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A Refinement of Schwarz–Pick Lemma for Higher Derivatives

Ern Gun Kwon [1], Jinkee Lee [2,*], Gun Kwon [3] and Mi Hui Kim [4]

1 Department of Mathematics Education, Andong National University, Andong 36729, Korea; egkwon@anu.ac.kr
2 Department of Mathematics, Pusan National University, Busan 46241, Korea
3 Department of Mechanical Engineering, Graduate School, Yeungnam University, Gyeongsan 38541, Korea; gunbbang80kg@gmail.com
4 Department of Mathematics, Graduate School, Andong National University, Andong 36729, Korea; mihezzang@naver.com
* Correspondence: jklee235@pusan.ac.kr

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Abstract: In this paper, a Schwarz–Pick estimate of a holomorphic self map f of the unit disc D having the expansion $f(w) = c_0 + c_n(w-z)^n + \ldots$ in a neighborhood of some z in D is given. This result is a refinement of the Schwarz–Pick lemma, which improves a previous result of Shinji Yamashita.

Keywords: Schwarz Lemma; maximum principle; Littlewood inequality

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1. Introduction

For the open unit disc D of the complex plane and the boundary ∂D of D, the following Schwarz–Pick lemma(see [1], Lemma 1.2) is well-known.

Theorem 1. *Let $f : D \longrightarrow D$ be holomorphic and $z_0 \in D$. Then,*

$$\left|\frac{f(z)-f(z_0)}{1-\overline{f(z_0)}f(z)}\right| \le \left|\frac{z-z_0}{1-\overline{z_0}z}\right|, \quad z \in D, \tag{1}$$

and

$$\frac{|f'(z_0)|}{1-|f(z_0)|^2} \le \frac{1}{1-|z_0|^2}. \tag{2}$$

Equality in (1) holds at some point $z \neq z_0$ or equality in (2) holds if and only if

$$f(z) = c\frac{z-a}{1-\bar{a}z}, \quad z \in D \tag{3}$$

for some $c \in \partial D$ and $a \in D$.

Among those interesting extensions of (2), there is a result of Shinji Yamashita(see [2], Theorem 1):

Theorem 2. *Let f be a function holomorphic and bounded, $|f| < 1$, in D, and let $z \in D$. Suppose that*

$$f(w) = c_0 + c_n(w-z)^n + c_{n+1}(w-z)^{n+1} + \ldots$$

in a neighborhood of z*, where* $n \geq 1$ *depends on* z *and* $c_n = 0$ *is possible. Then,*

$$\frac{(1-|z|^2)^n |f^{(n)}(z)|}{n!(1-|f(z)|^2)} \leq 1. \tag{4}$$

The inequality (4) is sharp in the sense that equality holds for the function

$$f(w) = e^{i\alpha} \left(\frac{w-z}{1-\bar{z}w} \right)^n \ (\alpha; \text{ a real constant})$$

of w*.*

For f holomorphic in D, $0 \leq r < 1$, and $0 \leq p \leq \infty$, as it is commonly used we denote $M_p(r, f)$ by the p-mean of f on ∂D, that is,

$$M_p(r,f) = \begin{cases} \exp\left(\int_{-\pi}^{\pi} \log |f(re^{i\theta})| \frac{d\theta}{2\pi} \right) & \text{if} \quad p = 0, \\ \left(\int_{-\pi}^{\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{\frac{1}{p}} & \text{if} \quad 0 < p < \infty, \\ \sup\limits_{|z|=r} |f(z)| & \text{if} \quad p = \infty. \end{cases}$$

If f is holomorphic, then $M_p(r, f)$ is an increasing function of $p : 0 \leq p \leq \infty$ as well as an increasing function of $r : 0 \leq r < 1$ (see [3]).

For $a \in D$, let φ_a be defined by

$$\varphi_a(z) = \frac{z+a}{1+\bar{a}z}, \quad z \in D.$$

φ_a satisfies $\varphi_a(\varphi_{-a}(z)) = z$ for all $z \in D$. It is well-known that $\varphi_a(\partial D) = \partial D$ and that the set of automorphisms, i.e., bijective biholomorphic mappings, of D consists of the mappings of the form $\alpha\varphi_a(z)$, where $a \in D$ and $|\alpha| = 1$.

Extending (2) in terms of $M_p(r, f)$, there is another result of Shinji Yamashita(see [4], Theorem 2):

Theorem 3. *Let* f *be a function holomorphic and bounded,* $|f| < 1$*, in* D *and let* $0 \leq p \leq \infty$*. Then*

$$\frac{(1-|w|^2)|f'(w)|}{1-|f(w)|^2} \leq \frac{1}{r} M_p(r, f_w) \leq 1 \tag{5}$$

for all $w \in D$ *and* $r : 0 < r < 1$*, where*

$$f_w(z) = \frac{f(\frac{z+w}{1+\bar{w}z}) - f(w)}{1 - \overline{f(w)} f(\frac{z+w}{1+\bar{w}z})}, \quad z \in D.$$

If the equality $r^{-1} M_p(r, f_w) = 1$ *holds in (5) for* $w \in D$ *and* $0 < r < 1$*, then* f *is of the form (3).*

Note that $n = 1$ in (4) reduces to (2) and that (5) refines (2). As the same manner, it is expected that there might be a refinement of Theorem 2 which reduces to Theorem 3 when $n = 1$. This is our objective of this note.

2. Result

The following is our corresponding result:

Theorem 4. *Let f be a function holomorphic and bounded, $|f| < 1$, in D and let $z \in D$. If*

$$f(w) = c_0 + c_n(w-z)^n + c_{n+1}(w-z)^{n+1} + \dots. \tag{6}$$

in a neighborhood of z, then

$$\left|\frac{f(w)-f(z)}{1-\overline{f(z)}f(w)}\right| \le \left|\frac{w-z}{1-\bar{z}w}\right|^n, \quad w \in D, \tag{7}$$

and

$$\frac{(1-|z|^2)^n |f^{(n)}(z)|}{n!(1-|f(z)|^2)} \le \frac{1}{r^n} M_p(r, f_z) \le 1 \tag{8}$$

for all $r : 0 < r < 1$, and for all $p : 0 \le p \le \infty$, where

$$f_z(w) = \frac{f \circ \varphi_z(w) - f(z)}{1 - \overline{f(z)} f \circ \varphi_z(w)}, \quad w \in D. \tag{9}$$

Equality in (7) holds at some point $w \in D$, $w \ne z$ if and only if

$$f(w) = \frac{\alpha\,(\varphi_{-z}(w))^n + c_0}{1 + \alpha \overline{c_0}\,(\varphi_{-z}(w))^n}, \quad w \in D \tag{10}$$

with $|\alpha| = 1$.

Equality in the first inequality or in the second inequality of (8) holds for some $r : 0 < r < 1$ and $p : 0 < p \le \infty$ if and only if f is of the form (10) (with $|\alpha| \le 1$ or $|\alpha| = 1$, respectively).

Remark 1. *(1) The case $n = 1$ of Theorem 4 should reduce to Theorem 1. Comparing (3) and (10), there should exist $z' \in D$ and $\beta : |\beta| = 1$ for which*

$$\frac{\alpha\varphi_{-z}(w) + c_0}{1 + \alpha\overline{c_0}\varphi_{-z}(w)} = \beta \frac{w - z'}{1 - \overline{z'}w} \tag{11}$$

for all $w \in D$. This can be verified as follows:

Since any automorphism, i.e., bijective holomorphic mapping, of D is of the form of the right-hand side of (11), it suffices to show that the left-hand side of (11), denote $\Phi(w)$, is an automorphism of D. That $\Phi(w)$ is holomorphic and into D is obvious. We show $\Phi(w)$ is bijective: If $\Phi(w_1) = \Phi(w_2)$, then $\varphi_{-z}(w_1) = \varphi_{-z}(w_2)$, and the injectivity of φ_{-z} shows $w_1 = w_2$. Thus, $\Phi(w)$ is injective. Next, for any $\zeta \in D$, by the surjectivity of φ_{c_0}, there exists $\eta \in D$ such that

$$\frac{\alpha\eta + c_0}{1 + \alpha\bar{c}_0\eta} = \zeta.$$

For this η, there is $\xi \in D$ such that $\eta = \varphi_{-z}(\xi)$, whence $\Phi(w)$ is surjective.

(2) Fix $z \in D$ and self-map f of D. Then, applying Littlewood's inequality (see [3,5,6]), it follows that

$$\left|\frac{f(w)-f(z)}{1-\overline{f(z)}f(w)}\right| \le \prod_j \left\{ |z_j| : f\left(\frac{w+z_j}{1+\bar{z}_j w}\right) = f(z) \right\} = \prod_{f(z_j)=f(z)} \left|\frac{w - z_j}{1 - \bar{z}_j w}\right|, \tag{12}$$

with equality holding only if f is an inner function. Equation (7) follows directly from (12).

In addition, the inequality

$$\frac{(1-|z|^2)^n |f^{(n)}(z)|}{n!(1-|f(z)|^2)} \le 1$$

of (8) can be obtained as a one stroke limit from (7):

$$1 \geq \left|\frac{f(w)-f(z)}{1-\overline{f(z)}f(w)}\right| \Big/ \left|\frac{w-z}{1-\bar{z}w}\right|^n = \left|\frac{f(w)-f(z)}{(w-z)^n}\right| \left|\frac{(1-\bar{z}w)^n}{1-\overline{f(z)}f(w)}\right| \longrightarrow \frac{|f^{(n)}(z)|(1-|z|^2)^n}{n!(1-|f(z)|^2)}$$

as $w \to z$ (by applying L'Hospital's rule).

The point of Theorem 4 lies in its connection with $M_p(r,\cdot)$ and in clarifying the condition of equality to make Yamashita type theorem complete.

After proving Theorem 4 in Section 3, applications of Theorem 4 to some coefficient problems will be given in Section 4.

3. Proof of Theorem 4

We may assume $c_n \neq 0$. (7) can be expressed as

$$|\varphi_{-f(z)} \circ f(w)| \leq |\varphi_{-z}(w)|^n, \quad w \in D.$$

By (6), $f(w) - f(z)$ has a zero of order n at $w = z$ so that

$$\frac{\varphi_{-f(z)} \circ f(w)}{\varphi_{-z}(w)^n}, \quad w \in D \tag{13}$$

is holomorphic in D whose modulus at $w \in \partial D$ is not greater than 1, so that the maximum principle gives (7).

Next, to verify inequality (8), take $\delta > 0$ such that (6) holds for $w : |w - z| < \delta$. Then, by (6),

$$\begin{aligned} f \circ \varphi_z(w) - f(z) =& c_n(\varphi_z(w) - z)^n + \dots \\ =& c_n \left(\frac{w}{1+\bar{z}w}\right)^n (1-|z|^2)^n + O(w^{n+1}) \end{aligned} \tag{14}$$

for $w : |w| < \frac{\delta}{1+|z|}$. This is because

$$|w| < \frac{\delta}{1+|z|} \implies |w| < \frac{\delta|1+\bar{z}w|}{1-|z|^2} \implies |\varphi_z(w) - z| < \delta.$$

Thus, $f_z(w)$ defined by (9) has a zero of order n at $w = 0$. Hence,

$$h(w) = \frac{1}{w^n} f_z(w), \quad w \in D, \tag{15}$$

is holomorphic in D. Since $h(0) \neq 0$ in a neighborhood of 0, $\log|h|$ is harmonic in the neighborhood, hence there exists r_0 such that

$$|h(0)| = \exp\left(\int_{-\pi}^{\pi} \log|h(re^{i\theta})| \frac{d\theta}{2\pi}\right) \tag{16}$$

for $r : r < r_0$.

On the other hand, by (15),

$$n!h(0) = \frac{d^n}{dw^n}(w^n h(w))\Big|_{w=0} = \frac{d^n}{dw^n} f_z(w)\Big|_{w=0}. \tag{17}$$

In order to calculate the final term of (17), let's put $F(w) = f \circ \varphi_z(w) - f(z)$ and $G(w) = 1 - \overline{f(z)} f \circ \varphi_z(w)$. Then,

$$\frac{d^n}{dw^n} f_z(w)\Big|_{w=0} = \sum_{j=0}^{n} \binom{n}{j} F^{(j)}(0)\ (G^{-1})^{(n-j)}(0).$$

By (14),

$$F^{(j)}(0) = \begin{cases} 0, & \text{if } j < n, \\ c_n n!(1-|z|^2)^n, & \text{if } j = n, \end{cases}$$

so that

$$\frac{d^n}{dw^n} f_z(w)\Big|_{w=0} = \frac{F^{(n)}(0)}{G(0)} = \frac{c_n n!(1-|z|^2)^n}{1-|f(z)|^2},$$

whence

$$h(0) = \frac{c_n(1-|z|^2)^n}{1-|f(z)|^2}. \tag{18}$$

Noting from (6) that $c_n = \frac{f^{(n)}(z)}{n!}$, we have, by (15), (16) and (18),

$$\frac{(1-|z|^2)^n \left|f^{(n)}(z)\right|}{n!(1-|f(z)|^2)} = \exp\left(\int_{-\pi}^{\pi} \log\left|\frac{1}{r^n} f_z(re^{i\theta})\right| \frac{d\theta}{2\pi}\right) = \frac{1}{r^n} M_0(r, f_z) \tag{19}$$

for $r < r_0$.

Now, the first inequality of (8) follows from the fact that $M_p(r,h)$ is an increasing function of $p : 0 \le p \le \infty$ and also an increasing function of $r : 0 < r < 1$.

In addition, since $M_p(r,h) \le M_\infty(r,h)$ and $|h| < 1$ by the maximum principle, the second inequality of (8) follows.

We next check the conditions of equality. Elementary calculation shows that

$$\frac{\varphi_{-f(z)} \circ f(w)}{\varphi_{-z}(w)^n} = \alpha \iff f(w) = \frac{\alpha\,(\varphi_{-z}(w))^n + c_0}{1 + \alpha\overline{c_0}\,(\varphi_{-z}(w))^n} \iff f_z(w) = \alpha w^n. \tag{20}$$

Thus, if equality in (7) holds, at some point $w \in D$, $w \ne z$; then, (13) is a constant function of modulus 1 by virtue of the maximum principle, which gives (10) with $|\alpha| = 1$ by (20). To see that $f(w)$ of (10) with $|\alpha| = 1$ gives the equality in (7) is straightforward also by (20).

If, for some p, $0 < p \le \infty$, and for some $r : 0 < r < 1$ the first inequality of (8) becomes equality, then, by (19), $M_0(r,h) = M_q(r,h)$ for $0 \le q \le p$, so that $|h(r\zeta)|$=constant, a.e. $\zeta \in \partial D$. Since h is holomorphic and $|h(0)| = |h(\rho\zeta)|$=constant, a.e. $\zeta \in \partial D$ for $\rho \le r$, it follows that h is a constant function. Letting $h = \alpha$ with $|\alpha| \le 1$ and solving this, as in (20), gives (10).

Finally, suppose the second inequality of (8) becomes equal for some $\rho_0 : 0 < \rho_0 < 1$ so that $M_p(\rho_0, h) = \frac{1}{\rho^n} M_p(\rho_0, f_z) = 1$. Then, $M_p(\rho, h) = 1$ for $\rho : \rho_0 \le \rho < 1$. Since $\log M_p(\rho, h)$ is a convex function of ρ (see [3]) and $\log M_p(\rho, h) = 0$ for $\rho_0 \le \rho < 1$, it follows that $\log M_p(\rho, h) \ge 0$ for $\rho \le \rho_0$ whence $\log M_p(\rho, h) = 0$ for all $\rho : 0 < \rho < 1$. Thus,

$$h(0) = \lim_{p \to 0} M_p(\rho, f) = \lim_{p \to 0} e^{\log M_p(\rho,h)} = 1.$$

Since h maps D into D, this forces, by the maximum principle, that $h(w)$ is a constant, $h(w) = \alpha$, with $|\alpha| = 1$. Hence, (20) gives (10).

Conversely, by (20), $f(w)$ of (10) with $|\alpha| \le 1$ makes h in (15) constant, so that the two inequalities in (8) become equalities.

4. Applications

Theorem 4 immediately gives the following estimate:

Corollary 1. *Let f be a function holomorphic and bounded, $|f| < 1$, in D and let $z \in D$. If*

$$f(w) = c_0 + c_n(w-z)^n + c_{n+1}(w-z)^{n+1} + \dots .$$

in a neighborhood of z, then

$$|c_n| \;\leq\; \frac{1-|c_0|^2}{r^n(1-|z|^2)^n} M_p(r, f_z) \;\leq\; \frac{1-|c_0|^2}{(1-|z|^2)^n} \tag{21}$$

for all $r : 0 < r < 1$, and for all $p : 0 \leq p \leq \infty$, where f_z is defined by (9).

Equality in the first inequality or in the second inequality of (21) holds for some $r : 0 < r < 1$ and $p : 0 < p \leq \infty$ if and only if f is of the form (10) (with $|\alpha| \leq 1$ or $|\alpha| = 1$, respectively).

For the case $z = 0$, we can also obtain

Corollary 2. *Let f be a function holomorphic and bounded, $|f| < 1$, in D and let $z \in D$. If*

$$f(w) = c_0 + c_1 w + c_2 w^2 + \cdots + c_n w^n + \cdots, \quad w \in D,$$

then

$$|c_n| \;\leq\; (1-|c_0|^2)\frac{M_p(r, g_0)}{r^n} \;\leq\; 1-|c_0|^2 \tag{22}$$

for all $r : 0 < r < 1$, and for all $p : 0 \leq p \leq \infty$, where

$$g_0(w) = \frac{g(w) - g(0)}{1 - \overline{g(0)}g(w)}, \quad w \in D$$

with

$$g(w) = \frac{1}{n}\sum_{k=1}^{n} f(e^{i2\pi k/n} w), \quad w \in D.$$

Equality in the first inequality or in the second inequality of (22) holds for some $r : 0 < r < 1$ and $p : 0 < p \leq \infty$ if and only if g is of the form

$$g(w) = \frac{\alpha w^n + c_0}{1 + \alpha \overline{c_0} w^n}, \quad w \in D \tag{23}$$

(with $|\alpha| \leq 1$ or $|\alpha| = 1$, respectively).

Proof of Corollary 2. As was frequently used (see [7] for example), we make use of the facts that $\sum_{k=1}^{n} e^{i2\pi jk/n} = n$ if j is a multiple of n, and 0 if j is otherwise. Noting that

$$g(w) = \frac{1}{n}\sum_{k=1}^{n} f(e^{i2\pi k/n} w) = c_0 + c_n w^n + c_{2n} w^{2n} + \cdots, \quad w \in D$$

is holomorphic and $|g| < 1$ in D, by Corollary 1 with $z = 0$, we have

$$|c_n| \;\leq\; \frac{1-|c_0|^2}{r^n} M_p(r, g_0) \;\leq\; 1-|c_0|^2$$

for all $r : 0 < r < 1$, and for all $p : 0 \leq p \leq \infty$.

In addition, by Corollary 1, equality in the first inequality or in the second inequality of (22) holds for some $r : 0 < r < 1$ and $p : 0 < p \leq \infty$ if and only if g is of the form (23) (with $|\alpha| \leq 1$ or $|\alpha| = 1$, respectively). $\square$

5. Conclusions

With imperative applications to particular situations, various forms of Schwarz Lemma have been called for. In this paper, we presented Schwarz-Pick Lemma for higher derivatives in connection with p-mean $M_p(r, f)$(see Theorem 4). It refined a previous result of Shinji Yamashita and clarified the condition of equality. As an immediate consequence, the result could be applied to refine well-known estimates for n-th Taylor coefficient of holomorphic self maps of D (see Corollary 1 and 2). We are expecting its further extensions and applications.

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