n) \geq 1$$

for all $n \in \mathbb{N}$. According to the triangular approach in assumption (i), one writes that

$$\alpha(\eta_m, \eta_n) \geq 1 \text{ for all } m, n \in \mathbb{N},\ m \neq n. \tag{3}$$

Suppose that there is $n_0 \in \mathbb{N}$ so that $\eta_{n_0} = \eta_{n_0+1}$. Then, η_{n_0} is a fixed point of Υ. Hence, suppose that $\eta_n \neq \eta_{n+1}$, i.e., $\nu_\lambda(\Upsilon\eta_{n-1}, \Upsilon\eta_n) > 0$ for all $n \in \mathbb{N}$.

We will show that

$$\lim_{n\to\infty} \nu_\lambda(\eta_n, \eta_{n+1}) = 0, \tag{4}$$

for all $\lambda > 0$. Since Υ is an α-ϑ-$\mathcal{F}$-contraction, we have

$$\mathcal{F}\big(\nu_\lambda(\eta_n, \eta_{n+1})\big) = \mathcal{F}\big(\nu_\lambda(\Upsilon\eta_{n-1}, \Upsilon\eta_n)\big) \leq \vartheta\big(\mathcal{F}(\nu_\lambda(\eta_{n-1}, \eta_n)\big),$$

which implies that

$$\mathcal{F}\big(\nu_\lambda(\eta_n, \eta_{n+1})\big) \leq \vartheta^n\big(\mathcal{F}(\nu_\lambda(\eta_0, \eta_1)\big) < \mathcal{F}(\nu_\lambda(\eta_0, \eta_1). \tag{5}$$

Taking the limit as $n \to \infty$ in Label (5) and using (ϑ_1), we have $\lim_{n\to\infty} \mathcal{F}(\nu_\lambda(\eta_n, \eta_{n+1})) = -\infty$. In view of $\mathcal{F} \in \Delta$, we get

$$\lim_{n\to\infty} \nu_\lambda(\eta_n, \eta_{n+1}) = 0.$$

Hence, Label (4) is proved for all $\lambda > 0$.

Next, we show that $\{\eta_n\}$ is a ν-Cauchy sequence in Ω, that is, there is some $\lambda > 0$ so that $\lim_{n,m} \nu_\lambda(\eta_{m_i}, \eta_{n_i}) = 0$.

Suppose there is $\varepsilon > 0$ for which for all $\lambda > 0$, we find $\{\eta_{m_i}\}$ and $\{\eta_{n_i}\}$ of $\{\eta_n\}$ so that n_i is the smallest index corresponding to

$$n_i > m_i > i \text{ and } \nu_\lambda(\eta_{m_i}, \eta_{n_i}) \geq \varepsilon, \tag{6}$$

for all $\lambda > 0$. This means that

$$\nu_\lambda(\eta_{m_i}, \eta_{n_i-1}) < \varepsilon. \tag{7}$$

From Label (6) and using the modular inequality, we get

$$\varepsilon \leq \eta_{4\lambda}(\eta_{m_i}, \eta_{n_i}) \leq s\eta_{2\lambda}(\eta_{m_i}, \eta_{m_i+1}) + s[s\nu_\lambda(\eta_{m_i+1}, \eta_{n_i+1}) + s\nu_\lambda(\eta_{n_i+1}, \eta_{n_i})].$$

Letting $i \to \infty$ and using Label (4), we get

$$\frac{\varepsilon}{s^2} \leq \limsup_{i\to\infty} \nu_\lambda(\eta_{m_i+1}, \eta_{n_i+1}). \tag{8}$$

We also have

$$\nu_\lambda(\eta_{m_i}, \eta_{n_i}) \leq s\nu_{\frac{\lambda}{2}}(\eta_{m_i}, \eta_{n_i-1}) + s\nu_{\frac{\lambda}{2}}(\eta_{n_i-1}, \eta_{n_i}).$$

Then, from (4) and (7), we get that

$$\limsup_{i\to\infty} \nu_\lambda(\eta_{m_i}, \eta_{n_i}) \leq s\varepsilon. \tag{9}$$

Because of (3), we can apply (13) to conclude that

$$\begin{aligned} \mathcal{F}(s^3 \cdot \nu_\lambda(\eta_{m_i+1}, \eta_{n_i+1})) &= \mathcal{F}(s^2 \cdot \nu_\lambda(\Upsilon\eta_{m_i}, \Upsilon\eta_{n_i})) \\ &\leq \vartheta(\mathcal{F}(\nu_\lambda(\eta_{m_i}, \eta_{n_i}))). \end{aligned} \tag{10}$$

Now, taking $i \to \infty$ in (10) and using $(\mathcal{F}_1)$, (8) and (9), we have

$$\begin{aligned} \mathcal{F}(s\varepsilon) = \mathcal{F}\left(s^3 \cdot \frac{\varepsilon}{s^2}\right) &\leq \mathcal{F}(s^3 \cdot \limsup_{i\to\infty} \nu_\lambda(\eta_{m_i+1}, \eta_{n_i+1})) \\ &\leq \vartheta(\mathcal{F}(\limsup_{i\to\infty} \nu_\lambda(\eta_{m_i}, \eta_{n_i}))) \\ &\leq \vartheta(\mathcal{F}(s\varepsilon)) < \mathcal{F}(s\varepsilon), \end{aligned}$$

which is a contradiction due to the property (ϑ_2).

Thus, $\{\eta_n\}$ is ν-Cauchy in the MbMS (Ω, ν_λ), that is, $\alpha - \nu$-complete, so since

$$\alpha(\eta_{n-1}, \eta_n) \geq 1$$

for all $n \in \mathbb{N}$, the sequence $\{\eta_n\}$ is ν-convergent to some $z \in \Omega$. Thus, there is some $\lambda > 0$ (without loss of generality, we choose $\lambda = \frac{\omega}{2} > 0$, where ω was given to ensure the regularity of (Ω, ν)), so that $\lim_{n\to\infty} \nu_\lambda(\eta_n, z) =: \lim_{n\to\infty} \nu_{\frac{\omega}{2}}(\eta_n, z) = 0$.

If $z \neq \Upsilon z$, then, using Lemma 1 and the $\alpha - \nu$-continuity of Υ, we have

$$\nu_\omega(z, \Upsilon z) \leq s[\nu_{\frac{\omega}{2}}(z, \Upsilon\eta_n) + \nu_{\frac{\omega}{2}}(\Upsilon\eta_n, \Upsilon z)].$$

Taking the limit as $n \to \infty$ and using again $\alpha - \nu$-continuity of Υ, we get that the the right-hand side goes to 0, so $\nu_\omega(z, \Upsilon z) = 0$. Using regularity of (Ω, ν), we obtain that $z = \Upsilon z$.

Let $\rho, \varrho \in \mathcal{F}ix(T)$ where $\rho \neq \varrho$ and $\alpha(\rho, \varrho) \geq 1$. We have

$$\mathcal{F}\big(\nu_\lambda(\Upsilon\rho, \Upsilon\varrho)\big) \leq \vartheta\big(\mathcal{F}\big(\nu_\lambda(\rho, \varrho)\big)\big) < \mathcal{F}\big(\nu_\lambda(\rho, \varrho)\big)\big).$$

It is a contradiction. We deduce that $\rho = \varrho$. Therefore, Υ has a unique fixed point. □

An MbMS (Ω, ν) is said to have the $\alpha - \nu$-sequential limit comparison property if, for each sequence $\{\eta_n\}$ in Ω so that $\alpha(\eta_n, \eta_{n+1}) \geq 1$ and ν-converges to $\rho \in \Omega$, one has $\alpha(\eta_n, \rho) \geq 1$ for all $n \in \mathbb{N}$.

Theorem 2. *Let $\alpha : \Omega \times \Omega \to [0, \infty)$ be a function and let (Ω, ν) be an $\alpha - \nu_\lambda$-complete MbMS. Let $\Upsilon : \Omega \to \Omega$ satisfy the following conditions:*

(i) *Υ is triangular α-admissible;*
(ii) *Υ is an $\alpha - \vartheta$-$\mathcal{F}$-contraction;*
(iii) *there is $\eta_0 \in \Omega$ such that $\alpha(\eta_0, \Upsilon\eta_0) \geq 1$;*
(iv) *(Ω, ν) enjoys the $\alpha - \nu$-sequential limit comparison property.*

Then, Υ has a fixed point. Furthermore, this fixed point is unique provided that $\alpha(\rho, \varrho) \geq 1$ for all $\rho, \varrho \in Fix(\Upsilon)$.

Proof. Let $\eta_0 \in \Omega$ be so that $\alpha(\eta_0, \Upsilon\eta_0) \geq 1$. As in the proof of Theorem 1, one has

$$\alpha(\eta_n, \eta_{n+1}) \geq 1 \quad \text{and } \eta_n \to \rho^* \text{ as } n \to \infty,$$

where $\eta_{n+1} = \Upsilon\eta_n$. Thus,

$$\alpha(\eta_{n+1}, \rho^*) \geq 1$$

for all $n \in \mathbb{N}$. For $\lambda = \omega > 0$, recall that (Ω, ν) is regular. By (13), we find that

$$\mathcal{F}\big(\nu_\omega(\Upsilon\eta_n, \Upsilon\rho^*)\big) \leq \vartheta\big(\mathcal{F}\big(\nu_\omega(\eta_n, \rho^*)\big)\big),$$

which implies

$$\mathcal{F}\big(\nu_\omega(\Upsilon\eta_n, \Upsilon\rho^*)\big) \leq \mathcal{F}\big(\nu_\omega(\eta_n, \rho^*)\big).$$

Using $(\mathcal{F}_1)$, we have

$$\nu_\omega(\eta_{n+1}, \Upsilon\rho^*) \leq \nu_\omega(\eta_n, \rho^*).$$

Suppose that $\rho^* \neq \Upsilon\rho^*$. By Lemma 1, we have

$$\frac{1}{s}\nu_\omega(\rho^*, \Upsilon\rho^*) \leq \limsup_{n\to\infty} \nu_\omega(\eta_{n+1}, \Upsilon\rho^*) \leq \limsup_{n\to\infty} \nu_\omega(\nu_n, \rho^*) = 0.$$

Thus, $\nu_\omega(\rho^*, \Upsilon\rho^*) = 0$. The regularity of (Ω, ν) implies that $\rho^* = \Upsilon\rho^*$. Thus, ρ^* is a fixed point of Υ. Its uniqueness comes as in Theorem 1. □

Taking $\vartheta(t) = t - \tau$ $(\tau > 0)$, an extension of Wardowski's result (Theorem 2.1 [30]) to the class of an MbMS is as follows.

Corollary 1. *Let (Ω, ν) be an $\alpha - \nu$-complete MbMS and $\Upsilon : \Omega \to \Omega$ be a self-mapping. Suppose that the following inequality:*

$$\tau + \mathcal{F}\big(s^3 \cdot \nu_\lambda(\Upsilon\rho, \Upsilon\varrho)\big) \leq \mathcal{F}\big(\nu_\lambda(\rho, \varrho)\big)$$

holds for all $\rho, \varrho \in \Omega$ with $\nu_\lambda(\Upsilon\rho, \Upsilon\varrho) > 0$, where $\tau > 0$. Then, Υ has a fixed point, if it satisfies the following conditions:

(iii) Υ *is $\alpha - \nu$-continuous, or*
(iii′) (Ω, ν) *enjoys the $\alpha - \nu$-sequential limit comparison property.*

By considering various functions $\mathcal{F} \in \Delta$ given in [30] and $\vartheta \in \Theta$, other results could be derived. We present an illustrative example.

Example 4. Let $\Omega = [0, \infty)$. Take the modular b-metric

$$\nu_\lambda(\rho, \varrho) = \begin{cases} \frac{(\rho^2+\varrho^2)^2}{\lambda}, & \text{if } \rho \neq \varrho, \\ 0, & \text{if } \rho = \varrho, \end{cases}$$

for all $\rho, \varrho \in \Omega$ and $\lambda > 0$. Define $\Upsilon : \Omega \to \Omega$, $\alpha : \Omega \times \Omega \to [0, \infty)$, $\vartheta : \mathbb{R} \to \mathbb{R}$ and $\mathcal{F} : (0, \infty) \to \mathbb{R}$ by

$$\Upsilon\rho = \begin{cases} 2\rho^2 + 1, & \text{if } \rho \in [0, 0.2), \\ \frac{1}{4}\rho^2, & \text{if } \rho \in [0.2, 1], \\ 3\rho - 1, & \text{if } \rho \in (1, 2), \\ 6\rho^{10} & \text{if } \rho \in [2, \infty), \end{cases}$$

,

$$\alpha(\rho, \varrho) = \begin{cases} 1, & \text{if } \rho, \varrho \in [0.2, 1], \\ 0, & \text{otherwise}, \end{cases}$$

$$\vartheta(\rho) = \begin{cases} \rho^3 - 1, & \text{if } \rho \in (-\infty, 1], \\ \rho - 1, & \text{otherwise}, \end{cases}$$

and $\mathcal{F}(t) = -\frac{1}{t} + t$, respectively.

Note that (Ω, ν) is a modular b-metric space with $s = 2$. Here, Υ is triangular α-admissible.

Let $\{\eta_n\}$ be in Ω so that $\alpha(\eta_n, \eta_{n+1}) \geq 1$ with $\eta_n \to \rho$ as $n \to \infty$, then $\eta_n \in [0.2, 1]$ for all $n \in \mathbb{N}$. Thus, $\rho \in [0.2, 1]$. This yields that $\alpha(\eta_n, \rho) \geq 1$ for all $n \in \mathbb{N}$. In addition, $\alpha(1, \Upsilon 1) \geq 1$. Moreover, $-\frac{1}{\frac{8\rho}{256}} + \frac{8\rho}{256} \leq (-\frac{1}{\rho} + \rho)^3 - 1$ for all $\rho \geq 0.2$. Now, for all ρ, ϱ with $\alpha(\rho, \varrho) \geq 1$, we have

$$\begin{aligned}
\mathcal{F}(s^3\nu_\lambda(\Upsilon\rho, \Upsilon\varrho)) &= -\frac{1}{s^3\nu_\lambda(\Upsilon\rho, \Upsilon\varrho)} + s^3\nu_\lambda(\Upsilon\rho, \Upsilon\varrho) \\
&= -\frac{1}{8\frac{(\Upsilon\rho^2+\Upsilon\varrho^2)^2}{\lambda}} + 8\frac{(\Upsilon\rho^2+\Upsilon\varrho^2)^2}{\lambda} \\
&= -\frac{1}{8\frac{((\frac{1}{4}\rho^2)^2+(\frac{1}{4}\varrho^2)^2)^2}{\lambda}} + 8\frac{((\frac{1}{4}\rho^2)^2+(\frac{1}{4}\varrho^2)^2)^2}{\lambda} \\
&\leq -\frac{1}{8\frac{((\frac{1}{4}\rho)^2+(\frac{1}{4}\varrho)^2)^2}{\lambda}} + 8\frac{((\frac{1}{4}\rho)^2+(\frac{1}{4}\varrho)^2)^2}{\lambda} \\
&\leq -\frac{1}{\frac{8\frac{(\rho^2+\varrho^2)^2}{\lambda}}{256}} + \frac{8\frac{(\rho^2+\varrho^2)^2}{\lambda}}{256} \\
&\leq -\frac{1}{\frac{8\nu_\lambda(\rho,\varrho)}{256}} + \frac{8\nu_\lambda(\rho,\varrho)}{256} \\
&\leq [-\frac{1}{\nu_\lambda(\rho,\varrho)} + \nu_\lambda(\rho,\varrho)]^3 - 1 \\
&= \vartheta(\mathcal{F}(\nu_\lambda(\rho,\varrho))).
\end{aligned}$$

Thus, Υ is an $\alpha - \vartheta$-$\mathcal{F}$ contraction. All the hypotheses of Theorem 2 are verified, so Υ has a fixed point.

A self-mapping Υ has the property P if $Fix(\Upsilon^n) = Fix(\Upsilon)$ for all $n \in \mathbb{N}$.

Theorem 3. *Let (Ω, ν) be an MbMS and $\Upsilon : \Omega \to \Omega$ be an $\alpha - \nu$-continuous self-mapping. Assume that there are $\vartheta \in \Theta$ and $\mathcal{F} \in \Delta$ such that*

$$\mathcal{F}\big(s^3\nu_\lambda(\Upsilon\rho, \Upsilon^2\rho)\big) \leq \vartheta\big(\mathcal{F}\big(\nu_\lambda(\rho, \Upsilon\rho)\big)\big) \tag{11}$$

for all $\rho \in \Omega$ with $\nu_\lambda(\Upsilon\rho, \Upsilon^2\rho) > 0$. If Υ is α-admissible and there exists $\eta_0 \in \Omega$ in order that $\alpha(\eta_0, \Upsilon\eta_0) \geq 1$, then Υ has the property P.

Proof. Let $n > 1$. Assume contrarily that $w \in Fix(\Upsilon^n)$ and $w \notin Fix(\Upsilon)$. Then, $\nu_{\lambda_0}(w, \Upsilon w) > 0$ for some $\lambda_0 > 0$. Now, we have

$$\begin{aligned}
\mathcal{F}(\nu_{\lambda_0}(w, \Upsilon w)) &= \mathcal{F}(\nu_{\lambda_0}(\Upsilon(\Upsilon^{n-1}w)), \Upsilon^2(\Upsilon^{n-1}w))) \\
&\leq \vartheta(\mathcal{F}(\nu_{\lambda_0}(\Upsilon^{n-1}w), \Upsilon^n w))) \\
&\leq \vartheta^2(\mathcal{F}(\nu_{\lambda_0}(\Upsilon^{n-2}w), \Upsilon^{n-1}w))) \leq \cdots \\
&\leq \vartheta^n(\nu_{\lambda_0}(w, \Upsilon w)).
\end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, we have $\mathcal{F}(\nu_{\lambda_0}(w, \Upsilon w)) = -\infty$. Hence, by $(\mathcal{F}_2)$, we get $\nu_{\lambda_0}(w, \Upsilon w) = 0$, which is a contradiction. Therefore, $Fix(\Upsilon^n) = Fix(\Upsilon)$ for all $n \in \mathbb{N}$. □

3. Results in Ordered Modular b-Metric Spaces

Let $(\Omega, \nu, \preceq)$ be a partially ordered MbMS and let Υ be a self-mapping on Ω.

Definition 11. *(i) (Ω, ν) is said to be $\preceq -\nu$-complete if every ν-Cauchy sequence $\{\rho_n\}$ in Ω with $\rho_n \preceq \rho_{n+1}$ for all $n \in \mathbb{N}$, ν-converges in Ω.*
(ii) Υ is said to be $\preceq -\nu$-continuous on (Ω, ν), if, for a ν-convergent sequence $\{\rho_n\}$ to some $\rho \in \Omega$ so that $\rho_n \preceq \rho_{n+1}$ for all $n \in \mathbb{N}$, we have $\{\Upsilon\rho_n\}$ is ν-convergent to $\Upsilon\rho$.
(iii) (Ω, ν) is said to have the $\preceq -\nu$-sequential limit comparison property if, for each sequence $\{\eta_n\}$ in Ω so that $\eta_n \preceq \eta_{n+1}$ and is ν-convergent to $\rho \in \Omega$, one has $\eta_n \preceq \rho$ for all $n \in \mathbb{N}$.

Now, we say that Υ is an $\preceq$-ϑ-$\mathcal{F}$-contraction if for all $\rho, \varrho \in \Omega$ with $\rho \preceq \varrho$ and $\nu_\lambda(\Upsilon\rho, \Upsilon\varrho) > 0$, we have

$$\mathcal{F}\big(s^3 \cdot \nu_\lambda(\Upsilon\rho, \Upsilon\varrho)\big) \leq \vartheta\big(\mathcal{F}(\nu_\lambda(\rho, \varrho))\big), \tag{12}$$

where $\mathcal{F} \in \Delta$ and $\vartheta \in \Theta$.

Using the above statements and applying Theorem 1, we have the following result.

Theorem 4. *Let $(\Omega, \nu, \preceq)$ be an $\preceq -\nu$-complete partially ordered MbMS. Assume that*

(i) *Υ is a $\preceq -\vartheta$-$\mathcal{F}$-contraction;*
(ii) *Υ is nondecreasing;*
(iii) *there is $\eta_0 \in \Omega$ so that $\eta_0 \preceq \Upsilon\eta_0$;*
(iv) *either Υ is $\preceq -\nu$-continuous, or $(\Omega, \nu, \preceq)$ possesses the $\preceq -\nu$-sequential limit comparison property.*

Then, Υ has a fixed point.

Again, we apply Theorem 2 to state the following result.

Theorem 5. *Let $(\Omega, \nu, \preceq)$ be an $\preceq -\nu$-complete partially ordered MbMS. Assume that*

(i) *the inequality (11) holds for all $\rho \in \Omega$ with $\nu_\lambda(\Upsilon\rho, \Upsilon^2\rho) > 0$.*
(ii) *Υ is nondecreasing;*
(iii) *there is $\eta_0 \in \Omega$ such that $\eta_0 \preceq \Upsilon\eta_0$;*
(iv) *either Υ is $\preceq -\nu$-continuous, or $(\Omega, \nu, \preceq)$ possesses the $\preceq -\nu$-sequential limit comparison property.*

Then, Υ has the property P.

4. Applications

In [17], Hussain and Salimi presented the relationship between modular metrics and fuzzy metrics and deduced certain fixed point results in triangular partially ordered fuzzy metric spaces.

Definition 12 ([32]). *A binary operation $\star : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if it satisfies the following assertions:*

$(T1)$ *$\star$ is commutative and associative;*
$(T2)$ *$\star$ is continuous;*
$(T3)$ *$a \star 1 = a$ for all $a \in [0,1]$;*
$(T4)$ *$a \star b \leq c \star d$ when $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0,1]$.*

Definition 13. *A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$ and for all $t, s > 0$,*

(i) *$M(x, y, t) > 0$;*
(ii) *$M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;*
(iii) *$M(x, y, t) = M(y, x, t)$;*

(iv) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$;
(v) $M(x,y,.) : (0,\infty) \to [0,1]$ *is continuous.*

The function $M(x,y,t)$ *denotes the degree of nearness between* x *and* y *with respect to* t*.*

Definition 14 ([33])**.** *A fuzzy b-metric space is an ordered triple* $(X,B,\star)$ *such that* X *is a nonempty set,* $\star$ *is a continuous t-norm and* B *is a fuzzy set on* $X \times X \times (0,\infty)$ *satisfying the following conditions, for all* $x,y,z \in X$ *and for all* $t,s > 0$*:*

$(F1)$ $B(x,y,t) > 0$;
$(F2)$ $B(x,y,t) = 1$ *if and only if* $x = y$;
$(F3)$ $B(x,y,t) = B(y,x,t)$;
$(F4)$ $B(x,y,t) \star B(y,z,s) \leq B(x,z,b(t+s))$ *where* $b \geq 1$;
$(F5)$ $B(x,y,\cdot) : (0,\infty) \to (0,1]$ *is left-continuous.*

Definition 15 ([33])**.** *Let* $(X,B,\star)$ *be a fuzzy b-metric space (in short, FbMS). Then,*

(i) *a sequence* $\{x_n\}$ *converges to* $x \in X$*, if and only if* $\lim_{n\to\infty} B(x_n,x,t) = 1$ *for all* $t > 0$;
(ii) *a sequence* $\{x_n\}$ *in* X *is a Cauchy sequence if and only if, for all* $\epsilon \in (0,1)$ *and for all* $t > 0$*, there exists* n_0 *such that* $B(x_n,x_m,t) > 1-\epsilon$ *for all* $m,n \geq n_0$;
(iii) *the fuzzy b-metric space is called complete if every Cauchy sequence converges to some* $x \in X$*.*

Definition 16 ([33])**.** *The fuzzy b-metric space* $(X,B,*)$ *is called triangular whenever*

$$\frac{1}{B(x,y,t)} - 1 \leq s[\frac{1}{B(x,z,t)} - 1 + \frac{1}{B(z,y,t)} - 1]$$

for all $x,y,z \in X$ *and for all* $t > 0$*.*

Motivated by Lemmas 33 and 34 of [34], we present the following.

Remark 2. *Let* $(X,B,*)$ *be a triangular fuzzy b-metric space. Define* $\nu : X \times X \times (0,\infty) \to [0,\infty)$ *by* $\nu(x,y,t) = s[\frac{1}{B(x,y,t)} - 1]$*. Then,* ν *is a modular b-metric.*

In view of Remark 2 and applying the results established in Section 2, we can deduce the following results in fuzzy b-metric spaces.

Definition 17. *Let* $(\Omega,B,*)$ *be an FbMS and* Υ *be a self-mapping on* Ω*. We say that* Υ *is a fuzzy* α*-*ϑ*-*$\mathcal{F}$*-contraction if, for all* $\rho,\varrho \in \Omega$ *with* $\alpha(\rho,\varrho) \geq 1$ *and* $B(\Upsilon\rho,\Upsilon\varrho,t) < 1$ $(t > 0)$*, we have*

$$\mathcal{F}(\frac{s^4}{B(\Upsilon\rho,\Upsilon\varrho,t)} - s^4) \leq \vartheta(\mathcal{F}(\frac{s}{B(\rho,\varrho,t} - s)), \tag{13}$$

where $\mathcal{F} \in \Delta$ *and* $\vartheta \in \Theta$*.*

In addition, note that Definition 9 could be derived for FbMS.

Theorem 6. *Let* $\alpha : \Omega \times \Omega \to [0,\infty)$ *be a function and let* $(\Omega,B,*)$ *be an* α*-*ν*-complete FbMS. Assume that* $\Upsilon : \Omega \to \Omega$ *is such that*

(i) Υ *is triangular* α*-admissible;*
(ii) Υ *is a fuzzy* α*-*ϑ*-*$\mathcal{F}$*-contraction;*
(iii) *there is* $\rho_0 \in \Omega$ *such that* $\alpha(\rho_0,\Upsilon\rho_0) \geq 1$;

(iv) Υ *is* $\alpha - B$*-continuous.*

Then, Υ *has a fixed point. In addition,* Υ *has a unique fixed point, provided that* $\alpha(\rho,\varrho) \geq 1$ *for all* $\rho,\varrho \in Fix(\Upsilon)$.

Proof. It follows from Theorem 1. □

Theorem 7. *Let* $\alpha : \Omega \times \Omega \to [0,\infty)$ *be a function and let* $(X, B, *)$ *be an* α*-*ν*-complete FbMS. Let* $\Upsilon : \Omega \to \Omega$ *satisfy the following conditions:*

(i) Υ *is triangular* α*-admissible;*
(ii) Υ *is a fuzzy* α*-*ϑ*-*$\mathcal{F}$*-contraction;*
(iii) there is $\eta_0 \in \Omega$ *such that* $\alpha(\eta_0, \Upsilon\eta_0) \geq 1$*;*
(iv) $(\Omega, B, *)$ *enjoys the* $\alpha - B$*-sequential limit comparison property.*

Then, Υ *has a fixed point. Furthermore, this fixed point is unique provided that* $\alpha(\rho,\varrho) \geq 1$ *for all* $\rho,\varrho \in Fix(\Upsilon)$.

Proof. It follows from Theorem 2. □

Theorem 8. *Let* $(X, B, *)$ *be an FbMS and* $\Upsilon : \Omega \to \Omega$ *be an* $\alpha - \nu$*-continuous self-mapping. Assume that there are* $\vartheta \in \Theta$ *and* $\mathcal{F} \in \Delta$ *such that*

$$\mathcal{F}\left(\frac{s^4}{B\left(\Upsilon\rho,\Upsilon^2\rho,t\right)} - s^4\right)\right) \leq \vartheta\left(\mathcal{F}\left(\frac{s}{B\left(\rho,\Upsilon\rho,t\right)} - s\right)\right) \tag{14}$$

for all $\rho \in \Omega$ *with* $B(\Upsilon\rho, \Upsilon^2\rho, t) < 1$ *. If* Υ *is* α*-admissible and there exists* $\eta_0 \in \Omega$ *in order that* $\alpha(\eta_0, \Upsilon\eta_0) \geq 1$*, then* Υ *has the property P.*

Proof. It follows from Theorem 3. □

Remark 3. *The analogue of Theorem 4 and Theorem 5 could be derived easily in the context of partially ordered fuzzy b-metric spaces.*

Now, we consider the following boundary value problem:

$$\begin{cases} y''(x) = f(x, y(x)), & x \in [0,1], \\ y(0) = y(1) = 0, \end{cases}$$

where $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function.

The above equation can be transformed to the following Fredholm integral equation:

$$y(x) = -\int_0^1 K(x,t) f(t, y(t))dt, \tag{15}$$

where the kernel is given by

$$K(x,t) = \begin{cases} t(1-x), & \text{if } t \in [0,x], \\ x(1-t), & \text{if } t \in [x,1]. \end{cases}$$

See [35] for details.

Now, to give an existence theorem for a solution of (15) that belongs to $X = C(I, \mathbb{R})$ (the set of continuous real functions defined on $I = [0,1]$), note that the space X endowed with the b-metric given by

$$d(x,y) = \max_{t \in I} \mid x(t) - y(t) \mid^2$$

for all $x, y \in X$ is a b-complete b-metric space ($s = 2^{2-1}$).

Take on X the partial order $\preceq$ given by

$$x \preceq y \Longleftrightarrow x(t) \leq y(t),$$

for all $x, y \in X$ and $t \in I$.

For $\rho \in X$, define

$$\|\rho\|_\infty = \sup_{t \in I} |\rho(t)|.$$

Here, $(X, \|\cdot\|_\infty)$ is a Banach space. The modular metric induced by this norm is

$$\mu_\lambda(\rho, \varrho) = \frac{\|\rho - \varrho\|_\infty}{\lambda} = \max_{t \in I} \frac{|\rho(t) - \varrho(t)|}{\lambda}, \quad \lambda > 0,$$

for all $\rho, \varrho \in X$. We consider the modular b-metric ν given by

$$\nu_\lambda(\rho, \varrho) = \frac{\|\rho - \varrho\|_\infty^2}{\lambda^2} = \max_{t \in I} \frac{|\rho(t) - \varrho(t)^2}{\lambda^2}.$$

Define $\Upsilon : X \to X$ by

$$\Upsilon\rho(x) = -\int_0^1 K(x,t) f(t, \rho(t)) dt, \quad \rho \in X, \ x \in I.$$

Clearly, a function $u \in X$ is a solution of (15) if and only if it is a fixed point of Υ.

Consider the following assumptions:

(C1) For all $u, v \in \mathbb{R}$ with $u \preceq v$ and for all $t \in I$,

$$\Big| f(t,u) - f(t,v) \Big|^2 \leq \frac{\|u - v\|_\infty^2}{8}.$$

(C2) There is $\eta_0 : I \to \mathbb{R}$ so that

$$\eta_0(x) \leq -\int_0^1 K(x,t) f(t, \eta_0(t)) dt, \quad x \in I.$$

(C3) $f(t,.) : \mathbb{R} \longrightarrow \mathbb{R}$ is nonincreasing for all $t \in [0,1]$.

Theorem 9. *Assume that above assumptions $(C1) - (C3)$ hold. Then, Equation (15) has a solution in X.*

Proof. First, by assumption $(C2)$, we have $\eta_0 \preceq \Upsilon\eta_0$. Clearly, Υ is $\preceq -\nu$-continuous and nondecreasing.

To show that all the assumptions of Theorem 4 are satisfied, it remains to prove that Υ is an $\preceq -\vartheta$-$\mathcal{F}$-contraction. Let $\rho, \varrho \in X$ with $\rho \preceq \varrho$. For each $x \in I$, we have

$$\begin{aligned}
&\left|\Upsilon\rho(x) - \Upsilon\varrho(x)\right|^2 = \Big| \int_0^1 K(x,t) f(t,\rho(t))dt - \int_0^1 K(x,t) f(t,\varrho(t))dt \Big|^2 \\
&= \int_0^1 (K(x,t)|f(t,\rho(t)) - f(t,\varrho(t))|)^2 \, dt \\
&\leq \Big[\int_0^1 |K(x,t)|^2 dt \Big] \Big[\int_0^1 |f(t,\rho(t))dt - f(t,\varrho(t))|^2 dt \Big] \\
&\leq \Big[\int_0^1 |K(x,t)|^2 dt \Big] \int_0^1 \frac{||\rho(t) - \varrho(t)||_\infty^2}{8} dt \\
&\leq \Big[\int_0^1 |K(x,t)|^2 dt \Big] \frac{||\rho - \varrho||_\infty^2}{8}.
\end{aligned}$$

Via a careful calculation, we get that

$$\int_0^1 |K(x,t)|^2 dt = \frac{(1-x)^2 x^3 + x^2(1-x)^3}{3}, \quad x \in [0,1].$$

We obtain that

$$\left|\Upsilon\rho(x) - \Upsilon\varrho(x)\right|^2 \leq \Big[\frac{(1-x)^2 x^3}{3} + \frac{x^2(1-x)^3}{3} \Big] \frac{||\rho - \varrho||_\infty^2}{8}. \tag{16}$$

Taking the supremum on $x \in [0,1]$, we deduce that

$$\left|\Upsilon\rho - \Upsilon\varrho\right|_\infty^2 \leq \frac{21}{1000} \frac{||\rho - \varrho||_\infty^2}{8}.$$

Now, one writes

$$\begin{aligned}
\ln\Big(\frac{s^3 \left|\Upsilon\rho - \Upsilon\varrho\right|_\infty^2}{\lambda^2}\Big) &= \ln\Big(\frac{8\left|\Upsilon\rho - \Upsilon\varrho\right|_\infty^2}{\lambda^2}\Big) \\
&\leq \ln\Big(\frac{21}{1000}\Big) + \ln\Big(\frac{||\rho - \varrho||_\infty^2}{\lambda^2}\Big) \\
&\leq \ln\Big(\frac{21}{1000}\Big) + \ln(\nu_\lambda(\rho, \varrho)).
\end{aligned}$$

That is,

$$\mathcal{F}\big(s^3 \cdot \eta(\Upsilon\rho, \Upsilon\varrho)\big) \leq \vartheta\big(\mathcal{F}(\nu_\lambda(\rho, \varrho)\big), \tag{17}$$

where $\mathcal{F}(t) = \ln t$ and $\vartheta(t) = t - \delta$ with $\delta = -\ln(\frac{21}{1000}) > 0$ (Example 3). Thus, all the hypotheses of Theorem 4 are fulfilled and we deduce the existence of $u \in X$ such that $u = \Upsilon u$. □

5. Conclusions

We presented some fixed point results for generalized $\mathcal{F}$-contractions in the setting of modular b-metric spaces. We also established some related results in fuzzy b-metric spaces. At the end, we resolved a Fredholm type integral equation.

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