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Existence Solution and Controllability of Sobolev Type Delay Nonlinear Fractional Integro-Differential System

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Abstract: Fractional integro-differential equations arise in the mathematical modeling of various physical phenomena like heat conduction in materials with memory, diffusion processes, etc. In this manuscript, we prove the existence of mild solution for Sobolev type nonlinear impulsive delay integro-differential system with fractional order $1 < q < 2$. We establish the sufficient conditions for the approximate controllability of Sobolev type nonlinear impulsive delay integro-differential system with fractional order $1 < q < 2$. In addition, we prove the exact null controllability of Sobolev type nonlinear impulsive delay integro-differential system with fractional order $1 < q < 2$. Finally, an example is given to illustrate the obtained results.

Keywords: Sobolev type delay nonlinear impulsive fractional integro-differential equations; resolvent operator; approximate controllability; null controllability

MSC: 34K37; 93B05; 35R12; 45J05

1. Introduction

Recently, fractional differential equations have been considered greatly by research community in various aspects due to its salient features for real world problems (see [1–7]). Controllability problems for different kinds of dynamical systems have been studied by several authors (see [8–15]) and references therein. Thus, the dynamical systems must be treated by the weaker concept of controllability, namely approximate controllability (see [16–21]). However, up to now, no work has been reported yet regarding the null controllability of Sobolev type nonlinear fractional delay integro-differential system with the impulsive condition of order $1 < q < 2$. Motivated by these facts, we study the existence of the mild solution for Sobolev-type impulsive fractional differential equations and we also discuss the sufficient conditions for approximate controllability and null controllability of the same problem.

2. Preliminaries

Let $L(X)$ be a Banach space of all bounded linear operators from X into X equipped with the norm $\|\cdot\|_{L(X)}$ and let $C(J, X)$ be the Banach space of all continuous functions from J to X equipped with the norm $\|\cdot\|_{C(J,X)}$. Moreover, $B_r(x, X)$ is the closed ball in X with center at x and radius $r > 0$.

Definition 1. ([22]). The linear space of all functions from $(-\infty, 0]$ into the Banach space X with a semi-norm $\|\cdot\|_\beta$ is known as phase space β .

The fundamental axioms on β are the following:

(A) If $x : (-\infty, T] \rightarrow X$, $T > 0$ is a continuous function on J and $x_0 \in \beta$ then for every $t \in J$, the following conditions hold:

- (i) $x_t \in \beta$,
- (ii) $\|x(t)\|_X \leq H\|x_t\|_\beta$,
- (iii) $\|x_t\|_\beta \leq K(t) \sup\{\|x(s)\|_X : 0 \leq s \leq t\} + N(t)\|x_0\|_\beta$,

where $H \geq 0$ is a constant, $K : [0, +\infty) \rightarrow [0, +\infty)$ is continuous, $N(\cdot) : [0, +\infty) \rightarrow [0, +\infty)$ is locally bounded, and $K(\cdot)$, $N(\cdot)$ are independent of $x(\cdot)$.

(B) For the function $x(\cdot)$ in (A), $t \rightarrow x_t$ is continuous function for $t \in [0, T]$.

(C) The space β is complete.

To obtain our results, we assume that the abstract fractional integro-differential problem

$$\begin{cases} {}^c D_0^q [\mu x(t)] = Ax(t) + \int_0^t \gamma(t-s)x(s)ds, & q \in (1, 2), \\ x(0) = x_0 \in X, \quad x'(0) = 0, \end{cases} \quad (1)$$

has an associated q -resolvent operator of bounded linear operator $(R_q(t))_{t \geq 0}$ on X .

The operators $A : D(A) \subset X \rightarrow X$ and $\mu : D(\mu) \subset X \rightarrow X$ satisfy the following hypotheses:

- (I) A and μ are closed linear operators,
- (II) $D(\mu) \subset D(A)$ and μ is bijective,
- (III) $\mu^{-1} : X \rightarrow D(\mu)$ is continuous.

Here, (I) and (II) together with the closed graph theorem imply the boundedness of the linear operator $A\mu^{-1} : X \rightarrow X$. Furthermore, $A\mu^{-1}$ generates a strongly continuous semigroup of bounded linear operators in X .

Definition 2. ([23]). A family of bounded linear operators $(R_q(t))_{t \geq 0}$ on X is said to be resolvent operator for Equation (1) if the following conditions are verified:

- (a) The function $R_q(\cdot) : [0, \infty) \rightarrow L(X)$ is strongly continuous and $R_q(0)x = x$ for all $x \in X$ and $q \in (1, 2)$.
- (b) For $x \in D(A)$, $R_q(\cdot)x \in C([0, \infty), X) \cap C([0, \infty), D(A))$,

$$\begin{aligned} {}^c D_0^q [R_q(t)\mu x] &= AR_q(t)x + \int_0^t \gamma(t-s)R_q(s)xds \\ &= R_q(t)Ax + \int_0^t R_q(t-s)\gamma(s)xds, \quad t \in J. \end{aligned} \quad (2)$$

In this paper, we have considered the following conditions.

(P1) The operator $A\mu^{-1} : D(A\mu^{-1}) \subset X \rightarrow X$ is a closed linear operator with $[D(A\mu^{-1})]$ dense in X .

Let $q \in (1, 2)$. For some $\phi_0 \in (0, \frac{\pi}{2}]$, for each $\phi < \phi_0$, there is a positive constant $c_0 = c_0(\phi)$ such that $\lambda \in \rho(A\mu^{-1})$ for each $\lambda \in \Sigma_{0,q,\nu} = \{\lambda \in \mathbb{C}; \lambda \neq 0, |\arg(\lambda)| < q\nu\}$, where $\nu = \phi + \frac{\pi}{2}$ and $\|R(\lambda, A\mu^{-1})\| \leq \frac{c_0}{|\lambda|}$ for all $\lambda \in \Sigma_{0,q,\nu}$.

(P2) For all $t \geq 0$, $A_1(t) : D(A_1(t)) \subseteq X \rightarrow X$ is a closed linear operator, $D(A) \subseteq D(A_1(t))$ and $A_1(\cdot)x$ is strongly measurable on $(0, \infty)$ for each $x \in D(A)$. There exists $b_0 \in L^1_{loc}(R^+)$ such that $\hat{b}_0(\lambda)$ exists for $\operatorname{Re}(\lambda) > 0$ and

$\|A_1(t)x\| \leq b_0(t)\|x\|$ for all $t > 0$ and $x \in D(A)$. Moreover, the operator $\tilde{A}_1 : \Sigma_{0, \frac{\pi}{2}} \rightarrow L([D(A)], X)$ has an analytical extension to $\Sigma_{0, \nu}$ such that $\|\tilde{A}_1(\lambda)x\| \leq \|\tilde{A}_1(\lambda)\|\|x\|$ for all $x \in D(A)$ and $\|\tilde{A}_1(\lambda)\| = O(\frac{1}{\lambda})$ as $|\lambda| \rightarrow \infty$.

(P3) There exists a subspace $D \subseteq D(A)$ dense in $[D(A)]$ and a positive constant c_1 such that $A(D) \subseteq D(A)$, $\tilde{A}_1(\lambda)(D) \subseteq D(A)$ and

$\|A\tilde{A}_1(\lambda)x\| \leq c_1\|x\|$ for every $x \in D$ for all $\lambda \in \Sigma_{0, \nu}$.

For $\tau > 0$ and $\theta \in (\frac{\pi}{2}, \nu)$ then $\Sigma_{\tau, \theta} = \{\lambda \in C; \lambda \neq 0, |\lambda| > \tau, |\arg(\lambda)| < \theta\}$.

For $\Gamma_{\tau, \theta}$, $\Gamma_{\tau, \theta}^i$, $i = 1, 2, 3$ are the paths

$$\Gamma_{\tau, \theta}^1 = \{te^{i\theta}; t \geq \tau\}, \quad \Gamma_{\tau, \theta}^2 = \{te^{i\tilde{\theta}}; -\theta \leq \tilde{\theta} \leq \theta\}, \quad \Gamma_{\tau, \theta}^3 = \{te^{-i\theta}; t \geq \tau\},$$

and $\Gamma_{\tau, \theta} = \bigcup_{i=1}^3 \Gamma_{\tau, \theta}^i$ oriented counterclockwise. In addition,

$$\rho_q(F_q) = \{\lambda \in C; F_q(\lambda) := \lambda^{q-1}(\lambda^q I - A\mu^{-1} - \tilde{A}_1(\lambda))^{-1} \in L(X)\}.$$

We now define the operator family $(R_q(t))_{t \geq 0}$ by

$$R_q(t) = \begin{cases} \frac{1}{2\pi i} \int_{\Gamma_{\tau, \theta}} e^{\lambda t} F_q(\lambda) d\lambda, & t < 0, \\ I, & t = 0. \end{cases}$$

Definition 3. ([23]). Let $q \in (1, 2)$, we define the family $(S_q(t))_{t \geq 0}$ by

$$S_q(t)x = \frac{1}{\Gamma(q-1)} \int_0^t (t-s)^{q-2} R_q(s)x ds, \quad t \geq 0$$

Lemma 1. ([23]). Assume that conditions (P1)–(P3) are fulfilled. Then, there exists a unique q -resolvent operator for the problem (2).

Lemma 2. ([23]). The function $R_q : [0, \infty) \rightarrow L(X)$ is strongly continuous.

Lemma 3. ([23]). If $R(\lambda_0^q, A\mu^{-1})$ is compact for some $\lambda_0^q \in \rho(A\mu^{-1})$, then $R_q(t)$ and $S_q(t)$ are compact for all $t > 0$.

We denote by $(-A\mu^{-1})^q$ the fractional power of operator $(-A\mu^{-1})$.

Lemma 4. ([23]). Suppose that the conditions (P1)–(P3) are satisfied. Let $q \in (1, 2)$ and $\beta \in (0, 1)$ such that there exists positive number C_1 such that:

$$\begin{aligned} \|(-A\mu^{-1})^q R_q(t)\| &\leq C_1 e^{rt} t^{-q\beta}, \\ \|(-A\mu^{-1})^q S_q(t)\| &\leq C_1 e^{rt} t^{q(1-\beta)-1}, \end{aligned}$$

for $t > 0$. If $x \in [D((-A\mu^{-1})^\beta)]$, then

$$(-A\mu^{-1})^\beta R_q(t)x = R_q(t)(-A\mu^{-1})^\beta x, \quad (-A\mu^{-1})^\beta S_q(t)x = S_q(t)(-A\mu^{-1})^\beta x.$$

3. Existence Solution

In this section, we investigate the existence of mild solution of Sobolev type of fractional integro-differential equation with finite delay and impulsive conditions in the following form:

$$\begin{cases} {}^c D_0^q [\mu x(t) + F(t, x_t)] = Ax(t) + \int_0^t \gamma(t-s)x(s)ds + G(t, x_t), t \in J = [0, T], t \neq t_k, \\ x_0 = \phi \in \beta, x'(0) = 0, \\ \Delta x(t_k) = I_k(x_{t_k}), k = 1, 2, 3, \dots, m, \end{cases} \quad (3)$$

where the state x takes values in a Banach space X , A , μ and $(\gamma(t))_{t \in J}$ are closed linear operators on X , ${}^c D_0^q$ represent the Caputo derivative of order $q \in (1, 2)$. The history $x_t : (-\infty, 0] \rightarrow X$ given by $x_t(\theta) = x(t + \theta)$ belongs to some abstract phase space β , defined later, F , G and I_k are appropriate functions, $0 < t_1 < t_2 < \dots < t_k < T$ are prefixed points and $\Delta x(t_k)$ is the jump of the solutions at impulse points t_k , which is defined by $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$.

To establish the result, we need the following hypotheses:

Hypothesis 1 (H1). $A\mu^{-1}$ is the infinitesimal generator of a resolvent operator $R_q(t)$ in X and there exists constant $M > 0$ such that

$$\|R_q(t)\|_{L(X)} \leq M \text{ and } \|S_q(t)\|_{L(X)} \leq M, t \in J \text{ and } \|(-A\mu^{-1})^\alpha S_q(t)\|_X \leq Mt^{q\alpha-1}, 0 < t \leq T.$$

Hypothesis 2 (H2). The functions $F, G : J \times \beta \rightarrow X$ and $I_k : \beta \rightarrow X$ are continuous and there exist positive constants L_F, L_G and L_k such that

$$\begin{aligned} \|(-A\mu^{-1})^\alpha F(t, \delta_1) - (-A\mu^{-1})^\alpha F(t, \delta_2)\|_X &\leq L_F \|\delta_1 - \delta_2\|_\beta, \delta_i \in \beta, t \in J, i = 1, 2, \\ \|(-A\mu^{-1})^\alpha F(t, \delta)\|_X &\leq L_F (\|\delta\|_\beta + 1), \\ \|G(t, \delta_1) - G(t, \delta_2)\|_X &\leq L_G \|\delta_1 - \delta_2\|_\beta, \delta_i \in \beta, t \in J, i = 1, 2, \\ \|I_k(\delta_1) - I_k(\delta_2)\|_X &\leq L_k \|\delta_1 - \delta_2\|_\beta, \delta_i \in \beta, i = 1, 2. \end{aligned}$$

Hypothesis 3 (H3). $\gamma(\cdot)x \in C(J, X)$ for every $x \in [D((A\mu^{-1})^{1-\alpha})]$ and there exist a function $\varrho(\cdot) \in L^1(J, \mathbb{R}^+)$ such that:

$$\|\gamma(s)S_q(t)\|_{L([D((A\mu^{-1})^\alpha)], X)} \leq M\varrho(s)t^{q\alpha-1}, 0 \leq s < t \leq T.$$

In addition, there exists a constant $L_S > 0$ such that $\|\varrho(s)\| \leq L_S$

Hypothesis 4 (H4). There exists a constant $r > 0$ such that

$$\begin{aligned} \{M\|\mu^{-1}\|[\|\mu\|\|\phi\|_\beta + L_F] + \|\mu^{-1}\|L_F + \|(-A)^{1-\beta}\|\|\mu^{-1}\|L_F M \frac{T^{q\beta}}{q\beta} + \|\mu^{-1}\|M \frac{T^{q\beta}}{q\beta} L_G \\ + \|(-A)^\beta\|\|\mu^{-1}\|L_S M \frac{T^{q\beta}}{q\beta} + \|\mu^{-1}\|ML_k\} \leq r. \end{aligned}$$

Definition 4. ([23]). We say that $x \in C(J, X)$ is a mild solution of the system (3) on J , if it satisfies

$$\begin{aligned} x(t) = & \mu^{-1}R_q(t)[\mu\phi(0) + F(0, \phi)] - \mu^{-1}F(t, x_t) - \int_0^t \mu^{-1}A\mu^{-1}S_q(t-s)F(s, x_s)ds \\ & - \mu^{-1} \int_0^t \int_0^s \gamma(s-\tau)S_q(t-s)F(\tau, x_\tau)d\tau ds + \mu^{-1} \int_0^t S_q(t-s)G(s, x_s)ds \\ & + \mu^{-1} \sum_{0 < t_k < t} R_q(t-t_k)I_k(x(t_k^-)), \quad t \in [0, T]. \end{aligned}$$

Theorem 1. If Hypotheses (H1)–(H4) are satisfied, then the system (3) has a unique mild solution on J provided that

$$\omega_1 := L_F \|(-A)^{-\beta}\| \|\mu^{-1}\| + \|\mu^{-1}\| ML_k + \|\mu^{-1}\| M \frac{T^{q\beta}}{q\beta} [L_F \|(-A)^{1-\beta}\| + L_F \|(-A)^{-\beta}\| L_S + L_G] < 1.$$

Proof. Consider the operator ψ on $C(J, X)$ defined as follows:

$$\begin{aligned} \psi x(t) = & \mu^{-1}R_q(t)[\mu\phi(0) + F(0, \phi)] - \mu^{-1}F(t, x_t) + \int_0^t T_q(t-s)\mu^{-1}[G(s, x_s)]ds \\ & - \int_0^t A\mu^{-1}T_q(t-s)F(s, x_s)ds - \int_0^t \mu^{-1}T_q(t-s) \int_0^s \gamma(s-v)F(v, x_v)dv ds + \sum_{0 < t_k < t} \mu^{-1}R_q(t-t_k)I_k(x(t_k^-)). \end{aligned}$$

□

We want to prove that the operator ψ has a fixed point.

First, we show that ψ maps B_r into itself. For $x \in B_r$,

$$\begin{aligned} \|\psi x(t)\| \leq & \|\mu^{-1}\| \|R_q(t)\| [\|\mu\| \|\phi(0) + F(0, \phi)\|] + \|\mu^{-1}\| \|F(t, x_t)\| \\ & + \|(-A)^{1-\beta}\| \int_0^t M(t-s)^{q\beta-1} \|(-A)^\beta F(s, x_s)\| ds \\ & + \|\mu^{-1}\| \|(-A)^\beta\| \int_0^t \int_0^s \varrho(s-v) M(t-s)^{q\beta-1} \|(-A)^{-\beta}\| F(v, x_v) \| dv ds \\ & + \|\mu^{-1}\| \int_0^t M(t-s)^{q\beta-1} \|G(s, x_s)\| ds + \|\mu^{-1}\| ML_K \\ \|\psi x(t)\| \leq & M \|\mu^{-1}\| [\|\mu\| \|\phi\|_\beta + L_F] + \|\mu^{-1}\| L_F + \|(-A)^{1-\beta}\| \|\mu^{-1}\| L_F M \frac{T^{q\beta}}{q\beta} + \|\mu^{-1}\| M \frac{T^{q\beta}}{q\beta} L_G \\ & + \|(-A)^\beta\| \|\mu^{-1}\| L_S M \frac{T^{q\beta}}{q\beta} + \|\mu^{-1}\| ML_k \leq r. \end{aligned}$$

Thus, ψ maps B_r into itself.

Next, we prove that ψ is contraction on B_r . For, $m, n \in B_r$, we obtain

$$\begin{aligned} \|\psi m(t) - \psi n(t)\| \leq & \|\mu^{-1}\| \|(-A)^{-\beta}\| \|(-A)^\beta F(t, m_t) - (-A)^\beta F(t, n_t)\| \\ & + \|\mu^{-1}\| \int_0^t \|(-A)^{1-\beta} S_q(t-s)\| \|(-A)^\beta F(s, m_s) - (-A)^\beta F(s, n_s)\| ds \\ & + \|(-A)^{-\beta}\| \|\mu^{-1}\| \int_0^t \int_0^s \varrho(s-v) S_q(t-s) \|(-A)^\beta F(v, m_v) - (-A)^\beta F(v, n_v)\| dv ds \\ & + \int_0^t \|S_q(t-s)\| \|\mu^{-1}\| \|G(s, m_s) - G(s, n_s)\| ds \\ & + \sum_{0 < t_k < t} \|R_q(t-t_k)\| L_k \|\mu^{-1}\| \|(m(t_k^-)) - (n(t_k^-))\| \\ \|\psi m(t) - \psi n(t)\| \leq & [L_F \|(-A)^{-\beta}\| \|\mu^{-1}\| + \|\mu^{-1}\| L_F \|(-A)^{1-\beta}\| M \frac{T^{q\beta}}{q\beta} \\ & + ML_F \|\mu^{-1}\| \|(-A)^{-\beta}\| \frac{T^{q\beta}}{q\beta} L_S + ML_G \|\mu^{-1}\| \frac{T^{q\beta}}{q\beta} + \|\mu^{-1}\| ML_k] \|n - m\|_Z. \end{aligned} \tag{4}$$

Then, ψ is a contraction mapping on B_r . Next, we will prove that ψ is completely continuous

Let, $(x^n)_{n \in \mathbb{N}}$ be a sequence in B_r and $x \in B_r$ such that $x_n \rightarrow x$, we want to prove that $\|x^n - x\| \rightarrow 0$ as $n \rightarrow \infty$

$$\begin{aligned} \|\psi x^n(t) - \psi x(t)\| &\leq \|\mu^{-1}\| \|F(t, (x_t^n)) - F(t, x_t)\| \\ &+ \int_0^t \|\mu^{-1}\| \|S_q(t-s)\| \left\{ \int_0^s \|\gamma(s-v)F(v, x_v^n) - \gamma(s-v)F(v, x_v)\| dv \right\} ds \\ &+ \int_0^t \|A\mu^{-1}\| \|S_q(t-s)\| \|F(s, (x_s^n)) - F(s, x_s)\| ds \\ &+ \int_0^t \|\mu^{-1}\| \|S_q(t-s)\| \left\{ \|G(s, x_s^n) - G(s, x_s)\| \right\} ds + \sum_{0 < t_k < t} \|\mu^{-1}\| \|R_q\| \mu \|(t-t_k)^{q-1}\| \left\{ \|I_k(x^n(t_k^-)) - I_k(x(t_k^-))\| \right\}. \end{aligned}$$

Since the functions γ , F and G are continuous, i.e., then, as $n \rightarrow \infty$, the following are satisfied:

$$\|F(v, x_v^n) - F(v, x_v)\| \rightarrow 0, \|F(t, x_t^n) - F(t, x_t)\| \rightarrow 0,$$

$$\|F(s, (x_s^n)) - F(s, x_s)\| \rightarrow 0, \|G(s, x_s^n) - G(s, x_s)\| \rightarrow 0,$$

$$\|I_k(x^n(t_k^-)) - I_k(x(t_k^-))\| \rightarrow 0,$$

$$\lim_{n \rightarrow \infty} \|\psi x^n - \psi x\| = 0.$$

Therefore, ψ is continuous.

Next, we show that $(\psi x)(t)$ is equicontinuous on J for any $x \in B_r$. Let $0 < t \leq b$ and $\epsilon > 0$ be sufficiently small; then,

$$\begin{aligned} \|(\psi x)(t+\epsilon) - (\psi x)(t)\| &\leq \|\mu^{-1}\| \left[\|\mu\phi(0) + F(0, \phi)\| \|R_q(t+\epsilon) - R_q(t)\| \right. \\ &\quad \left. + \|\mu^{-1}\| \|F(t+\epsilon, x_{t+\epsilon}) - F(t, x_t)\| \right] \\ &\quad + \|\mu^{-1}\| \int_0^{t+\epsilon} \|A\mu^{-1}\| \|S_q(t+\epsilon-s) - S_q(t-s)\| \|F(s, x_s)\| ds \\ &\quad + \|\mu^{-1}\| \int_0^{t+\epsilon} \|S_q(t+\epsilon-s) - S_q(t-s)\| \left\{ \int_0^s \|\gamma(s-\tau)F(\tau, x_\tau)\| d\tau \right\} ds \\ &\quad + \|\mu^{-1}\| \int_0^{t+\epsilon} \|S_q(t+\epsilon-s) - S_q(t-s)\| \|G(s, x_s)\| ds \\ &\quad + \|\mu^{-1}\| \sum_{0 < t_k < t+\epsilon} \|R_q(t+\epsilon)I_k(x(t_k^+))\| - \sum_{0 < t_k < t} \|R_q(t-t_k)I_k(x(t_k^+))\|. \end{aligned} \quad (5)$$

It is known that the right-hand side of (5) tends to zero as $\epsilon \rightarrow 0$. Hence, $(\psi x)(t)$ is completely continuous on J . By using a fixed point theorem, ψ has a unique fixed point $x(t)$ on J . Therefore, system (3) has a unique mild solution on J .

4. Approximate Controllability

We will establish a set of sufficient conditions for approximate controllability of impulsive delay fractional differential equation in the following form:

$$\begin{cases} {}^c D_0^q [\mu x(t) + F(t, x_t)] = Ax(t) + \int_0^t \gamma(t-s)x(s)ds + G(t, x_t) + Bu(t), & t \in J = [0, T], \quad t \neq t_k, \\ x_0 = \phi \in \beta, \quad x'(0) = 0, \\ \Delta x(t_k) = I_k(x_{t_k}), \quad k = 1, 2, 3, \dots, m, \end{cases} \quad (6)$$

where the control function $u(\cdot) \in L^2(J, U)$, the Banach space of admissible control functions with U a Banach space and B is a bounded linear operator from U into X .

Definition 5. We say that $x \in C(J, X)$ is a mild solution of system (6) if it satisfies

$$\begin{aligned} x(t) &= \mu^{-1} R_q(t) [\mu\phi(0) + F(0, \phi)] - \mu^{-1} F(t, x_t) - \mu^{-1} \int_0^t A\mu^{-1} S_q(t-s) F(s, x_s) ds \\ &- \mu^{-1} \int_0^t \int_0^s \gamma(s-\tau) S_q(t-s) F(\tau, x_\tau) d\tau ds + \mu^{-1} \int_0^t S_q(t-s) [G(s, x_s) + Bu(s)] ds \\ &+ \mu^{-1} \sum_{0 < t_k < t} R_q(t-t_k) I_k(x(t_k^-)), \quad t \in J. \end{aligned}$$

In order to study the approximate controllability for system (6), we introduce the following linear fractional differential system:

$$\begin{cases} {}^c D_0^q [\mu x(t)] = Ax(t) + Bu(t), t \in J = [0, T], \\ x_0 = \phi \in \beta, x'(0) = 0. \end{cases} \quad (7)$$

We define the operators associated with (6) as follows:

$$\begin{aligned} \Gamma_0^T &= \int_0^T (T-s)^{q-1} S_q(T-s) B B^* \mu^{-1} S_q^*(T-s) ds, \\ R(\lambda, \Gamma_0^T) &= (\lambda I + \Gamma_0^T)^{-1}, \lambda > 0. \end{aligned} \quad (8)$$

Let $x(T, x_0, u)$ be the state value of (6) at terminal state T , corresponding to the control u and the initial value x_0 . Denote by $R(T, x_0) = \{x(T, x_0, u) : u \in L^2(J, U)\}$ the reachable set of system (6) at terminal time T , its closure in X is denoted by $\overline{R(T, x_0)}$.

Definition 6. System (6) is said to be approximately controllable on the interval $[0, T]$ if $\overline{R(T, x_0)} = X$.

Lemma 5. ([24]). The linear system (7) is approximate controllable on $[0, T]$ if and only if the operator $\lambda R(\lambda, \Gamma_0^T) = \lambda(\lambda I + \Gamma_0^T)^{-1} \rightarrow 0$ as $\lambda \rightarrow 0$ in the strong operator topology.

To establish the result, we need the following addition.

Hypothesis 5 (H5). There exist a constant $r > 0$ such that

$$M \|\mu^{-1}\| \psi_1 + \|\mu^{-1}\| L_F \psi_2 + \psi_3 + \|\mu^{-1}\| M L_k + \frac{\|B\| \|B^*\| M \|\mu^{-1}\|}{\lambda} \zeta_1 \leq r,$$

where

$$\begin{aligned} \psi_1 &= [\|\mu\| \|\phi\|_p + L_F], \psi_2 = [1 + \frac{T^{q\beta}}{q\beta} M], \\ \psi_3 &= \|\mu^{-1}\| M \frac{T^{q\beta}}{q\beta} L_G + \|(-A)^\beta\| \|\mu^{-1}\| L_S M \frac{T^{q\beta}}{q\beta}, \\ \zeta_1 &= \{\|h\| + M \|\mu^{-1}\| [\|\mu\| \|\phi\|_\beta + L_F] + \|\mu^{-1}\| L_F + \|(-A)^{1-\beta}\| \|\mu^{-1}\| L_F M \frac{T^{q\beta}}{q\beta} + \|\mu^{-1}\| M \frac{T^{q\beta}}{q\beta} L_G \\ &\quad + \|(-A)^\beta\| \|\mu^{-1}\| L_S M \frac{T^{q\beta}}{q\beta} + \|\mu^{-1}\| M L_k\} \\ \zeta_2 &= \{\|\mu^{-1}\| L_F + M \|\mu^{-1}\| \frac{T^{q\beta}}{q\beta} [\|(-A)^{1-\beta}\| L_F + L_S L_F] + \|\mu^{-1}\| \frac{T^{q\beta}}{q\beta} L_G + \|\mu^{-1}\| M L_k + \\ &\quad \frac{\|B\| \|B^*\| M \|\mu^{-1}\|}{\lambda} \zeta_1\}. \end{aligned}$$

Theorem 2. If hypotheses (H1)–(H3) and (H5) are satisfied, then system (6) has a mild solution on J provided that

$$\zeta_2 < 1.$$

Proof. Consider the operator ψ^* on $C(J, X)$ as follows:

$$\begin{aligned} (\psi^* x)(t) &= \mu^{-1} R_q(t) [\mu \phi(0) + F(0, \phi)] - \mu^{-1} F(t, x_t) - \mu^{-1} \int_0^t A \mu^{-1} S_q(t-s) F(s, x_s) ds \\ &\quad - \mu^{-1} \int_0^t \int_0^s \gamma(s-\tau) S_q(t-s) F(\tau, x_\tau) d\tau ds + \mu^{-1} \int_0^t S_q(t-s) [G(s, x_s) + Bu(s)] ds \\ &\quad + \mu^{-1} \sum_{0 < t_k < t} R_q(t-t_k) I_k(x(t_k^-)), \end{aligned}$$

where

$$\begin{aligned} u(t) = & B^* S_q^*(T-t)(\lambda I + \Gamma_0^T)^{-1} \{h - \mu^{-1} R_q(T)[\mu \phi(0) + F(0, \phi)] + \mu^{-1} F(T, x_T) \\ & + \mu^{-1} \int_0^T A \mu^{-1} S_q(T-s) F(s, x_s) ds \\ & + \mu^{-1} \int_0^T \int_0^s \gamma(s-\tau) S_q(T-s) F(\tau, x_\tau) d\tau ds - \mu^{-1} \int_0^T S_q(T-s) G(s, x_s) ds \\ & - \mu^{-1} \sum_{0 < t_k < T} R_q(T-t_k) I_k(x(t_k^-))\}. \end{aligned}$$

□

We want to prove that the operator ψ^* has a fixed point. This fixed point is then a mild solution of system (6). We show that ψ^* maps B_r into itself, $x \in B_r$,

$$\begin{aligned} \|\psi^* x\| \leq & \|\mu^{-1} R_q(t)[\mu \phi(0) + F(0, \phi)]\| + \|\mu^{-1} F(t, x_t)\| + \int_0^t \|\mu^{-1}\| \|A \mu^{-1} S_q(t-s) F(s, x_s)\| ds \\ & + \int_0^t \int_0^s \|\mu^{-1}\| \|\gamma(s-\tau) S_q(t-s) F(\tau, x_\tau)\| d\tau ds + \int_0^t \|\mu^{-1}\| \|S_q(t-s)[G(s, x_s) + Bu(s)]\| ds \\ & + \sum_{0 < t_k < t} \|\mu^{-1}\| \|R_q(t-t_k) I_k(x(t_k^-))\| \\ \leq & M \|\mu^{-1}\| [\|\mu\| \|\phi\|_\beta + L_F] + \|\mu^{-1}\| L_F + \|(-A)^{1-\beta}\| \|\mu^{-1}\| L_F M \frac{T^{q\beta}}{q\beta} + \|\mu^{-1}\| M \frac{T^{q\beta}}{q\beta} L_G \\ & + \|(-A)^\beta\| \|\mu^{-1}\| L_S M \frac{T^{q\beta}}{q\beta} + \|\mu^{-1}\| M L_k + \frac{\|B\| \|B^*\| M \|\mu^{-1}\|}{\lambda} \\ \times \{ & \|h\| + M \|\mu^{-1}\| [\|\mu\| \|\phi\|_\beta + L_F] + \|\mu^{-1}\| L_F + \|(-A)^{1-\beta}\| \|\mu^{-1}\| L_F M \frac{T^{q\beta}}{q\beta} + \|\mu^{-1}\| M \frac{T^{q\beta}}{q\beta} L_G \\ & + \|(-A)^\beta\| \|\mu^{-1}\| L_S M \frac{T^{q\beta}}{q\beta} + \|\mu^{-1}\| M L_k \} \\ = & M \|\mu^{-1}\| \psi_1 + \|\mu^{-1}\| L_F \psi_2 + \psi_3 + \|\mu^{-1}\| m L_k + \frac{\|B\| \|B^*\| M \|\mu^{-1}\|}{\lambda} \zeta_1 \leq r. \end{aligned}$$

Thus, ψ^* maps B_r into itself.

Next, for $x, y \in B_r$, we obtain

$$\begin{aligned} \|\psi^* x(t) - \psi^* y(t)\| \leq & \|\mu^{-1}\| [\|F(t, x_t) - F(t, y_t)\|] \\ & + \|\mu^{-1}\| \|(-A)^{1-\beta}\| \int_0^t \|S_q(t-s)\| \|(-A)^\beta F(s, x_s) - F(s, y_s)\| ds \\ & + \|\mu^{-1}\| \int_0^t \int_0^s \|\gamma(s-\tau)\| \|S_q(t-s)\| \|F(\tau, x_\tau) - F(\tau, y_\tau)\| d\tau ds \\ & + \|\mu^{-1}\| \int_0^t \|S_q(t-s)\| [\|G(s, x_s) - G(s, y_s)\|] ds + \sum_{0 < t_k < t} \|\mu^{-1}\| \|R_q(t-t_k)\| \|I_k(x(t_k^-)) - I_k(y(t_k^-))\| \\ & + \|\mu^{-1}\| \int_0^t \|S_q(t-s)\| \|B\| \|B^*\| (\lambda I + \Gamma_0^T)^{-1} \times \{\|\mu^{-1}\| [\|F(t, x_t) - F(t, y_t)\|] \\ & + \|\mu^{-1}\| \|(-A)^{1-\beta}\| \int_0^t \|S_q(T-s)\| \|(-A)^\beta F(s, x_s) - F(s, y_s)\| ds \\ & + \|\mu^{-1}\| \int_0^T \int_0^s \|\gamma(s-\tau)\| \|S_q(T-s)\| \|F(\tau, x_\tau) - F(\tau, y_\tau)\| d\tau ds \\ & + \|\mu^{-1}\| \int_0^T \|S_q(T-s)\| [\|G(s, x_s) - G(s, y_s)\|] ds + \sum_{0 < t_k < T} \|\mu^{-1}\| \|R_q(T-t_k)\| \|I_k(x(t_k^-)) - I_k(y(t_k^-))\|\} \end{aligned} \quad (9)$$

$$\begin{aligned} \leq & \{\|\mu^{-1}\| L_F + M \|\mu^{-1}\| \frac{T^{q\beta}}{q\beta} [\|(-A)^{1-\beta}\| L_F + L_S L_F] \\ & + \|\mu^{-1}\| \frac{T^{q\beta}}{q\beta} L_G + \|\mu^{-1}\| M L_k + \frac{\|B\| \|B^*\| M \|\mu^{-1}\|}{\lambda} \zeta_1\} \|x(t) - y(t)\|. \end{aligned}$$

This implies that

$$\|(\psi^*x) - (\psi^*y)\| \leq \zeta_2 \|x - y\|.$$

Then, ψ^* is a contraction mapping and hence there exist a unique fixed point $x \in B_r$ such that $\psi^*x(t) = x(t)$. Hence, any fixed point of ψ^* is a mild solution of (6) on J .

Theorem 3. Assume that hypotheses (H1)–(H5) hold. Further, if the functions

$F : J \times X \times X \rightarrow X, G : J \times X \rightarrow X$ are uniformly bounded and the resolvent operator R_q and S_q compact operators, then system (6) is approximately controllable on J .

Proof. Let $x^\lambda(\cdot)$ be a fixed point of ψ^* in B_r . Any fixed point of ψ^* is a mild solution of (6) on J under the control

$$u^\lambda(t) = B^*S_q(T-t)R(\lambda, \Gamma_0^T)P(x^\lambda), \quad (10)$$

where

$$p(x^\lambda) = h - \mu^{-1}R_q(T)[\mu\phi(0) + F(0, \phi)] + \mu^{-1}F(T, x_T) + \mu^{-1} \int_0^T A\mu^{-1}S_q(T-s)F(s, x_s)ds \\ + \mu^{-1} \int_0^T \int_0^s \gamma(s-\tau)S_q(T-s)F(\tau, x_\tau)d\tau ds - \mu^{-1} \int_0^T S_q(T-s)G(s, x_s)ds - \mu^{-1} \sum_{0 < t_k < T} R_q(T-t_k)I_k(x(t_k^-)) \} \quad (11)$$

and satisfies

$$x^\lambda(T) = h - \lambda(\lambda I + \Gamma_0^T)^{-1}p(x^\lambda). \quad (12)$$

Thus, the Lebesgue dominated convergence theorem and the compactness of $S_q(t)$ yield

$$\|x^\lambda(T) - h\| \leq \|\lambda(\lambda I + \Gamma_0^T)^{-1}\| \|h - \mu^{-1}R_q(T)[\mu\phi(0) + F(0, \phi)]\| + \|\lambda(\lambda I + \Gamma_0^T)^{-1}A\mu^{-1}F(T, x_T)\| \\ + \|\lambda(\lambda I + \Gamma_0^T)^{-1} \int_0^T \int_0^s \gamma(s-\tau)S_q(T-s)\mu^{-1}F(\tau, x_\tau)d\tau ds\| \\ + \|\lambda(\lambda I + \Gamma_0^T)^{-1} \int_0^T S_q(T-s)\mu^{-1}G(s, x_s)ds\| \\ + \|\lambda(\lambda I + \Gamma_0^T)^{-1}\| \|\mu^{-1}\| \sum_{0 < t_k < T} \|R_q(t-t_k)\| \|I_k(x(t_k^-))\| \} \rightarrow 0, \text{ as } \lambda \rightarrow 0 \quad (13)$$

Then, system (6) is approximate controllable. $\square$

5. Exact Null Controllability

In this section, we investigate the exact null controllability of fractional nonlinear differential equation of the system (6).

To study the exact null controllability of (6), we consider the fractional linear system

$$\begin{cases} {}^c D_0^q[\mu x(t)] = Ax(t) + G(t) + F(t) + Bu(t), \quad t \in J = [0, T], \\ x_0 = \phi \in \beta, \quad x'(0) = 0, \end{cases} \quad (14)$$

associated with system (6).

Define the operator $L_0^T u = \int_0^T \mu^{-1}S_q(T-s)Bu(s)ds : L^2(J, U) \rightarrow X, u \in L^2(J, U)$, where $L_0^T u$ has a bounded inverse operator $(L_0)^{-1}$ with values in $L^2(J, U) / \ker(L_0^T)$ and, $N_0^T(x_0, F, G) = \mu^{-1}R_q(T)x_0 + \int_0^T \mu^{-1}S_q(T-s)[G(s) + F(s)]ds : X \times L^2(J, U) \rightarrow X$.

Definition 7. ([25].) The linear system (14) is said to be exactly null controllable on J if $\text{Im}L_0^T \supset \text{Im}N_0^T$.

Definition 8. ([26].) System (6) is said to be exactly null controllable if there is a $u \in L^2(J, U)$, such that the solution x of the system (6) satisfies $x(T) = 0$.

Definition 9. ([26].) Suppose that the linear system is exactly null controllable on $[0, T]$. Then, the linear operator $H := L_0^{-1}N_0^T : X \times L^2(J, X) \rightarrow L^2(J, U)$ is bounded.

In this section, we need the following assumption.

Hypothesis 6 (H6). The linear system (14) is exactly null controllable on $[0, T]$. Through this section, set

$$\begin{aligned} d_1 &= \|(-A)^{1-\beta}\| \|\mu^{-1}\| L_F M \frac{T^{q\beta}}{q\beta} + \|\mu^{-1}\| M \frac{T^{q\beta}}{q\beta} L_G + \|(-A)^\beta\| \|\mu^{-1}\| L_S M \frac{T^{q\beta}}{q\beta} + \|\mu^{-1}\| M L_k, \\ d_2 &= \|(-A)^{1-\beta}\| \|\mu^{-1}\| L_F M \frac{T^{q\beta}}{q\beta} + \|\mu^{-1}\| M \frac{T^{q\beta}}{q\beta} L_G, \\ d_3 &= \|(-A)^\beta\| \|\mu^{-1}\| L_S M \frac{T^{q\beta}}{q\beta} + \|\mu^{-1}\| M L_k, \\ \lambda_1 &= M \|\mu^{-1}\| \frac{T^{q\beta}}{q\beta} [\|(-A)^{1-\beta}\| L_F + L_S L_F] + \|\mu^{-1}\| \frac{T^{q\beta}}{q\beta} L_G + \|\mu^{-1}\| M L_k, \\ \lambda_2 &= (\|\mu\| \|\phi\|_p + L_F) M \|\mu^{-1}\| + L_F \|\mu^{-1}\| + d_1. \end{aligned}$$

Now, we are able to state and prove our main results.

Theorem 4. Assume assumptions (H1)–(H3) and (H6) are satisfied. Then, the system (6) is exactly null controllable provided that:

$$[\|\mu^{-1}\| L_F + \lambda_1 + \|B\| \|H\| \lambda_2 \frac{T^{q\beta}}{q\beta}] < 1.$$

Proof. For an arbitrary x , define the operator φ on $C(J, X)$ as follows:

$$\begin{aligned} (\varphi x)(t) &= \mu^{-1} R_q(t) [\mu \phi(0) + F(0, \phi)] - \mu^{-1} F(t, x_t) - \mu^{-1} \int_0^t A \mu^{-1} S_q(t-s) F(s, x_s) ds \\ &\quad - \mu^{-1} \int_0^t \int_0^s \gamma(s-\tau) S_q(t-s) F(\tau, x_\tau) d\tau ds + \mu^{-1} \int_0^t S_q(t-s) [G(s, x_s) + Bu(s)] ds \\ &\quad + \mu^{-1} \sum_{0 < t_k < t} R_q(t-t_k) I_k(x(t_k^-)), \quad t \in [0, T], \end{aligned}$$

where

$$\begin{aligned} u(t) &:= -(L_0^{-1}) \{ \mu^{-1} R_q(T) [\mu \phi(0) + F(0, \phi)] - \mu^{-1} F(T, x_T) \\ &\quad - \int_0^T \mu^{-1} A \mu^{-1} S_q(T-s) F(s, x_s) ds - \mu^{-1} \int_0^T \int_0^s \gamma(s-\tau) S_q(T-s) F(\tau, x_\tau) d\tau ds \\ &\quad + \mu^{-1} \int_0^T S_q(T-s) G(s, x_s) ds + \mu^{-1} \sum_{0 < t_k < T} R_q(T-t_k) I_k(x(t_k^-)) \}, \quad t \in [0, T]. \end{aligned}$$

It will be shown that the operator φ from $C(J, X)$ into itself has a fixed point. $\square$

Step 1. The control $u(\cdot) = -H(x_0, F, G)$ is bounded on B_r .

Indeed,

$$\begin{aligned} \|u\| &\leq \|(L_0^{-1})\| \{ \|\mu^{-1}\| \|R_q(T)\| [\|\mu\| \|\phi(0) + F(0, \phi)\|] + \|\mu^{-1}\| \|F(T, x_T)\| \\ &\quad + \|\mu^{-1}\| \int_0^T \|A \mu^{-1}\| \|S_q(T-s)\| \|F(s, x_s)\| ds \\ &\quad + \|\mu^{-1}\| \int_0^T \int_0^s \|\gamma(s-\tau)\| \|S_q(T-s)\| \|F(\tau, x_\tau)\| d\tau ds + \|\mu^{-1}\| \int_0^T \|S_q(T-s)\| \|G(s, x_s)\| ds \\ &\quad + \|\mu^{-1}\| \sum_{0 < t_k < T} \|R_q(T-t_k)\| \|I_k(x(t_k^-))\| \} \\ \|u\| &\leq \|H\| (\|\mu\| \|\phi\|_\beta + L_F) M \|\mu^{-1}\| + L_F \|\mu^{-1}\| \\ &\quad + \|(-A)^{1-\beta}\| \|\mu^{-1}\| L_F M \frac{T^{q\beta}}{q\beta} + \|\mu^{-1}\| M \frac{T^{q\beta}}{q\beta} L_G + \|(-A)^\beta\| \|\mu^{-1}\| L_S M \frac{T^{q\beta}}{q\beta} + \|\mu^{-1}\| M L_k. \end{aligned}$$

Step 2. There exist $r > 0$ such that φ sends B_r into itself.

If $x(t) \in B_r, t \in [0, T]$, we have

$$\begin{aligned} \|(\varphi x)(t)\| &\leq \|\mu^{-1}\| \|R_q(t)\| [\|\mu\| \|\phi(0) + F(0, \phi)\|] + \|\mu^{-1}\| \|F(t, x_t)\| \\ &\quad + \|\mu^{-1}\| \int_0^t \|A\mu^{-1}\| \|S_q(t-s)\| \|F(s, x_s)\| ds \\ &\quad + \|\mu^{-1}\| \int_0^t \int_0^s \|\gamma(s-\tau)\| \|S_q(t-s)\| \|F(\tau, x_\tau)\| d\tau ds \\ &\quad + \|\mu^{-1}\| \int_0^t \|S_q(t-s)\| [\|G(s, x_s)\| + \|B\| \|H(x_0, F, G)\|] ds \\ &\quad + \|\mu^{-1}\| \sum_{0 < t_k < t} \|R_q(t-t_k)\| \|I_k(x(t_k^-))\|, \\ \|(\varphi x)(t)\| &\leq M \|\mu^{-1}\| [\|\mu\| \|\phi\|_\beta + L_F] + \|\mu^{-1}\| L_F + \|(-A)^{1-\beta}\| \|\mu^{-1}\| L_F M \frac{T^{q\beta}}{q\beta} + \|\mu^{-1}\| M \frac{T^{q\beta}}{q\beta} L_G \\ &\quad + \|B\| \|\mu^{-1}\| M \frac{T^{q\beta}}{q\beta} \|H\| \{(\|\mu\| \|\phi\|_p + L_F) M \|\mu^{-1}\| + L_F \|\mu^{-1}\| \\ &\quad + \|(-A)^{1-\beta}\| \|\mu^{-1}\| L_F M \frac{T^{q\beta}}{q\beta} + \|\mu^{-1}\| M \frac{T^{q\beta}}{q\beta} L_G + \|(-A)^\beta\| \|\mu^{-1}\| L_S M \frac{T^{q\beta}}{q\beta} + \|\mu^{-1}\| M L_k\} \\ &\quad + \|(-A)^\beta\| \|\mu^{-1}\| L_S M \frac{T^{q\beta}}{q\beta} + \|\mu^{-1}\| M L_k \leq r. \end{aligned}$$

Hence, φ maps B_r into itself.

Step 3. We prove $(\varphi x)(t)$ is continuous on J for any $x \in X$.

Let $0 < t \leq T$ and $\epsilon > 0$ be sufficiently small, then,

$$\begin{aligned} \|(\varphi x)(t+\epsilon) - (\varphi x)(t)\| &\leq \|\mu^{-1}\| [\|\mu\phi(0) + F(0, \phi)\|] \|R_q(t+\epsilon) - R_q(t)\| \\ &\quad + \|\mu^{-1}\| [\|F(t+\epsilon, x_{t+\epsilon}) - F(t, x_t)\|] \\ &\quad + \|\mu^{-1}\| \int_0^{t+\epsilon} \|A\mu^{-1}\| \|S_q(t+\epsilon-s) - S_q(t-s)\| \|F(s, x_s)\| ds \\ &\quad + \|\mu^{-1}\| \int_0^{t+\epsilon} \|S_q(t+\epsilon-s) - S_q(t-s)\| \|\gamma(s-\tau)\| \|F(\tau, x_\tau)\| d\tau ds \\ &\quad + \|\mu^{-1}\| \int_0^{t+\epsilon} \|S_q(t+\epsilon-s) - S_q(t-s)\| \|G(s, x_s)\| ds \\ &\quad + \|\mu^{-1}\| \int_0^{t+\epsilon} \|S_q(t+\epsilon-s)\| [\|B\| \|u\|] ds - \int_0^t \|S_q(t-s)\| [\|B\| \|u\|] ds \\ &\quad + \|\mu^{-1}\| \|\sum_{0 < t_k < t+\epsilon} R_q(t+\epsilon-t_k) I_k(x(t_k^+)) - \sum_{0 < t_k < t} R_q(t-t_k) I_k(x(t_k^+))\|. \end{aligned} \tag{15}$$

Clearly, from the continuity of the operators R_q and S_q , the right-hand side of (15) tends to zero as $\epsilon \rightarrow 0$.

Hence, $(\varphi x)(t)$ is continuous on $[0, T]$.

Step 4. We prove that $(\varphi x)(t)$ is a contraction on X .

Let $x, y \in X$ for $t \in (0, T]$ be fixed; then,

$$\begin{aligned} \|(\varphi x)(t) - (\varphi y)(t)\| &\leq \|\mu^{-1}\| [\|F(t, x) - F(t, y)\|] \\ &\quad + \|\mu^{-1}\| \|(-A)^{1-\beta}\| \int_0^t \|S_q(t-s)\| \|(-A)^\beta F(s, x) - F(s, y)\| ds \\ &\quad + \|\mu^{-1}\| \int_0^t \int_0^s \|\gamma(s-\tau)\| \|S_q(t-s)\| \|F(\tau, x_\tau) - F(\tau, y_\tau)\| d\tau ds \end{aligned}$$

$$\begin{aligned}
& + \|\mu^{-1}\| \int_0^t \|S_q(t-s)\| \|G(s, x_s) - G(s, y_s)\| ds + \sum_{0 < t_k < t} \|\mu^{-1}\| \|R_q(t-t_k)\| \|I_k(x(t_k^-)) - I_k(y(t_k^-))\| \\
& + \|\mu^{-1}\| \int_0^t \|S_q(t-s)[BH(y(0), F, G)(s)]\| ds - \int_0^t \|S_q(t-s)[BH(x(0), F, G)(s)]\| ds \\
& \leq \{\|\mu^{-1}\| L_F + \lambda_1 + \|B\| \|H\| \lambda_2 \frac{T^{q\beta}}{q\beta}\} \|x - y\| \\
& \leq \|x - y\|.
\end{aligned}$$

Hence, φ is a contraction on $C(J, X)$. From the Banach fixed point theorem, φ has a unique fixed point. Therefore, system (6) is exact null controllable on J .

6. Application

Consider the following fractional delay partial differential equation of fractional order:

$$\begin{cases} {}^c D_{0t}^q [v(t, x) - v_{xx}(t, x) + K_1(t - \tau, v(t - \tau, x))] = \frac{\partial^2}{\partial x^2} v(t, x) + \xi(t, x) + K_2(t - \tau, v(t - \tau, x)) \\ + \int_0^t (t-s)^2 e^{-\rho(t-s)} \frac{\partial^2}{\partial x^2} v(s, x) ds, \quad 0 \leq x \leq \pi, \quad t \in J, \\ v(t, 0) = v(t, \pi) = 0, \quad t \in J, \\ v(t, x) = \phi(t, x), \quad t \in J, \quad 0 \leq x \leq \pi \\ v(t_k^+, x) - v(t_k^-, x) = I_k v(t_k, x), \quad k = 1, 2, \dots, m, \end{cases} \quad (16)$$

where ${}^c D_{0t}^q$ is a Caputo fractional derivative of order of order $1 < q < 2$. The operators $A : D(A) \subset X \rightarrow Y$ and $\mu : D(\mu) \subset X \rightarrow Y$ introduced as follows:

$$Av = v_{xx}, \quad \mu v = v - v_{xx},$$

where each domain $D(A)$ and $D(\mu)$ is given by

$$\{v \in X : v, v_x \text{ are absolutely continuous, } v_{xx} \in X, v(0) = v(\pi) = 0\}.$$

Then, A and μ can be written respectively as:

$$Av = \sum_{n=1}^{\infty} (n^2) \langle v, v_n \rangle v_n, \quad v \in D(A),$$

$$\mu v = \sum_{n=1}^{\infty} (1 + n^2) \langle v, v_n \rangle v_n, \quad v \in D(\mu),$$

$$A\mu^{-1}v = \sum_{n=1}^{\infty} \frac{n^2}{n^2 + 1} \langle v, v_n \rangle v_n,$$

$$\mu^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \langle v, v_n \rangle v_n,$$

$$v_n(z) = \sqrt{\frac{2}{\pi}} \sin(nz), \quad n = 1, 2, 3, \dots$$

The operator $A\mu^{-1}$ is self-adjoint and has the eigenvalues $\lambda_n = -n^2\pi^2, n \in \mathbb{N}$, with the corresponding normalized eigenvectors $e_n(\xi) = \sqrt{2} \sin(n\pi\xi)$.

We define the bounded operator $B : U \rightarrow X$ by $Bu = \chi(t, y), 0 \leq y \leq \pi, u \in U$.

In addition, we define the following functions:

$$x(t)z = x(t, z), F(t, x_t) = K_1(t - \tau, v(t - \tau, x)), G(t, x_t) = K_2(t - \tau, v(t - \tau, x)), \xi(t, x) = Bu(t).$$

In addition,

$$\gamma(t-s)x(s) = (t-s)^2 e^{-\rho(t-s)} \frac{\partial^2}{\partial x^2} v(s, x).$$

Now, the conditions (H1)–(H3) and (H5) are satisfied.

Hence, all the hypotheses of Theorems 2 and 3 are satisfied and

$$\begin{aligned} \zeta_2 =: & \{ \|\mu^{-1}\|_{L_F} + M \|\mu^{-1}\| \frac{T^{q\beta}}{q\beta} [\|(-A)^{1-\beta}\|_{L_F} + L_S L_F] \\ & + \|\mu^{-1}\| \frac{T^{q\beta}}{q\beta} L_G + \|\mu^{-1}\| M L_k + \frac{\|B\| \|B^*\| M \|\mu^{-1}\|}{\lambda} \zeta_1 \} < 1. \end{aligned}$$

Thus, the fractional delay partial differential equation with impulsive condition (16) is approximately controllable on J .

7. Conclusions

This paper dealt with a Sobolev type nonlinear fractional delay integro-differential system with an impulsive condition. By using fractional calculus and fixed point theorems with the resolvent operator, we proved the existence of a mild solution for a Sobolev type nonlinear fractional delay integro-differential system with an impulsive condition. In addition, we established the necessary conditions for approximate controllability and null controllability of a Sobolev type nonlinear fractional delay integro-differential system with an impulsive condition. In the end, we provided an example in a fractional integro-partial differential equation to illustrate our results.

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