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Statistical Properties and Different Methods of Estimation for Type I Half Logistic Inverted Kumaraswamy Distribution

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Abstract: In this paper, we introduce and study a new three-parameter lifetime distribution constructed from the so-called type I half-logistic-G family and the inverted Kumaraswamy distribution, naturally called the type I half-logistic inverted Kumaraswamy distribution. The main feature of this new distribution is to add a new tuning parameter to the inverted Kumaraswamy (according to the type I half-logistic structure), with the aim to increase the flexibility of the related inverted Kumaraswamy model and thus offering more precise diagnostics in data analyses. The new distribution is discussed in detail, exhibiting various mathematical and statistical properties, with related graphics and numerical results. An exhaustive simulation was conducted to investigate the estimation of the model parameters via several well-established methods, including the method of maximum likelihood estimation, methods of least squares and weighted least squares estimation, and method of Cramer-von Mises minimum distance estimation, showing their numerical efficiency. Finally, by considering the method of maximum likelihood estimation, we apply the new model to fit two practical data sets. In this regards, it is proved to be better than recent models, also derived to the inverted Kumaraswamy distribution.

Keywords: half-logistic distribution; inverted Kumaraswamy distribution; estimation methods; data analysis

MSC: 60E05; 62E15; 62F10

1. Introduction

Over the last years, numerous families of distributions have been introduced, providing new opportunities in terms of statistical modeling and data analysis. One of the most promising of them is the so-called type I half-logistic-G (TIHL-G) family introduced by [1]. The TIHL-G family is characterized by the cumulative distribution function (cdf) given by

$$F(x;\lambda,\xi) = \frac{1-[1-G(x;\xi)]^{\lambda}}{1+[1-G(x;\xi)]^{\lambda}}, \quad x \in \mathbb{R},$$

where $G(x;\xi)$ is a baseline cdf of a continuous distribution depending on a parameter vector ξ and $\lambda > 0$. The definition of $F(x;\lambda,\xi)$ is derived to the T-X technique of [2] applied with the half-logistic distribution as main generator, that is, $F(x;\lambda,\xi) = \int_0^{-\log[1-G(x;\xi)]} f_0(t;\lambda)dt$, where $f_0(t;\lambda) = 2\lambda e^{-\lambda t}/(1+e^{-\lambda t})^2$, $t > 0$, is the probability density function (pdf) of the half-logistic distribution. Among the qualities of the TIHL-G family, we would like to mention the tractability of the corresponding functions and the fact that the additional parameters λ can significantly increase the flexibility of the mode and the tails of the former distribution characterized by $G(x;\xi)$. That explains why several special members have been recently studied in detail. We refer the reader to [3] for the type I half-logistic Burr XII distribution (also called extended Bur XII distribution), [4] for the type I half-logistic generalized Weibull distribution, [5] for the type I half-logistic Lomax distribution, and [6] for the type I half-logistic exponential distribution. Also, with the use of an additional parameter, the type I half-logistic-G family was recently generalized by [7].

In parallel, several new flexible continuous distributions was studied. In particular, [8] recently introduced a new two-parameter lifetime distribution, called the inverted Kumaraswamy (IK) distribution. It is characterized by the cdf given by

$$G(x;a,b) = \left[1-(1+x)^{-a}\right]^b, \quad x > 0,$$

where $a, b > 0$. The basics of the IK distribution is as follows: It is the distribution of the random variable $V = 1/U - 1$, where U is a random variable following the standard Kumaraswamy distribution introduced by [9], i.e., with the cdf given by $G_*(x;a,b) = 1-(1-x^a)^b$, $x \in (0,1)$. Among the features of this distribution, it covers many well-established distributions as the Lomax (Pareto type II) distribution when $b = 1$, the beta type II (inverted beta) distribution when $a = 1$, the log-logistic (Fisk) distribution when $a = b = 1$, the inverted Weibull distribution when $b \to +\infty$ and the generalized exponential distribution when $a \to +\infty$. Also, the IK distribution demonstrates a great flexibility in terms of curvature of the distribution functions, specially on the mode and the tails of the distribution.

In this paper, we define a new three-parameter lifetime distribution by mixing the TIHL-G family and the IK distribution. Thus, it can be viewed as a new special member of the TIHL-G family benefiting from the qualities of the IK distribution, naturally called the type I half-logistic inverted Kumaraswamy (TIHLIK) distribution. This study is devoted to both its theoretical and practical features, with an emphasis on the applied side. Indeed, a substantial part is devoted to the estimation of the model parameters via various methods, with all the details. Nice performance of the estimates are observed, with discussions. Then, we show that the related TIHLIK model shows better fits for some data sets when compared to recent rivals, also defined with the IK model as baseline. The required computations are carried out in the R-language introduced by R Development Core Team [10]. Beyond the data analysis, the TIHLIK distribution (and its symmetric version around 0) can find applications in many other applied domains. For instance, it can be used to construct new prior distributions in a Bayesian setting and new mixtures of distributions in a discriminant analysis framework. Modern developments in these directions can be found in [11,12].

The rest of the paper is divided into six sections. Section 2 presents the TIHLIK distribution. Some mixture representations of the main functions in terms of Lomax distribution functions are given in Section 3. In Section 4, we attempt to derive the main mathematical and statistical properties of the TIHLIK distribution. Section 5 is devoted to the estimation of the model parameters, with a simulation study. Applications are given in Section 6. Section 7 provides concluding remarks.

2. The TIHLIK Distribution

By mixing the TIHL-G family and the IK distribution, the cdf of the TIHLIK distribution is given by

$$F(x;\lambda,a,b)=\frac{1-\left\{1-[1-(1+x)^{-a}]^{b}\right\}^{\lambda}}{1+\left\{1-[1-(1+x)^{-a}]^{b}\right\}^{\lambda}},\quad x>0. \tag{1}$$

Upon differentiation, the corresponding pdf is given by

$$f(x;\lambda,a,b)=\frac{2\lambda ab(1+x)^{-a-1}\,[1-(1+x)^{-a}]^{b-1}\left\{1-[1-(1+x)^{-a}]^{b}\right\}^{\lambda-1}}{\left[1+\left\{1-[1-(1+x)^{-a}]^{b}\right\}^{\lambda}\right]^{2}},\quad x>0. \tag{2}$$

Also, the hazard rate function (hrf) of the TIHLIK distribution is given by

$$\begin{aligned}h(x;\lambda,a,b)&=\frac{f(x;\lambda,a,b)}{1-F(x;\lambda,a,b)}\\&=\frac{\lambda ab(1+x)^{-a-1}\,[1-(1+x)^{-a}]^{b-1}}{\left\{1-[1-(1+x)^{-a}]^{b}\right\}\left[1+\left\{1-[1-(1+x)^{-a}]^{b}\right\}^{\lambda}\right]},\quad x>0\end{aligned}$$

and the corresponding cumulative hazard rate function (chrf) is given by

$$\begin{aligned}H(x;\lambda,a,b)&=-\log\left[1-F(x;\lambda,a,b)\right]\\&=-\log(2)-\lambda\log\left\{1-[1-(1+x)^{-a}]^{b}\right\}+\log\left[1+\left\{1-[1-(1+x)^{-a}]^{b}\right\}^{\lambda}\right],\quad x>0.\end{aligned}$$

In order to illustrate the flexibility of the TIHLIK distribution, Figures 1 and 2 present plots of the above pdfs and hrfs. We would like to mention that the values for the parameters λ, a, and b have been taken arbitrarily until we get a wide variety of shapes for the involved functions.

We observe that the pdf is left, right skewed, symmetrical, and reverse J shaped while the hrf is increasing, decreasing, and upside-down bathtub shaped.

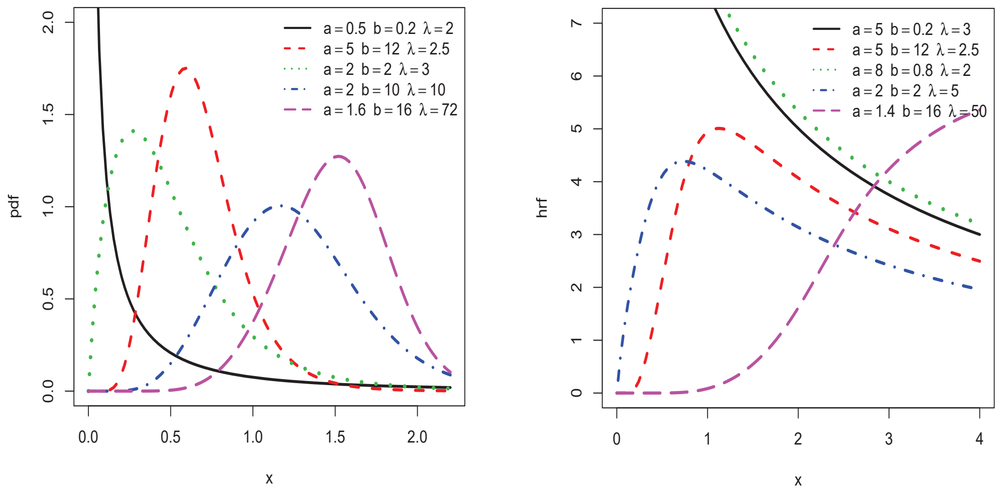

Figure 1. Plots of some probability density functions (pdfs) and hazard rate functions (hrfs) of the type I half-logistic inverted Kumaraswamy (TIHLIK) distribution.

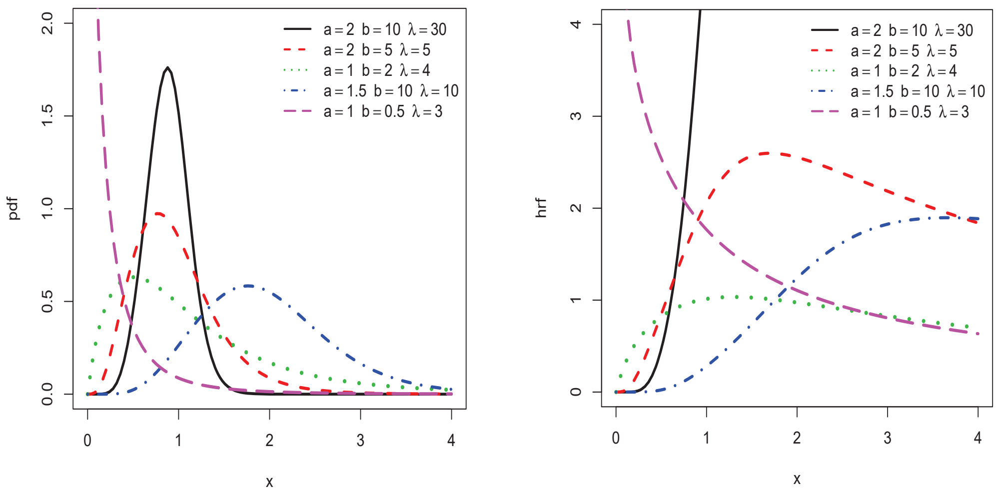

Figure 2. Plots of some pdfs and hrfs of the TIHLIK distribution.

3. Mixture Representations

In this section, we will express the pdf and cdf of the TIHLIK distribution in terms of pdfs and cdfs of the well-established Lomax distribution (also called the Pareto Type II distribution).

Proposition 1. *We have the following mixture representation for $F(x;\lambda,a,b)$:*

$$F(x;\lambda,a,b) = -1 + \sum_{k,\ell,m=0}^{+\infty} \alpha_{k,\ell,m} S_m(x;a),$$

where $\alpha_{k,\ell,m} = 2\binom{\lambda k}{\ell}\binom{b\ell}{m}(-1)^{k+\ell+m}$ (with $\binom{v}{u} = v(v-1)\ldots(v-u+1)/u!$) and $S_m(x;a)$ denotes the survival function of the Lomax distribution with parameters am and 1, i.e., $S_m(x;a) = (1+x)^{-am}$.

Proof. Since $\left\{1-\left[1-(1+x)^{-a}\right]^b\right\}^\lambda \in (0,1)$, the power series formula gives

$$F(x;\lambda,a,b) = -1 + 2\frac{1}{1+\left\{1-\left[1-(1+x)^{-a}\right]^b\right\}^\lambda} = -1 + 2\sum_{k=0}^{+\infty}(-1)^k\left\{1-\left[1-(1+x)^{-a}\right]^b\right\}^{\lambda k}.$$

Since $\left[1-(1+x)^{-a}\right]^b \in (0,1)$ and $(1+x)^{-a} \in (0,1)$, it follows from the binomial formula applied two times in a row that

$$\begin{aligned}\left\{1-\left[1-(1+x)^{-a}\right]^b\right\}^{\lambda k} &= \sum_{\ell=0}^{+\infty}\binom{\lambda k}{\ell}(-1)^\ell\left[1-(1+x)^{-a}\right]^{b\ell} \\ &= \sum_{\ell,m=0}^{+\infty}\binom{\lambda k}{\ell}\binom{b\ell}{m}(-1)^{\ell+m}(1+x)^{-am}.\end{aligned}$$

By putting the above equalities together, we obtain

$$F(x;\lambda,a,b) = -1 + 2\sum_{k,\ell,m=0}^{+\infty}\binom{\lambda k}{\ell}\binom{b\ell}{m}(-1)^{k+\ell+m}(1+x)^{-am}.$$

This ends the proof of Proposition 1. □

Corollary 1. *Upon differentiation of $F(x;\lambda,a,b)$, we obtain the following mixture representation for $f(x;\lambda,a,b)$:*

$$f(x;\lambda,a,b)=\sum_{k,\ell=0}^{+\infty}\sum_{m=1}^{+\infty}\beta_{k,\ell,m}f_m(x;a),$$

where $\beta_{k,\ell,m}=-\alpha_{k,\ell,m}=2\binom{\lambda k}{\ell}\binom{b\ell}{m}(-1)^{k+\ell+m+1}$ and $f_m(x;a)$ denotes the pdf of the Lomax distribution with parameters am and 1, i.e., $f_m(x;a)=am(1+x)^{-am-1}$.

In Corollary 1, we would like to mention that the sum of m begins with 1 since $f_0(x)=0$, which remains an important detail for the coming technical developments involving the distributional properties of the Lomax distribution.

The following result considers a mixture representation for the exponentiated $F(x;\lambda,a,b)$.

Proposition 2. *Let ζ be a positive integer. Then, we have the following mixture representation:*

$$F(x;\lambda,a,b)^{\zeta}=\sum_{k=0}^{\zeta}\sum_{\ell,m,q=0}^{+\infty}v^{(\zeta)}_{k,\ell,m,q}S_q(x;a),$$

where $v^{(\zeta)}_{k,\ell,m,q}=\binom{\zeta}{k}\binom{-k}{\ell}\binom{\lambda\ell}{m}\binom{bm}{q}(-1)^{\zeta-k+m+q}2^k$ and $S_q(x;a)$ denotes the survival function of the Lomax distribution with parameters aq and 1.

Proof. Owing to the (standard) binomial formula, we get

$$\begin{aligned}F(x;\lambda,a,b)^{\zeta}&=\left\{-1+2\frac{1}{1+\left\{1-\left[1-(1+x)^{-a}\right]^b\right\}^{\lambda}}\right\}^{\zeta}\\&=\sum_{k=0}^{\zeta}\binom{\zeta}{k}(-1)^{\zeta-k}2^k\left[1+\left\{1-\left[1-(1+x)^{-a}\right]^b\right\}^{\lambda}\right]^{-k}.\end{aligned}$$

Since $\left\{1-\left[1-(1+x)^{-a}\right]^b\right\}^{\lambda}\in(0,1)$, $\left[1-(1+x)^{-a}\right]^b\in(0,1)$ and $(1+x)^{-a}\in(0,1)$, the general binomial formula applied three times in a row gives

$$\begin{aligned}\left[1+\left\{1-\left[1-(1+x)^{-a}\right]^b\right\}^{\lambda}\right]^{-k}&=\sum_{\ell=0}^{+\infty}\binom{-k}{\ell}\left\{1-\left[1-(1+x)^{-a}\right]^b\right\}^{\lambda\ell}\\&=\sum_{\ell,m=0}^{+\infty}\binom{-k}{\ell}\binom{\lambda\ell}{m}(-1)^m\left[1-(1+x)^{-a}\right]^{bm}\\&=\sum_{\ell,m,q=0}^{+\infty}\binom{-k}{\ell}\binom{\lambda\ell}{m}\binom{bm}{q}(-1)^{m+q}(1+x)^{-aq}.\end{aligned}$$

We obtain the desired result by combining the above equalities together, ending the proof of Proposition 2. □

4. Mathematical and Statistical Properties

This section deals with the mathematical and statistical properties of the TIHLIK distribution. Hereafter, we consider a random variable X following the TIHLIK distribution, i.e., with the cdf given by Equation (1) and the pdf given by Equation (2).

4.1. Shapes and Asymptotes

When $x \to 0$, we have

$$F(x;\lambda,a,b) \sim \frac{\lambda}{2}a^b x^b, \quad f(x;\lambda,a,b) \sim \frac{\lambda}{2}a^b b x^{b-1}, \quad h(x;\lambda,a,b) \sim \frac{\lambda}{2}a^b b x^{b-1}.$$

From these equivalences, when $b \in (0,1)$, we get $f(x;\lambda,a,b) \to +\infty$, when $b = 1$, we obtain $f(x;\lambda,a,b) \to \lambda a/2$ and when $b > 1$, we have $f(x;\lambda,a,b) \to 0$. The same holds for $h(x;\lambda,a,b)$, under the same conditions. When $x \to +\infty$, we have

$$F(x;\lambda,a,b) \sim 1 - 2b^\lambda x^{-a\lambda}, \quad f(x;\lambda,a,b) \sim 2b^\lambda a\lambda x^{-a\lambda-1}, \quad h(x;\lambda,a,b) \sim a\,\lambda x^{-1}.$$

So, $f(x;\lambda,a,b) \to 0$ and $h(x;\lambda,a,b) \to 0$ in all circumstances.

The shapes of $f(x;\lambda,a,b)$ also depend on the critical point(s) of the function, given by the solution(s) of the following equation: $\{\log[f(x;\lambda,a,b)]\}' = 0$, i.e.,

$$a\,(b-1)\frac{(1+x)^{-a-1}}{1-(1+x)^{-a}} - (a+1)(1+x)^{-1} - (\lambda-1)ab\frac{(1+x)^{-a-1}\left[1-(1+x)^{-a}\right]^{b-1}}{1-\left[1-(1+x)^{-a}\right]^b}$$
$$+\,2\lambda ab\frac{(1+x)^{-a-1}\left[1-(1+x)^{-a}\right]^{b-1}\left\{1-\left[1-(1+x)^{-a}\right]^b\right\}^{\lambda-1}}{1+\left\{1-\left[1-(1+x)^{-a}\right]^b\right\}^{\lambda}} = 0.$$

In a same way, the shapes of $h(x;\lambda,a,b)$ also depend on the critical point(s) of the function, given by the solution(s) of the following equation: $\{\log[h(x;\lambda,a,b)]\}' = 0$, i.e., $\{\log[f(x;\lambda,a,b)]\}' + h(x;\lambda,a,b) = 0$, i.e.,

$$a\,(b-1)\frac{(1+x)^{-a-1}}{1-(1+x)^{-a}} - (a+1)(1+x)^{-1} + \frac{ab(1+x)^{-a-1}\left[1-(1+x)^{-a}\right]^{b-1}}{1-\left[1-(1+x)^{-a}\right]^b}$$
$$+\,\lambda ab\frac{(1+x)^{-a-1}\left[1-(1+x)^{-a}\right]^{b-1}\left\{1-\left[1-(1+x)^{-a}\right]^b\right\}^{\lambda-1}}{1+\left\{1-\left[1-(1+x)^{-a}\right]^b\right\}^{\lambda}} = 0.$$

These equations provide some mathematical backgrounds to Figures 1 and 2.

4.2. Quantile Function

The quantile function (qf) of X is given by

$$Q(y;\lambda,a,b) = \left\{1-\left(1-\left[\frac{1-y}{1+y}\right]^{1/\lambda}\right)^{1/b}\right\}^{-1/a} - 1, \quad y \in (0,1).$$

The median of X is given by $M = Q(1/2;\lambda,a,b)$. The other quartiles can be defined in a similar manner.

Simulated values from the TIHLIK distribution can be performed by using the following result. For any random variable U following the uniform distribution $U(0,1)$, $x_U = Q(U;\lambda,a,b)$ follows the TIHLIK distribution.

Upon differentiation of $Q(y;\lambda,a,b)$, the corresponding quantile density function is given by

$$q(y;\lambda,a,b) =$$
$$\frac{2}{ab\lambda}\frac{1}{(1+y)^2}\left[\frac{1-y}{1+y}\right]^{1/\lambda-1}\left(1-\left[\frac{1-y}{1+y}\right]^{1/\lambda}\right)^{1/b-1}\left\{1-\left(1-\left[\frac{1-y}{1+y}\right]^{1/\lambda}\right)^{1/b}\right\}^{-1/a-1},$$
$$y\in(0,1).$$

4.3. Ordinary Moments

We begin the study of the ordinary moments of X by an existence result.

Proposition 3. *Let r be a positive integer. Then, the r-th ordinary moment of X, i.e., $\mu'_r = E(X^r)$, exists if, and only, if $a\lambda > r$.*

Proof. The proof is centered around the equivalence results presented in Section 4.1. When $x \to 0$, we have $x^r f(x;\lambda,a,b) \sim (\lambda/2)a^b b x^{r+b-1}$ and, for any $\epsilon > 0$, by the Riemann integral criteria, $\int_0^\epsilon x^{r+b-1}dx$ exists since $r+b-1 > -1$. When $x \to +\infty$, we have $x^r f(x;\lambda,a,b) \sim 2b^\lambda a\lambda x^{r-a\lambda-1}$ and, for any $\epsilon > 0$, by the Riemann integral criteria , $\int_\epsilon^{+\infty} x^{r-a\lambda-1}dx$ exists if, and only if, $a\lambda > r$. This ends the proof of Proposition 3. □

Then, when $a\lambda > r$, the r-th ordinary moment of X is defined by

$$\mu'_r = \int_0^{+\infty} x^r f(x;\lambda,a,b)dx.$$

This integral can be evaluated by any mathematical software.

If $a\min(\lambda,1) > r$, an alternative expression, with possible gain in precision in terms of errors, is given by using Corollary 1, i.e.,

$$\mu'_r = \sum_{k,\ell=0}^{+\infty}\sum_{m=1}^{+\infty}\beta_{k,\ell,m}\int_0^{+\infty} x^r f_m(x;a)dx = \sum_{k,\ell=0}^{+\infty}\sum_{m=1}^{+\infty}\beta_{k,\ell,m}\frac{\Gamma(am-r)\Gamma(r+1)}{\Gamma(am)},$$

where $\Gamma(s) = \int_0^{+\infty} t^s e^{-t}dt$, $s > 0$. After some algebra, one can remark that $\Gamma(am-r)\Gamma(r+1)/\Gamma(am) = r!/\prod_{u=1}^{r}(am-u)$. For practical purpose, one can consider finite limit for the sums, say a large integer as 40.

As consequence, if $a\min(\lambda,1) > 2$, the mean and the variance of X can be expressed as, respectively, $\mu = \mu'_1$ and $\sigma^2 = \mu'_2 - \mu^2$, i.e.,

$$\mu = \sum_{k,\ell=0}^{+\infty}\sum_{m=1}^{+\infty}\beta_{k,\ell,m}\frac{1}{am-1}, \quad \sigma^2 = \left\{\sum_{k,\ell=0}^{+\infty}\sum_{m=1}^{+\infty}\beta_{k,\ell,m}\frac{2}{(am-1)(am-2)}\right\} - \mu^2.$$

Table 1 presents the numerical values of some moments (and variance) of X for some values of the parameters (respecting the condition $a\lambda > 4$).

Table 1. Some moments and variance of X for the following parameters values respecting the order (a,b,λ); (i): $(2,2,3)$, (ii): $(2,3,3)$, (iii): $(3,3,3)$ (iv): $(3,3,5)$ (v): $(3,6,5)$, and (vi): $(8,6,5)$.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)
μ'_1	0.6117	0.8889	0.5127	0.3666	0.6484	0.2026
μ'_2	0.6083	1.1716	0.3594	0.1716	0.49172	0.0461
μ'_3	0.9516	2.2852	0.3326	0.0982	0.4289	0.0116
μ'_4	2.5250	0.2515	0.4065	0.06761	0.42739	0.0031
σ^2	0.2342	0.3814	0.09652	0.0372	0.0712	0.0050

4.4. Skewness and Kurtosis

Assuming that $a\lambda > 4$, the first four moments of X can be used to determine the measures of skewness and kurtosis of X defined by, respectively,

$$\gamma_1 = \frac{\mu_3' - 3\mu_2'\mu + 2\mu^3}{\sigma^3}, \quad \gamma_2 = \frac{\mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4}{\sigma^4}.$$

To relax the assumption $a\lambda > 4$, one can consider measures of skewness and kurtosis based on the qf, as the Bowley skewness and Moors kurtosis defined by, respectively,

$$\gamma_1^* = \frac{Q(6/8;\lambda,a,b) + Q(2/8;\lambda,a,b) - 2Q(4/8;\lambda,a,b)}{Q(6/8;\lambda,a,b) - Q(2/8;\lambda,a,b)}$$

and

$$\gamma_2^* = \frac{Q(7/8;\lambda,a,b) - Q(5/8;\lambda,a,b) + Q(3/8;\lambda,a,b) - Q(1/8;\lambda,a,b)}{Q(6/8;\lambda,a,b) - Q(2/8;\lambda,a,b)}.$$

They was introduced by [13,14], respectively.

The plots of γ_1^* and γ_2^* are shown in Figures 3–8, for different parameter ranges. We see smooth non-monotonic variation of these measures, attesting a significant effect of a, b, and λ on them.

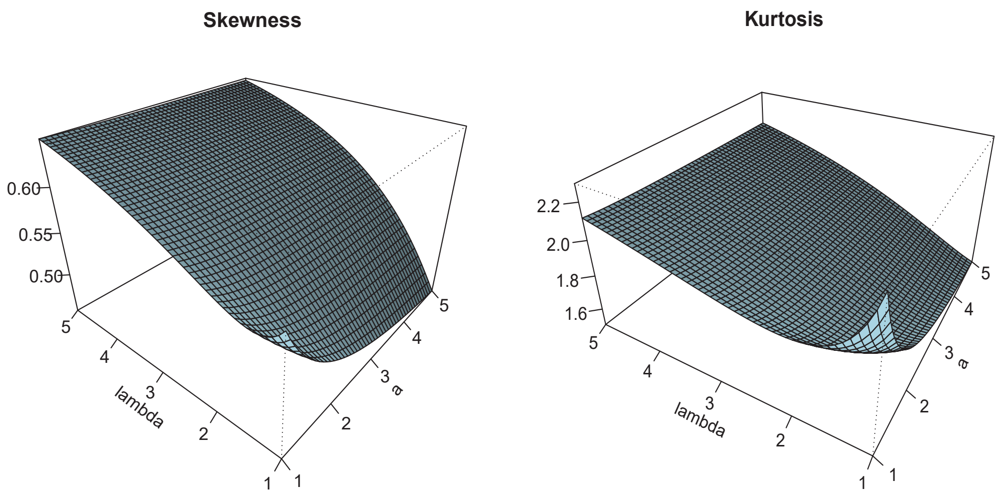

Figure 3. Plots of γ_1^* and γ_2^* for $\lambda, a \in (1,5)$ and $b = 0.3$.

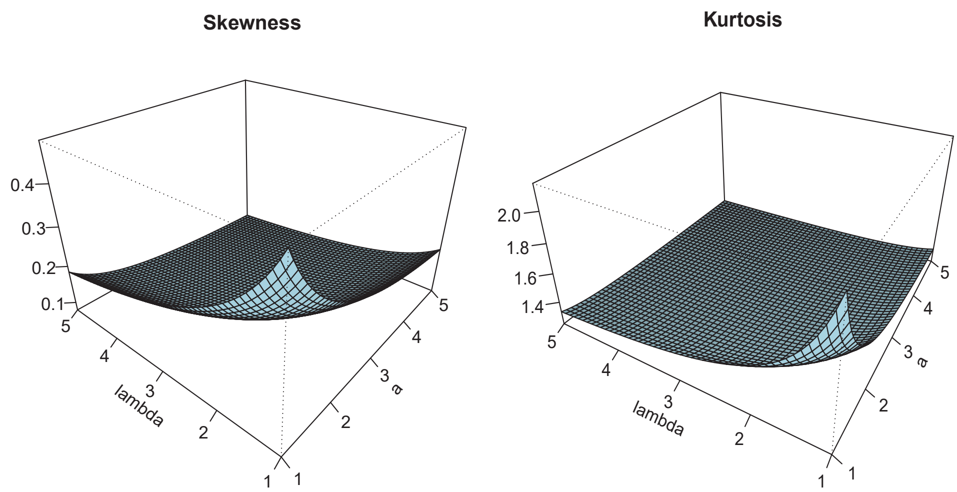

Figure 4. Plots of γ_1^* and γ_2^* for $\lambda, a \in (1,5)$ and $b = 3$.

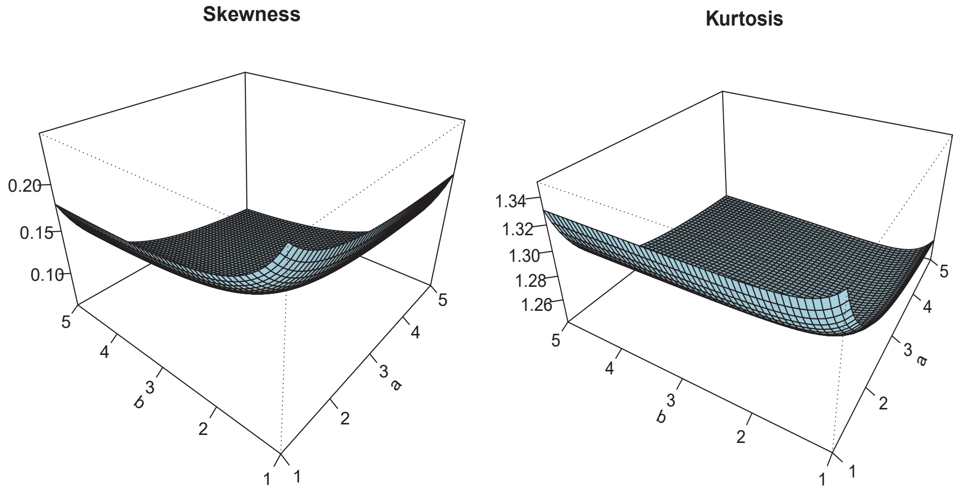

Figure 5. Plots of γ_1^* and γ_2^* for $a, b \in (1, 5)$ and $\lambda = 5$.

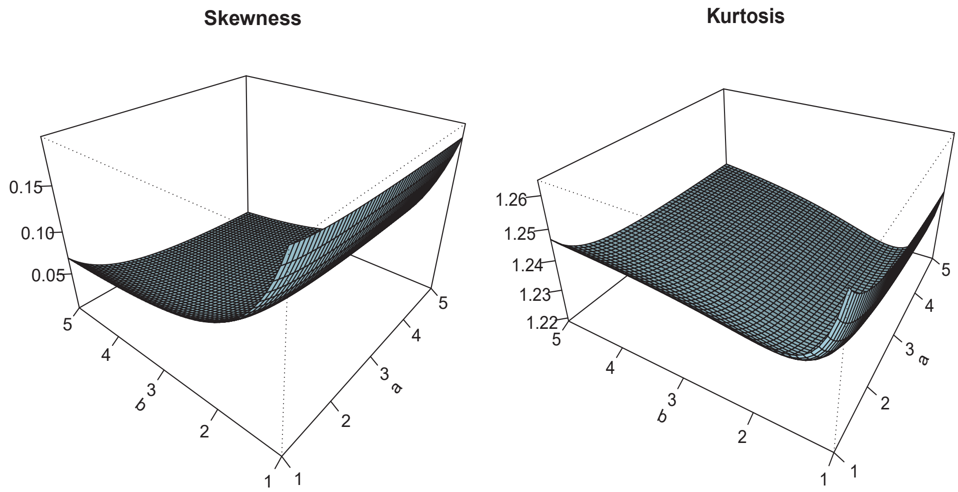

Figure 6. Plots of γ_1^* and γ_2^* for $a, b \in (1, 5)$ and $\lambda = 20$.

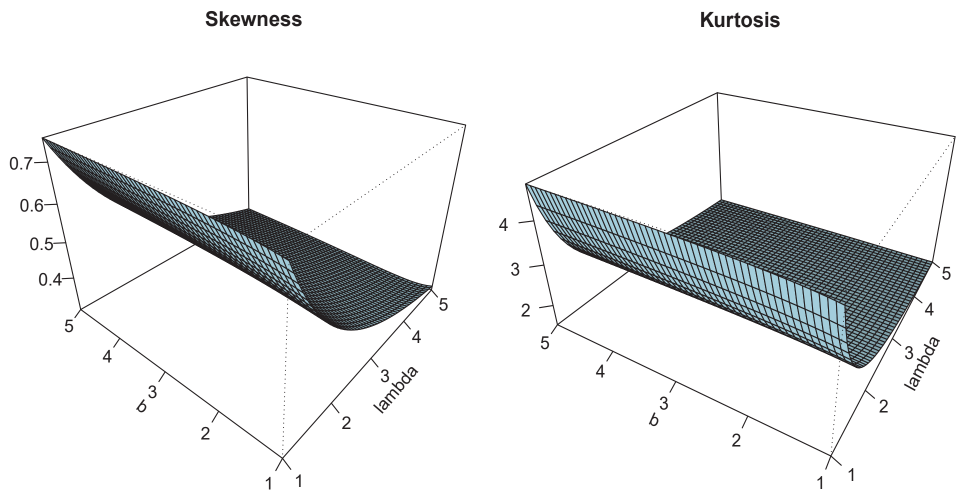

Figure 7. Plots of γ_1^* and γ_2^* for $\lambda, b \in (1, 5)$ and $a = 0.5$.

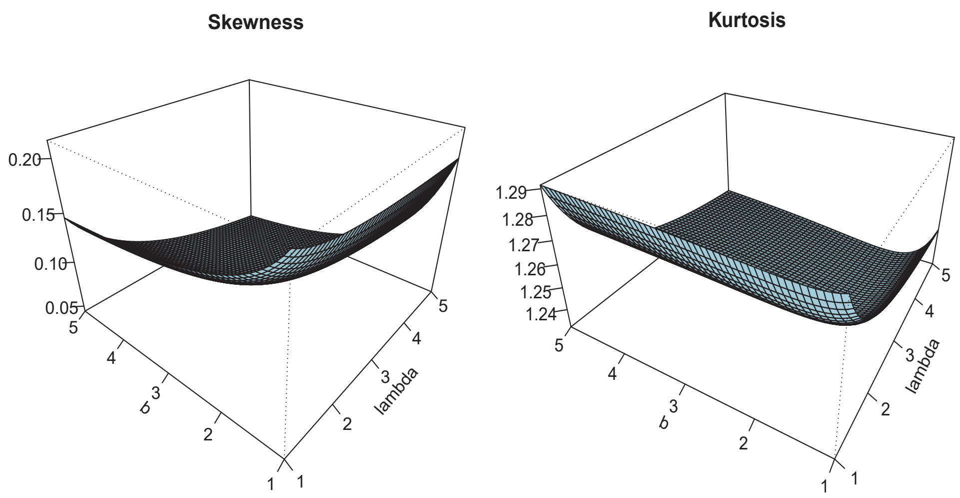

Figure 8. Plots of γ_1^* and γ_2^* for $\lambda, b \in (1, 5)$ and $a = 10$.

4.5. Incomplete Moments

Let us now studied the incomplete moments of X. For any $t > 0$, the r-th incomplete moment of X exists and it is given by

$$\mu_r'(t) = E(X^r 1_{\{X \leq t\}}) = \int_0^t x^r f(x; \lambda, a, b) dx,$$

where $1_{\{X \leq t\}}$ denotes the random variable such that $1_{\{X \leq t\}} = 1$ if $\{X \leq t\}$ is realized, and 0 otherwise. It follows from Corollary 1 that

$$\mu_r'(t) = \sum_{k,\ell=0}^{+\infty} \sum_{m=1}^{+\infty} \beta_{k,\ell,m} \int_0^t x^r f_m(x; a) dx.$$

For $t \in (0, 1)$, we have the following sum expression for the integral term:

$$\int_0^t x^r f_m(x; a) dx = am \int_0^t x^r (1 + x)^{-am-1} dx = am \sum_{k=0}^{+\infty} \binom{-am-1}{k} \frac{t^{r+k+1}}{r+k+1}.$$

The first incomplete moment is given by $\mu_1'(t)$. It is useful to define other important quantities, as the mean deviation of X about μ given by

$$\delta_1 = E(|X - \mu|) = \int_0^{+\infty} |x - \mu| f(x; \lambda, a, b) dx = 2\mu F(\mu; \lambda, a, b) - 2\mu_1'(\mu),$$

the mean of X about M given by

$$\delta_2 = E(|X - M|) = \int_0^{+\infty} |x - M| f(x; \lambda, a, b) dx = \mu - 2\mu_1'(M),$$

the Bonferroni curve given by

$$B(y) = \frac{1}{y\mu} \mu_1' \left[Q(y; \lambda, a, b) \right] = \frac{1}{y\mu} \mu_1' \left[\left\{ 1 - \left(1 - \left[\frac{1-y}{1+y} \right]^{1/\lambda} \right)^{1/b} \right\}^{-1/a} - 1 \right], \quad y \in (0, 1)$$

and the Lorenz curves given by $L(y) = yB(y)$, $y \in (0, 1)$.

4.6. Stress Strength Parameter

This subsection is devoted to the stress strength parameter, as described in [15], in the context of the TIHLIK distribution. It is defined by $R = P(X_2 < X_1)$, when X_1 and X_2 are two independent random variables following the TIHLIK distribution with the parameters λ_1, a_1 and b_1, and λ_2, a_2, and b_2, respectively.

Proposition 4. *Under the setting described above, we have*

$$R = -1 + \sum_{k,\ell,m,u,v=0}^{+\infty} \sum_{w=1}^{+\infty} v_{k,\ell,m,u,v,w} \frac{a_1 w}{a_1 w + a_2 m},$$

where $v_{k,\ell,m,u,v,w} = 4\binom{\lambda_2 k}{\ell}\binom{b_2 \ell}{m}\binom{\lambda_1 u}{v}\binom{b_1 v}{w}(-1)^{k+\ell+m+u+v+w+1}$.

Proof. Owing to Propositions 1 and Corollary 1, we have

$$\begin{aligned} R &= \int_0^{+\infty} F(x;\lambda_2,a_2,b_2) f(x;\lambda_1,a_1,b_1) dx \\ &= -1 + \sum_{k,\ell,m,u,v=0}^{+\infty} \sum_{w=1}^{+\infty} \alpha^{(2)}_{k,\ell,m} \beta^{(1)}_{u,v,w} \int_0^{+\infty} S_m(x;a_2) f_w(x;a_1) dx, \end{aligned}$$

where $\alpha^{(2)}_{k,\ell,m} = 2\binom{\lambda_2 k}{\ell}\binom{b_2 \ell}{m}(-1)^{k+\ell+m}$ and $\beta^{(1)}_{u,v,w} = 2\binom{\lambda_1 u}{v}\binom{b_1 v}{w}(-1)^{u+v+w+1}$. Then, one can notice that $v_{k,\ell,m,u,v,w} = \alpha^{(2)}_{k,\ell,m}\beta^{(1)}_{u,v,w}$ and that the integral term can be expressed as

$$\int_0^{+\infty} S_m(x;a_2) f_w(x;a_1) dx = a_1 w \int_0^{+\infty} (1+x)^{-a_1 w - a_2 m - 1} dx = \frac{a_1 w}{a_1 w + a_2 m}.$$

By putting the above equalities together, we end the proof of Proposition 4. □

Finally, we would like to mention that, when $\lambda_1 = \lambda_2$, $a_1 = a_2$, and $b_1 = b_2$, we have $R = 1/2$.

4.7. Order Statistics

Order statistics was first studied by [16] in the context of the standard normal distribution. In a more general way, order statistics naturally arise for modeling a wide variety of phenomenas, mainly in reliability and life testing. Here, we provide some useful results involving the order statistics of the TIHLIK distribution. Let $X_1, \ldots, X_n$ be n independent random having the TIHLIK distribution as common distribution and $X_{(i)}$ be the i-th order statistic defined by the i-th random variable such that, by arranging $X_1, \ldots, X_n$ in increasing order, we have $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$. Then, a well-known result ensures that the cdf of $X_{(i)}$ is given by

$$F_{(i)}(x;\lambda,a,b) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \frac{(-1)^j}{j+i} \binom{n-i}{j} F(x;\lambda,a,b)^{j+i}, \quad x > 0.$$

By applying Proposition 2, we obtain the following mixture representation:

$$F_{(i)}(x;\lambda,a,b) = \sum_{j=0}^{n-i} \sum_{k=0}^{j+i} \sum_{\ell,m,q=0}^{+\infty} \phi_{i,j,k,\ell,m,q} S_q(x;a), \tag{3}$$

where $\phi_{i,j,k,\ell,m,q} = \{n!/[(i-1)!(n-i)!]\}\,[(-1)^j/(j+i)]\binom{n-i}{j} v^{(j+i)}_{k,\ell,m,q}$.

Also, it is well-known that the pdf of $X_{(i)}$ is given by

$$f_{(i)}(x;\lambda,a,b) = \frac{n!}{(i-1)!(n-i)!} f(x;\lambda,a,b)\,[1-F(x;\lambda,a,b)]^{n-i}\,F(x;\lambda,a,b)^{i-1}.$$

Upon differentiation of Equation (3), one can express $f_{(i)}(x;\lambda,a,b)$ as a mixture of pdfs of the Lomax distribution, i.e.,

$$f_{(i)}(x;\lambda,a,b) = \sum_{j=0}^{n-i}\sum_{k=0}^{j+i}\sum_{\ell,m,q=0}^{+\infty} \psi_{i,j,k,\ell,m,q} f_q(x;a), \tag{4}$$

where $\psi_{i,j,k,\ell,m,q} = -\phi_{i,j,k,\ell,m,q}$ and $f_q(x;a)$ denotes the pdf of the Lomax distribution with parameters aq and 1. From this expression, one can derive some structural properties of $X_{(i)}$ (ordinary moments, incomplete moments…).

Let us now specially focus on the crucial first order statistic given by $X_{(1)} = \inf(X_1,\ldots,X_n)$. The cdf and pdf of $X_{(1)}$ are, respectively, given by

$$F_{(1)}(x;\lambda,a,b) = 1-[1-F(x;\lambda,a,b)]^n = 1-2^n \frac{\left\{1-[1-(1+x)^{-a}]^b\right\}^{\lambda n}}{\left[1+\left\{1-[1-(1+x)^{-a}]^b\right\}^{\lambda}\right]^n}, \quad x>0$$

and

$$\begin{aligned} f_{(1)}(x;\lambda,a,b) &= n\,[1-F(x;\lambda,a,b)]^{n-1} f(x;\lambda,a,b) \\ &= n2^n\lambda ab(1+x)^{-a-1}\left[1-(1+x)^{-a}\right]^{b-1} \frac{\left\{1-[1-(1+x)^{-a}]^b\right\}^{\lambda n-1}}{\left[1+\left\{1-[1-(1+x)^{-a}]^b\right\}^{\lambda}\right]^{n+1}}, \quad x>0. \end{aligned}$$

The corresponding mixture expressions in terms of Lomax distribution functions are given by Equations (3) and (4), respectively, by taking $i=1$.

Let us now derive the asymptotic distribution of $X_{(1)}$. It follows from the equivalence results of Section 4.1 that

$$\lim_{\epsilon\to 0}\frac{F(\epsilon x;\lambda,a,b)}{F(\epsilon;\lambda,a,b)} = \lim_{\epsilon\to 0}\frac{(\lambda/2)a^b(\epsilon x)^b}{(\lambda/2)a^b\epsilon^b} = x^b.$$

Hence, since $F(0;\lambda,a,b)=0$, it follows from [17] (Theorem 8.3.6) that the asymptotic distribution of $X_{(1)}$ is the Weibull distribution with parameter b, i.e., there exist two sequence of real numbers $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} P(u_n(X_{(1)}-v_n)\le x) = 1-e^{-x^b}$.

5. Estimation with Simulation

This section is devoted to some statistical features of the TIHLIK model, assuming that λ, a, and b are unknown. The estimation of λ, a, and b is performed by the several recognized methods of estimation. Hereafter, $x_1,\ldots,x_n$ denote n observed values from X, and $x_{(1)},\ldots,x_{(n)}$ their ascending ordering values, i.e., $x_{(1)}\le\ldots\le x_{(n)}$.

5.1. Method of Maximum Likelihood Estimation

The method of maximum likelihood estimation in the context of the TIHLIK model is described below. We refer the reader to [18] for the general details. The maximum likelihood estimates (MLEs) of λ, a, and b can be obtained by maximizing, with respect to λ, a, and b, the likelihood function given by $L(\lambda, a, b) = \prod_{i=1}^{n} f(x_i; \lambda, a, b)$ or, alternatively, the log-likelihood function for (λ, a, b) given by $\ell(\lambda, a, b) = \log[L(\lambda, a, b)]$, i.e.,

$$\begin{aligned}\ell(\lambda, a, b) &= n\log(2) + n\log(\lambda) + n\log(a) + n\log(b) - (a+1)\sum_{i=1}^{n}\log(1+x_i) \\ &\quad + (b-1)\sum_{i=1}^{n}\log\left[1-(1+x_i)^{-a}\right] + (\lambda-1)\sum_{i=1}^{n}\log\left\{1-\left[1-(1+x_i)^{-a}\right]^{b}\right\} \\ &\quad - 2\sum_{i=1}^{n}\log\left[1+\left\{1-\left[1-(1+x_i)^{-a}\right]^{b}\right\}^{\lambda}\right].\end{aligned}$$

Thus, the MLEs are obtained by solving the following equations simultaneously: $\partial\ell(\lambda, a, b)/\partial\lambda = 0$, $\partial\ell(\lambda, a, b)/\partial a = 0$ and $\partial\ell(\lambda, a, b)/\partial b = 0$, where

$$\begin{aligned}\frac{\partial\ell(\lambda, a, b)}{\partial\lambda} &= \frac{n}{\lambda} + \sum_{i=1}^{n}\log\left\{1-\left[1-(1+x_i)^{-a}\right]^{b}\right\} \\ &\quad - 2\sum_{i=1}^{n}\frac{\left\{1-[1-(1+x_i)^{-a}]^{b}\right\}^{\lambda}\log\left\{1-[1-(1+x_i)^{-a}]^{b}\right\}}{1+\left\{1-[1-(1+x_i)^{-a}]^{b}\right\}^{\lambda}},\end{aligned}$$

$$\begin{aligned}\frac{\partial\ell(\lambda, a, b)}{\partial a} &= \frac{n}{a} - \sum_{i=1}^{n}\log(1+x_i) + (b-1)\sum_{i=1}^{n}\frac{(1+x_i)^{-a}\log(1+x_i)}{1-(1+x_i)^{-a}} \\ &\quad - (\lambda-1)b\sum_{i=1}^{n}\frac{\log(1+x_i)(1+x_i)^{-a}\left[1-(1+x_i)^{-a}\right]^{b-1}}{1-\left[1-(1+x_i)^{-a}\right]^{b}} \\ &\quad + 2b\lambda\sum_{i=1}^{n}\frac{\log(1+x_i)(1+x_i)^{-a}\left[1-(1+x_i)^{-a}\right]^{b-1}\left\{1-[1-(1+x_i)^{-a}]^{b}\right\}^{\lambda-1}}{1+\left\{1-[1-(1+x_i)^{-a}]^{b}\right\}^{\lambda}}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\ell(\lambda, a, b)}{\partial b} &= \frac{n}{b} + \sum_{i=1}^{n}\log\left[1-(1+x_i)^{-a}\right] - (\lambda-1)\sum_{i=1}^{n}\frac{\left[1-(1+x_i)^{-a}\right]^{b}\log\left[1-(1+x_i)^{-a}\right]}{1-\left[1-(1+x_i)^{-a}\right]^{b}} \\ &\quad + 2\lambda\sum_{i=1}^{n}\frac{\log\left[1-(1+x_i)^{-a}\right]\left[1-(1+x_i)^{-a}\right]^{b}\left\{1-[1-(1+x_i)^{-a}]^{b}\right\}^{\lambda-1}}{1+\left\{1-[1-(1+x_i)^{-a}]^{b}\right\}^{\lambda}}.\end{aligned}$$

These equations can be solved numerically by using any mathematical software. When n is large enough, under some conditions of regularity, the subjacent distribution of the MLEs can be approximated by normal distributions, with variance given as the corresponding component of the inverse of the observed information matrix computed at those MLEs. Owing to these distributional results, confidence intervals and statistical tests for λ, a, and b can be defined analytically.

5.2. Methods of Least Squares and Weighted Least Squares Estimation

We now consider the methods of least squares and weighted least squares estimation introduced by [19]. The least square estimates (LSEs) of λ, a, and b can be determined by minimizing, with respect to λ, a, and b, the following function:

$$LS(\lambda,a,b)=\sum_{i=1}^{n}\left[F(x_{(i)};\lambda,a,b)-\frac{i}{n+1}\right]^2=\sum_{i=1}^{n}\left[\frac{1-\left\{1-\left[1-(1+x_{(i)})^{-a}\right]^b\right\}^{\lambda}}{1+\left\{1-\left[1-(1+x_{(i)})^{-a}\right]^b\right\}^{\lambda}}-\frac{i}{n+1}\right]^2.$$

Thus, the LSEs are obtained by solving the following equations simultaneously: $\partial LS(\lambda,a,b)/\partial\lambda=0$, $\partial LS(\lambda,a,b)/\partial a=0$, and $\partial LS(\lambda,a,b)/\partial b=0$, where

$$\frac{\partial LS(\lambda,a,b)}{\partial\lambda}=2\sum_{i=1}^{n}\eta_i^{(1)}(\lambda,a,b)\left[\frac{1-\left\{1-\left[1-(1+x_{(i)})^{-a}\right]^b\right\}^{\lambda}}{1+\left\{1-\left[1-(1+x_{(i)})^{-a}\right]^b\right\}^{\lambda}}-\frac{i}{n+1}\right],$$

$$\frac{\partial LS(\lambda,a,b)}{\partial a}=2\sum_{i=1}^{n}\eta_i^{(2)}(\lambda,a,b)\left[\frac{1-\left\{1-\left[1-(1+x_{(i)})^{-a}\right]^b\right\}^{\lambda}}{1+\left\{1-\left[1-(1+x_{(i)})^{-a}\right]^b\right\}^{\lambda}}-\frac{i}{n+1}\right]$$

and

$$\frac{\partial LS(\lambda,a,b)}{\partial b}=2\sum_{i=1}^{n}\eta_i^{(3)}(\lambda,a,b)\left[\frac{1-\left\{1-\left[1-(1+x_{(i)})^{-a}\right]^b\right\}^{\lambda}}{1+\left\{1-\left[1-(1+x_{(i)})^{-a}\right]^b\right\}^{\lambda}}-\frac{i}{n+1}\right],$$

where

$$\eta_i^{(1)}(\lambda,a,b)=-2\frac{\left\{1-\left[1-(1+x_{(i)})^{-a}\right]^b\right\}^{\lambda}\log\left\{1-\left[1-(1+x_{(i)})^{-a}\right]^b\right\}}{\left[1+\left\{1-\left[1-(1+x_{(i)})^{-a}\right]^b\right\}^{\lambda}\right]^2}, \tag{5}$$

$$\eta_i^{(2)}(\lambda,a,b)=2b\lambda\frac{\log(1+x_{(i)})(1+x_{(i)})^{-a}\left[1-(1+x_{(i)})^{-a}\right]^{b-1}\left\{1-\left[1-(1+x_{(i)})^{-a}\right]^b\right\}^{\lambda-1}}{\left[1+\left\{1-\left[1-(1+x_{(i)})^{-a}\right]^b\right\}^{\lambda}\right]^2} \tag{6}$$

and

$$\eta_i^{(2)}(\lambda,a,b)=2\lambda\frac{\log\left[1-(1+x_{(i)})^{-a}\right]\left[1-(1+x_{(i)})^{-a}\right]^b\left\{1-\left[1-(1+x_{(i)})^{-a}\right]^b\right\}^{\lambda-1}}{\left[1+\left\{1-\left[1-(1+x_{(i)})^{-a}\right]^b\right\}^{\lambda}\right]^2}. \tag{7}$$

The weighted least square estimates (WLSEs) of λ, a, and b can be determined by minimizing, with respect to λ, a, and b, the following function:

$$\begin{aligned} LSW(\lambda, a, b) &= \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{(i)}; \lambda, a, b) - \frac{i}{n+1} \right]^2 \\ &= \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[\frac{1 - \left\{1 - \left[1 - (1+x_{(i)})^{-a}\right]^b\right\}^\lambda}{1 + \left\{1 - \left[1 - (1+x_{(i)})^{-a}\right]^b\right\}^\lambda} - \frac{i}{n+1} \right]^2. \end{aligned}$$

Thus, the WLSEs can be determined by solving the following equations simultaneously: $\partial WLS(\lambda, a, b)/\partial\lambda = 0$, $\partial WLS(\lambda, a, b)/\partial a = 0$, and $\partial WLS(\lambda, a, b)/\partial b = 0$, which are similar to those previously presented, with the weight sequence plug-in in the right place.

5.3. Method of Cramer-von Mises Minimum Distance Estimation

Another famous estimation method is the method of Cramer-von Mises minimum distance estimation introduced by [20]. By applying it in the context of the TIHLIK model, the Cramer-von Mises minimum distance estimates (CVEs) of λ, a, and b can be obtained by minimizing, with respect to λ, a, and b, the following function:

$$\begin{aligned} C(\lambda, a, b) &= \frac{1}{12n} + \sum_{i=1}^{n} \left[F(x_{(i)}; \lambda, a, b) - \frac{2i-1}{2n} \right]^2 \\ &= \frac{1}{12n} + \sum_{i=1}^{n} \left[\frac{1 - \left\{1 - \left[1 - (1+x_{(i)})^{-a}\right]^b\right\}^\lambda}{1 + \left\{1 - \left[1 - (1+x_{(i)})^{-a}\right]^b\right\}^\lambda} - \frac{2i-1}{2n} \right]^2. \end{aligned}$$

Thus, the CVEs are obtained by solving the following equations simultaneously: $\partial C(\lambda, a, b)/\partial\lambda = 0$, $\partial C(\lambda, a, b)/\partial a = 0$, and $\partial C(\lambda, a, b)/\partial b = 0$, where

$$\frac{\partial C(\lambda, a, b)}{\partial\lambda} = 2\sum_{i=1}^{n} \eta_i^{(1)}(\lambda, a, b) \left[\frac{1 - \left\{1 - \left[1 - (1+x_{(i)})^{-a}\right]^b\right\}^\lambda}{1 + \left\{1 - \left[1 - (1+x_{(i)})^{-a}\right]^b\right\}^\lambda} - \frac{2i-1}{2n} \right],$$

$$\frac{\partial C(\lambda, a, b)}{\partial a} = 2\sum_{i=1}^{n} \eta_i^{(2)}(\lambda, a, b) \left[\frac{1 - \left\{1 - \left[1 - (1+x_{(i)})^{-a}\right]^b\right\}^\lambda}{1 + \left\{1 - \left[1 - (1+x_{(i)})^{-a}\right]^b\right\}^\lambda} - \frac{2i-1}{2n} \right]$$

and

$$\frac{\partial C(\lambda, a, b)}{\partial b} = 2\sum_{i=1}^{n} \eta_i^{(3)}(\lambda, a, b) \left[\frac{1 - \left\{1 - \left[1 - (1+x_{(i)})^{-a}\right]^b\right\}^\lambda}{1 + \left\{1 - \left[1 - (1+x_{(i)})^{-a}\right]^b\right\}^\lambda} - \frac{2i-1}{2n} \right],$$

$\eta_i^{(1)}(\lambda, a, b)$, $\eta_i^{(2)}(\lambda, a, b)$, and $\eta_i^{(3)}(\lambda, a, b)$ are given by Equations (5)–(7), respectively.

5.4. Methods of Anderson–Darling and Right-Tail Anderson–Darling Estimation

The method of Anderson–Darling estimation was introduced by [21]. Here, the Anderson–Darling estimates (ADEs) of λ, a, and b can be determined by minimizing, with respect to λ, a, and b, the following function:

$$A(\lambda,a,b) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log[F(x_{(i)};\lambda,a,b)] - H(x_{(n+1-i)};\lambda,a,b)\right\}.$$

This function is expressible, but we omit it to save space. Thus, the Anderson–Darling estimates are obtained by solving the following equations simultaneously: $\partial A(\lambda,a,b)/\partial\lambda = 0$, $\partial A(\lambda,a,b)/\partial a = 0$, and $\partial A(\lambda,a,b)/\partial b = 0$, where

$$\frac{\partial A(\lambda,a,b)}{\partial\lambda} = -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\eta_i^{(1)}(\lambda,a,b)}{F(x_{(i)};\lambda,a,b)} - \frac{\eta_{n+1-i}^{(1)}(\lambda,a,b)}{1-F(x_{(n+1-i)};\lambda,a,b)}\right],$$

$$\frac{\partial A(\lambda,a,b)}{\partial a} = -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\eta_i^{(2)}(\lambda,a,b)}{F(x_{(i)};\lambda,a,b)} - \frac{\eta_{n+1-i}^{(2)}(\lambda,a,b)}{1-F(x_{(n+1-i)};\lambda,a,b)}\right]$$

and

$$\frac{\partial A(\lambda,a,b)}{\partial b} = -\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\eta_i^{(3)}(\lambda,a,b)}{F(x_{(i)};\lambda,a,b)} - \frac{\eta_{n+1-i}^{(3)}(\lambda,a,b)}{1-F(x_{(n+1-i)};\lambda,a,b)}\right],$$

$\eta_i^{(1)}(\lambda,a,b)$, $\eta_i^{(2)}(\lambda,a,b)$, and $\eta_i^{(3)}(\lambda,a,b)$ are given by Equations (5)–(7), respectively.

Similarly, the right-tail Anderson–Darling estimates (RTADEs) of λ, a, and b, can be obtained by minimizing, with respect to λ and α, the following function:

$$R(\lambda,a,b) = \frac{n}{2} - 2\sum_{i=1}^{n}\log[F(x_{(i)};\lambda,a,b)] + \frac{1}{n}\sum_{i=1}^{n}(2i-1)H(x_{(n+1-i)};\lambda,a,b).$$

Thus, the RTADEs can be determined by solving the following equations simultaneously: $\partial R(\lambda,a,b)/\partial\lambda = 0$, $\partial R(\lambda,a,b)/\partial a = 0$, and $\partial R(\lambda,a,b)/\partial b = 0$, where

$$\frac{\partial R(\lambda,a,b)}{\partial\lambda} = -2\sum_{i=1}^{n}\frac{\eta_i^{(1)}(\lambda,a,b)}{F(x_{(i)};\lambda,a,b)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\eta_{n+1-i}^{(1)}(\lambda,a,b)}{1-F(x_{(n+1-i)};\lambda,a,b)},$$

$$\frac{\partial R(\lambda,a,b)}{\partial a} = -2\sum_{i=1}^{n}\frac{\eta_i^{(2)}(\lambda,a,b)}{F(x_{(i)};\lambda,a,b)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\eta_{n+1-i}^{(2)}(\lambda,a,b)}{1-F(x_{(n+1-i)};\lambda,a,b)}$$

and

$$\frac{\partial R(\lambda,a,b)}{\partial b} = -2\sum_{i=1}^{n}\frac{\eta_i^{(3)}(\lambda,a,b)}{F(x_{(i)};\lambda,a,b)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\eta_{n+1-i}^{(3)}(\lambda,a,b)}{1-F(x_{(n+1-i)};\lambda,a,b)}.$$

5.5. Simulation Study

In this section, we come up with a numerical study to compare the behavior of the different estimates presented above. We generate $N = 1000$ random samples of size $n = 30, 50$, and 100 from the TIHLIK distribution. Four sets of the parameters are assigned as: Set1: ($a = 1.5, \lambda = 2, b = 2$), Set2: $(a = 2, \lambda = 2, b = 2)$, Set3: $(a = 1.5, \lambda = 3, b = 2)$, and Set4: $(a = 1.5, \lambda = 3, b = 3)$. The MLE, LSE, WLSE, CVE, and RTADE of λ, b, and a are determined, along with their mean estimate Est. $= (1/N)\sum_{i=1}^{N} \hat{\epsilon}_i$ and their mean square errors (MSEs), i.e., $MSE = (1/N)\sum_{i=1}^{N}(\hat{\epsilon}_i - \epsilon)^2$, where, for a given method, ϵ corresponds to λ, a, or b and $\hat{\epsilon}_i$ denotes the considered estimates for ϵ obtained by using the i-th random sample. All the numerical values are set in Tables 2–5.

Table 2. Estimates and mean square errors (MSEs) of TIHLIK model for maximum likelihood (ML), least squares (LS), weighted least square (WLS), Cramer-von Mises minimum distance (CV), and right-tail Anderson–Darling (RTAD) estimates for the Set1, i.e., $a = 1.5$, $\lambda = 2$, $b = 2$.

	MLE		**LSE**		**WLSE**		**CVE**		**RTADE**	
n	**Est.**	**MSEs**	**Est.**	**MSEs**	**Est.**	**MSEs**	**Est.**	**MSEs**	**Est.**	**MSEs**
	1.628	0.043	1.473	0.259	1.581	2.084	1.562	0.459	1.473	0.319
30	1.950	0.102	2.296	0.763	2.329	0.991	2.395	0.888	2.450	1.580
	2.237	0.492	2.109	0.551	2.126	1.188	2.351	1.018	2.171	0.851
	1.603	0.026	1.550	0.263	1.616	0.362	1.575	0.311	1.547	0.189
50	1.895	0.070	2.110	0.588	2.067	0.600	2.201	0.708	2.144	0.639
	2.144	0.282	2.049	0.348	2.094	0.362	2.169	0.534	2.088	0.386
	1.601	0.015	1.505	0.137	1.610	0.242	1.541	0.151	1.492	0.128
100	1.900	0.026	2.162	0.475	2.031	0.498	2.172	0.516	2.241	0.536
	2.124	0.095	2.009	0.120	2.046	0.148	2.068	0.131	2.058	0.139

Table 3. Estimates and MSEs of TIHLIK model for ML, LS, WLS, CV, and RTAD estimates for the Set2, i.e., $a = 2$, $\lambda = 2$, $b = 2$.

	MLE		**LSE**		**WLSE**		**CVE**		**RTADE**	
n	**Est.**	**MSEs**	**Est.**	**MSEs**	**Est.**	**MSEs**	**Est.**	**MSEs**	**Est.**	**MSEs**
	1.808	0.074	1.688	0.324	1.834	0.646	1.775	0.310	1.757	0.297
30	2.344	0.187	2.672	1.429	2.618	1.718	2.766	1.604	2.646	1.264
	2.058	0.226	1.951	0.407	2.052	0.752	2.142	0.572	2.020	0.273
	1.789	0.073	1.657	0.320	1.747	0.399	1.630	0.320	1.824	0.227
50	2.325	0.153	2.726	1.146	2.719	1.400	2.893	1.450	2.524	0.850
	2.022	0.207	1.941	0.167	2.016	0.213	2.006	0.258	2.034	0.247
	1.803	0.053	1.672	0.273	1.715	0.385	1.621	0.301	1.786	0.202
100	2.325	0.131	2.710	1.142	2.714	1.383	2.833	1.286	2.482	0.728
	2.025	0.111	1.942	0.144	1.956	0.124	1.960	0.116	1.968	0.100

Table 4. Estimates and MSEs of TIHLIK model for ML, LS, WLS, CV, and RTAD estimates for the Set3, i.e., $a = 1.5$, $\lambda = 3$, $b = 2$.

	MLE		**LSE**		**WLSE**		**CVE**		**RTADE**	
n	**Est.**	**MSEs**	**Est.**	**MSEs**	**Est.**	**MSEs**	**Est.**	**MSEs**	**Est.**	**MSEs**
	1.869	0.186	1.715	0.419	1.753	0.469	1.778	0.355	1.835	0.304
30	2.435	0.410	2.834	0.939	2.940	2.095	2.942	0.944	2.699	1.218
	2.278	0.547	2.193	0.716	2.186	0.861	2.305	0.614	2.258	0.846
	1.856	0.152	1.616	0.215	1.827	0.458	1.685	0.233	1.851	0.345
50	2.394	0.407	2.876	0.627	2.743	1.126	2.973	0.759	2.621	0.836
	2.186	0.201	2.047	0.265	2.219	0.409	2.193	0.336	2.254	0.421
	1.850	0.132	1.581	0.155	1.804	0.410	1.674	0.197	1.721	0.173
100	2.388	0.393	2.957	0.598	2.709	1.072	2.934	0.683	2.747	0.661
	2.147	0.109	2.032	0.119	2.092	0.133	2.145	0.205	2.095	0.118

Table 5. Estimates and MSEs of TIHLIK model for ML, LS, WLS, CV, and RTAD estimates for the Set4, i.e., $a = 1.5, \lambda = 3, b = 3$.

	MLE		LSE		WLSE		CVE		RTADE	
n	**Est.**	**MSEs**	**Est.**	**MSEs**	**Est.**	**MSEs**	**Est.**	**MSEs**	**Est.**	**MSEs**
	1.810	0.110	1.751	0.358	1.753	0.554	1.775	0.326	1.839	0.399
30	2.400	0.503	2.673	1.243	2.925	1.393	2.959	1.702	2.746	2.197
	3.456	0.973	3.171	1.150	3.330	1.014	3.515	1.673	3.350	1.164
	1.833	0.124	1.561	0.165	1.671	0.344	1.742	0.252	1.737	0.253
50	2.409	0.447	3.021	1.218	2.925	1.063	2.916	1.430	2.849	1.066
	3.408	0.633	3.074	0.740	3.183	0.747	3.502	1.379	3.398	0.862
	1.825	0.111	1.586	0.199	1.692	0.283	1.539	0.112	1.681	0.173
100	2.357	0.454	2.964	0.698	2.836	0.906	3.123	0.632	2.848	0.730
	3.346	0.307	3.090	0.570	3.169	0.362	3.117	0.340	3.260	0.379

The MSE of a, λ, and b for all methods of estimation decreases as n increases. Tables 2–5 show that MLEs get the least MSE of a, λ, and b in all situations, even if some extra bias are observable.

6. Applications to Practical Data Sets

This section provides applications of the TIHLIK model to two practical data sets. We compare the TIHLIK model with some recent efficiency models: those corresponding to the IK distribution, generalized inverted Kumaraswamy (GIK) distribution proposed by [22], Marshall–Olkin extended inverted Kumaraswamy (MOEIK) distribution introduced by [23], and Topp–Leone generalized inverted Kumaraswamy (TLGIK) distribution developed by [24]. The corresponding pdfs are presented below.

- The pdf of the MOEIK distribution is given by

$$f_{MOEIK}(x;\lambda,\alpha,\beta) = \lambda\alpha\beta\frac{(1+x)^{-\alpha-1}\left[1-(1+x)^{-\alpha}\right]^{\beta-1}}{\left[1-(1-\lambda)\left\{1-\left[1-(1+x)^{-\alpha}\right]^{\beta}\right\}\right]^{2}}, \quad x>0,$$

  with $\lambda, \alpha, \beta > 0$.
- The pdf of the GIK distribution is given by

$$f_{GIK}(x;\alpha,\beta,\theta) = \alpha\beta\theta x^{\theta-1}(1+x^{\theta})^{-\alpha-1}\left[1-(1+x^{\theta})^{-\alpha}\right]^{\beta-1}, \quad x>0,$$

  where $\alpha, \beta, \theta > 0$.
- The pdf of the TLGIK distribution is given by

$$f_{TLGIK}(x;\lambda,\alpha,\beta,\theta) = 2\alpha\beta\lambda\theta x^{\lambda-1}(1+x^{\lambda})^{-\alpha-1}\left[1-(1+x^{\lambda})^{-\alpha}\right]^{\beta-1}\times \\ \left\{1-\left[1-(1+x^{\lambda})^{-\alpha}\right]^{\beta}\right\}\left[1-\left\{1-\left[1-(1+x^{\lambda})^{-\alpha}\right]^{\beta}\right\}^{2}\right]^{\theta-1}, \quad x>0,$$

  where $\lambda, \alpha, \beta, \theta > 0$.

Since they have no analytical expressions, the MLEs of the model parameters are computed using an iterative optimization technique (the so-called limited-memory Broyden–Fletcher–Goldfarb–Shanno algorithm allowing bound constraints on variables). The well-known goodness-of-fit measures minus log-likelihood ($-\hat{\ell}$), Akaike information criterion (AIC), Bayesian information criterion (BIC), Anderson–Darling (A^*), and Cramer-von Mises (W^*) values are computed. The rule is clear: the lower the values of these criteria, the better the fit. The value for the Kolmogorov Smirnov (KS) statistic along with its p-value are also provided. We recall that the required computations were carried out via the R software.

The first data set (data set 1). The first practical life data consist of 30 observations of precipitation (in inches) collected in March in Saint Paul, Minneapolis. It was originally reported by [25]. The data are: 0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

The second data set (data set 2). The second data set is obtained from [26]. The data are: 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.40, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92.

A first analysis of the data is given in Table 6, with descriptive statistics of the two considered data sets. Figures 9 and 10 present the boxplots and total test time (TTT) plots for data sets 1 and 2, respectively. In particular, we see that both TTT plots are convex, indicating that the subjacent hrf are increasing, as the hrf of the TIHLIK distribution can be for some values of the parameters. We refer the reader to [27] for more facts about the TTT plot. The use of the TIHLIK model for these data sets is pertinent at the first glance; this will be refined below.

Tables 7 and 8 show the MLEs of the considered models along with the corresponding standard errors for data set 1 and 2, respectively. The goodness-of-fit measures for the considered models are given in Tables 9 and 10 for data set 1 and 2, respectively. Probability–Probability (P–P) plots and Quantile–Quantile (Q–Q) plots for the estimated TIHLIK model are presented in Figures 11 and 12 for data sets 1 and 2, respectively. We can observe the nice adjustments of the scatter plots by the line of equation $y = x$, illustrating that the TIHLIK model is adequate for the considered data. In order to observe the obtained fits, plots of the estimated pdfs over the histograms corresponding to data set 1 and 2 are shown in Figures 13 and 14, respectively.

Table 6. Descriptive statistics for data sets 1 and 2, respectively.

	n	Mean	Median	Standard Deviation	Skewness	Kurtosis
Data set 1	30	1.68	1.47	1.00	1.03	0.93
Data set 2	50	2.95	2.94	0.64	0.4	1.02

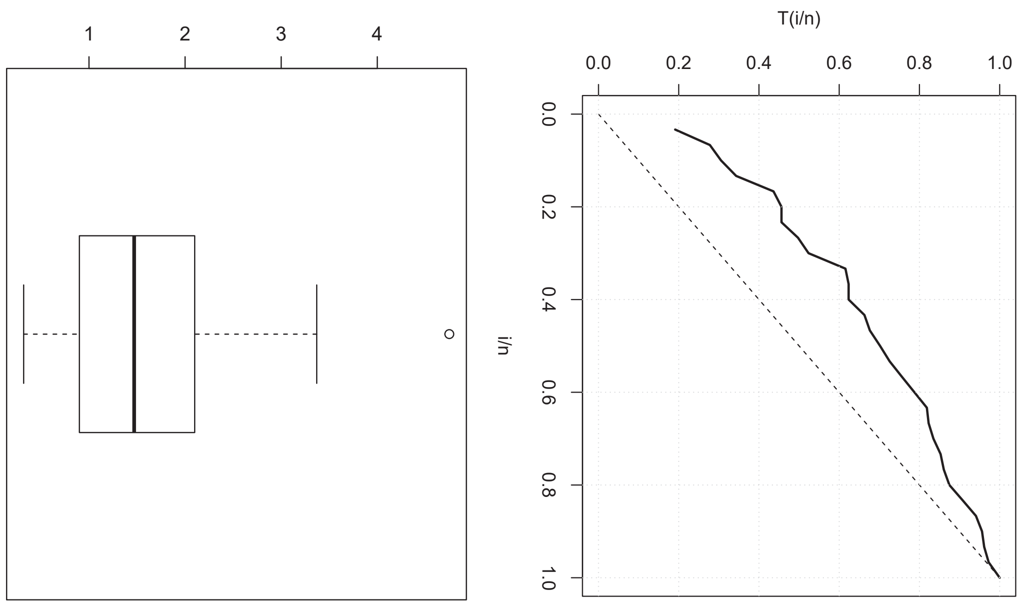

Figure 9. Total test time (TTT) plot and boxplot for data set 1.

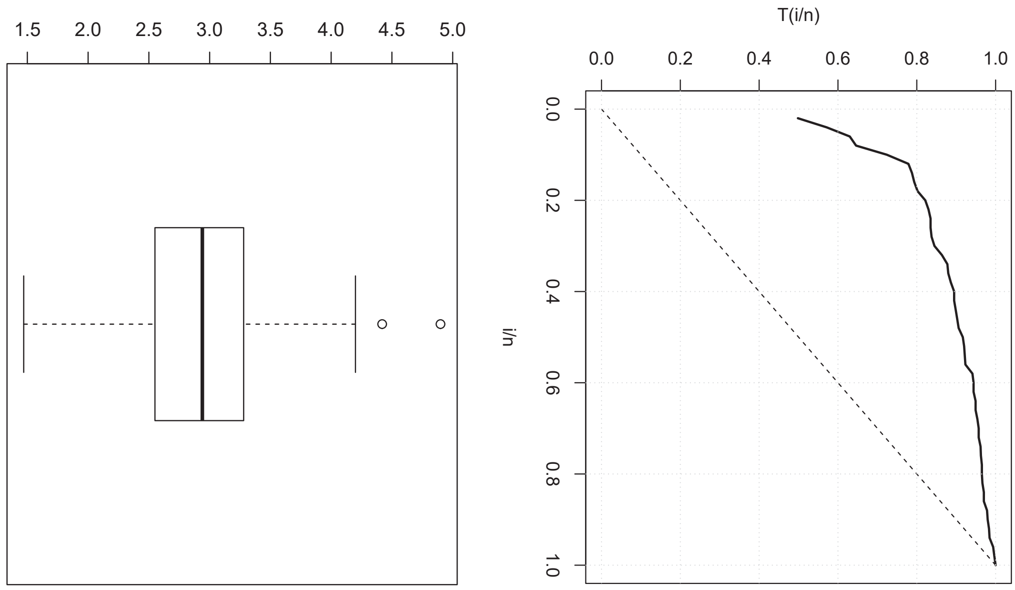

Figure 10. TTT plot and boxplot for data set 2.

Table 7. MLEs along with their standard errors (in parentheses) for data set 1.

Model	a	b	λ	α	β	θ
TIHLIK	0.6888	3.3437	13.5822	-	-	-
	(0.4690)	(0.2944)	(0.6816)	-	-	-
TLGIK	-	-	1.2334	1.9057	21.3625	0.2051
	-	-	(0.9359)	(1.5024)	(3.8678)	(0.3058)
MOEIK	-	-	6.8751	4.3210	6.6101	-
	-	-	(3.5056)	(0.9386)	(2.9989)	-
GIK	-	-		3.9515	1.9557	1.4199
	-	-		(3.7014)	(1.8132)	(1.0215)
IK	2.9874	8.5913	-	-	-	-
	(0.4730)	(3.1230)	-	-	-	-

Table 8. MLEs along with their standard errors (in parentheses) for data set 2.

Model	a	b	λ	α	β	θ
TIHLIK	1.7115	33.0611	30.6921	-	-	-
	(0.5412)	(1.3753)	(0.8705)	-	-	-
TLGIK	-	-	1.7205	2.0776	63.2200	1.1643
	-	-	(0.8203)	(1.1502)	(4.8377)	(1.1403)
MOEIK	-	-	29.7926	5.3374	41.9828	-
	-	-	(17.8294)	(0.4872)	(20.7064)	-
GIK	-	-		8.7194	15.7859	0.3478
	-	-		(6.3860)	(4.2844)	(0.2559)
IK	3.1501	51.0099	-	-	-	-
	(0.2359)	(14.4844)	-	-	-	-

Table 9. The $-\hat{\ell}$, AIC , BIC, W*, A*, KS, and p-value values for data set 1.

Model	$-\hat{\ell}$	**AIC**	**BIC**	**W***	**A***	**KS**	***p*-Value (KS)**
TIHLIK	38.1741	82.3483	83.2714	0.0144	0.1055	0.0575	0.9999
TLGIK	38.5302	85.0605	90.6653	0.0302	0.1943	0.0932	0.9566
MOEIK	38.3427	82.6855	86.8891	0.0196	0.1349	0.0679	0.9871
GIK	39.3172	84.6345	88.8381	0.0514	0.3226	0.1097	0.8628
IK	39.4255	83.2954	85.6534	0.0561	0.3505	0.1143	0.8279

Table 10. The $-\hat{\ell}$, AIC , BIC, W*, A*, KS, and p-value values for data set 2.

Model	$-\hat{\ell}$	AIC	BIC	W*	A*	KS	p-Value (KS)
TIHLIK	47.3670	100.7342	106.4702	0.0616	0.4354	0.0879	0.8338
TLGIK	51.8215	111.6431	119.2911	0.1941	1.2270	0.1153	0.5189
MOEIK	61.4733	128.9467	134.6828	0.0737	0.5004	0.2589	0.0205
GIK	61.0390	128.0782	133.81421	0.3163	1.9228	0.1926	0.0488
IK	68.3343	140.6686	144.4927	0.1668	1.0619	0.2548	0.0030

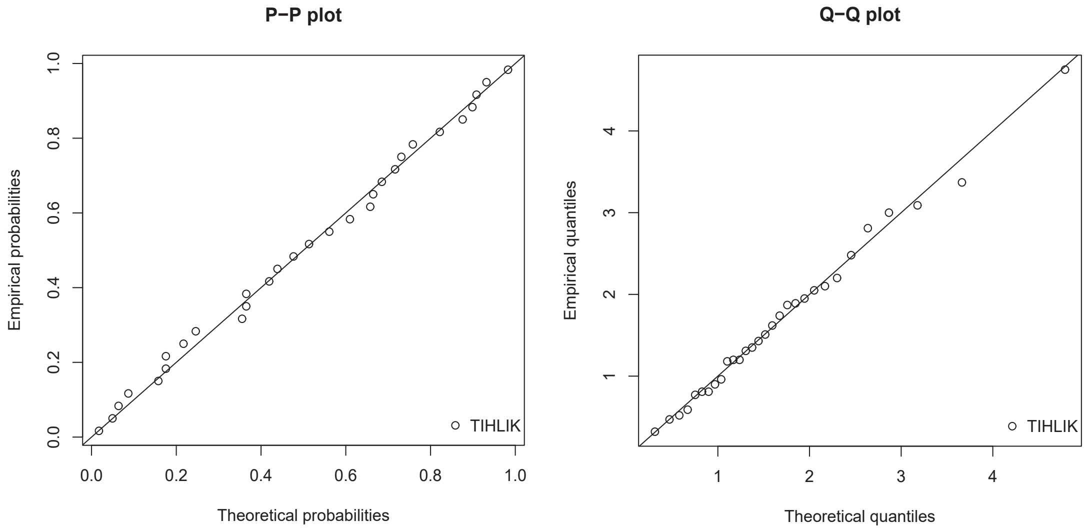

Figure 11. Probability–Probability (P–P) plots and Quantile–Quantile (Q–Q) plot of the estimated TIHLIK model for data set 1.

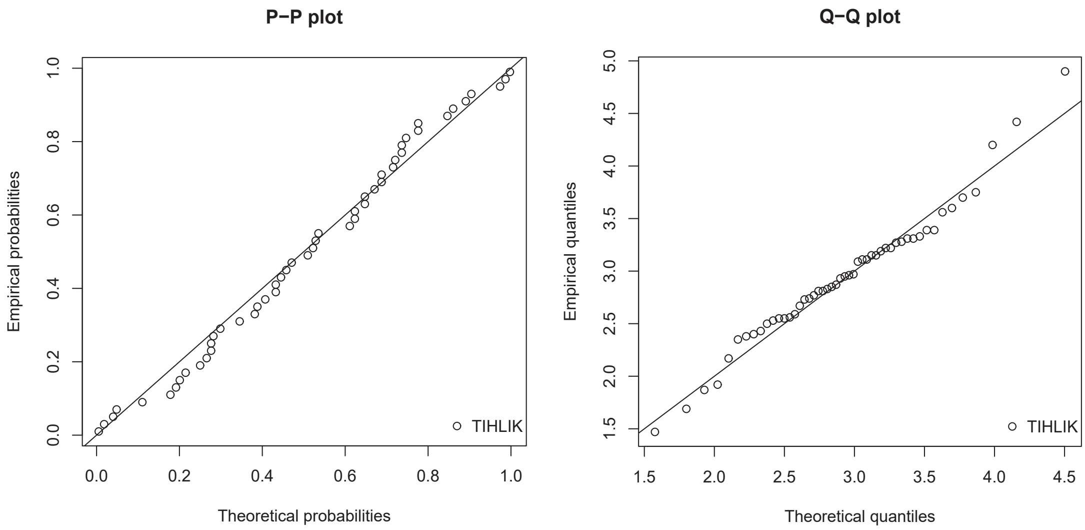

Figure 12. P–P plot and Q–Q plot of the estimated TIHLIK model for data set 2.

By observing the values of the AIC, BIC, A*, and W* in Tables 9 and 10, since they are the smallest for the TIHLIK model, we conclude that it provides the best fit for the considered data sets. This is also confirmed with the p-value of the KS test. The superiority of the TIHLIK model comparing to the others is particularly flagrant for data set 2 in view of the fits in Figure 14. Last but not least, the TIHLIK model has only three parameters; it is thus less complex than the TLGIK and MOEIK models having both one more parameter.

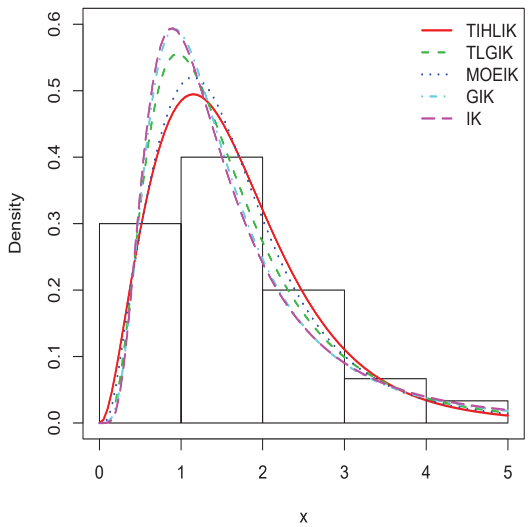

Figure 13. Plots of the estimated pdfs for data set 1.

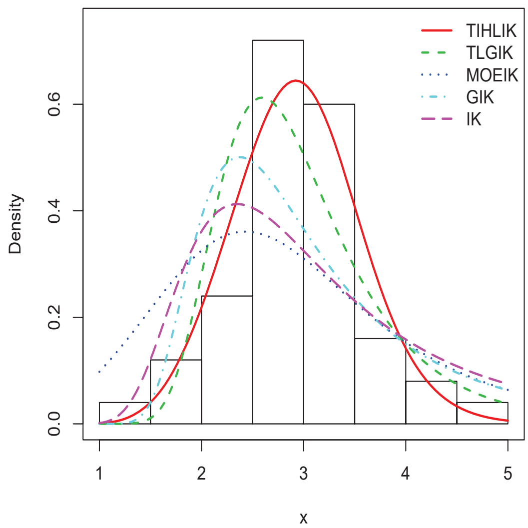

Figure 14. Plots of the estimated pdfs for data set 2.

7. Conclusions

In this paper, we introduce a new three-parameter lifetime distribution called the type I half-logistic inverted Kumaraswamy (TIHLIK) distribution, following the methodology described in Appendix A. Its main mathematical and statistical properties are derived, including mixture representations of crucial functions, shapes and asymptotes, quantile function, ordinary moments, skewness and kurtosis, incomplete moments, stress strength parameter, and order statistics. Several methods are investigated to estimate the TIHLIK model parameters, with efficiency supported by a simulation study considering varying sample sizes. We use two practical data sets to show that the new model can provide adequate fits as compared to four rivals, also based on the inverted Kumaraswamy distribution. In this setting, the smallest values for AIC, BIC, W^*, A^*, and KS are obtained for the TIHLIK model. We thus believe that it can be of interest for statisticians looking for precision in fitting various data sets extracted from sophisticated experiments, among others.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

The main lines of the study, including contributions, are presented in the diagram of Figure A1.

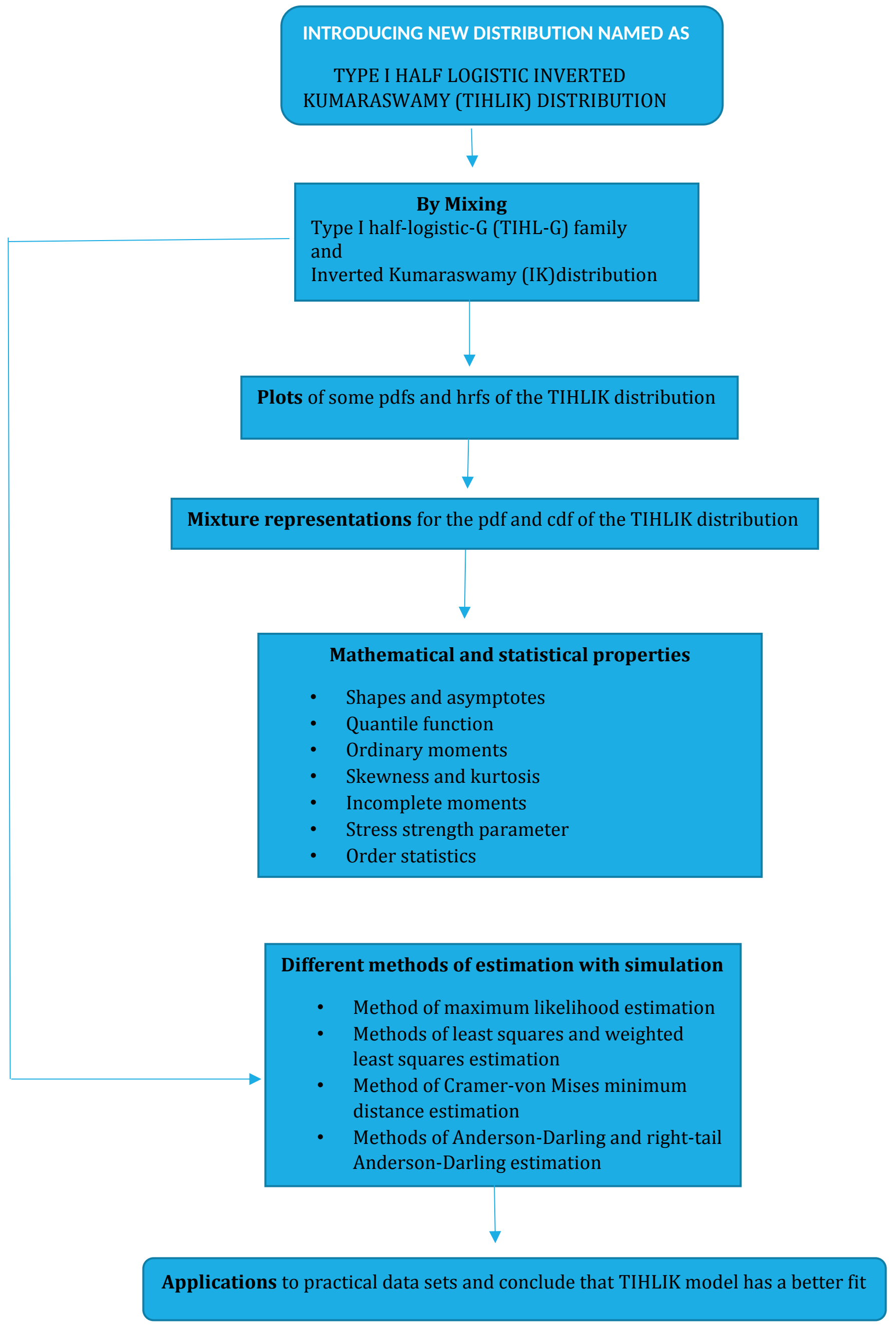

Figure A1. Diagram summarizing the methodology and the main lines of the paper.

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