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On a Certain Subclass of Analytic Functions Involving Integral Operator Defined by Polylogarithm Function

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Abstract: In the present paper, we have introduced a new subclass of analytic functions involving integral operator defined by polylogarithm function. Necessary and sufficient conditions are obtained for this class. Further distortion theorem, linear combination and results on partial sums are investigated.

Keywords: univalent function; analytic function; polylogarithm function; coefficient estimates

MSC: 30C45; 30A20

1. Introduction

The class A consists of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. Let S denote the subclass of A, which consists of functions of the form (1) that are univalent and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ in U.

Many authors have investigated the properties of subclasses of S and their results have several applications in engineering, hydrodynamics and signal theory. Some of their results have to do with the starlikeness and convexity properties of subclasses of univalent functions S. One of the significant problems in geometric function theory are the extremal problems, which create an effective method for finding the existence of analytic functions with certain natural properties.

Extremal problems play an important role in geometric function theory, for finding coefficient bounds, sharp estimates, and an extremal function. The theory of analytic univalent functions is a powerful tool in the study of many problems related to the time evolution of the free boundary of a viscous fluid for planar flows in Hele-Shaw cells under injection. The results we obtained here may have prospective application in other branches of mathematics, both pure and applied.

In addition, in [1] Silverman introduced the class T of analytic functions with negative coefficients consisting of functions f of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad (z \in U). \tag{2}$$

The Hadamard product (or convolution) of two power series $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ is given by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k .$$

Let $\Phi_\delta(a; z)$ denote the well-known generalization of the Riemann zeta and polylogarithm functions, or simply the δ^{th} order polylogarithm function, given by

$$\Phi_\delta(c; z) = \sum_{k=1}^{\infty} \frac{z^k}{(k+c)^\delta} ,$$

where any term with $k + c = 0$ is excluded (see Lerch [2]). Using the definition of the Gamma function (for details see [3], p. 27), a simple transformation produces the integral formula

$$\Phi_\delta(c; z) = \frac{1}{\Gamma(\delta)} \int_0^1 z \left(log \frac{1}{t} \right)^{\delta-1} \frac{t^c}{1 - tz} dt ,$$

where $Re(c) > -1$ and $Re(\delta) > 1$. For more details about polylogarithm function, see Ponnusamy [4] and Ponnusamy and Sabapathy [5].

For $f \in A$ of the form (1), Al-Shaqsi [6] defined the following integral operator:

$$\begin{aligned} \mathscr{I}_c^\delta f(z) &= (1+c)^\delta \Phi_\delta(c; z) * f(z) \\ &= -\frac{(1+c)^\delta}{\Gamma(\delta)} \int_0^1 t^{c-1} \left(log \frac{1}{t} \right)^{\delta-1} f(tz) dt , \end{aligned} \tag{3}$$

where $c > 0$, $\delta > 1$ and $z \in U$.

In [6], Al-Shaqsi noted that the operator defined by (3) can be expressed by series expansion as below:

$$\mathscr{I}_c^\delta f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+c}{k+c} \right)^\delta a_k z^k. \tag{4}$$

From (3), it is clear that

$$z \left(\mathscr{I}_c^\delta f(z) \right)' = (c+1) \mathscr{I}_c^{\delta-1} f(z) - c \mathscr{I}_c^\delta f(z) ,$$

and

$$z^2 \left(\mathscr{I}_c^\delta f(z) \right)'' = (c+1)^2 \mathscr{I}_c^{\delta-2} f(z) - (2c+1)(c+1) \mathscr{I}_c^{\delta-1} f(z) + c(c+1) \mathscr{I}_c^\delta f(z) .$$

Authors like Murugusundaramoorthy and Janani [7], Dziok and Murugusundaramoorthy [8], Obradovic and Joshi [9], Padmanabhan [10] and Ramesha et al. [11] have investigated various subclasses of analytic functions. Motivated by aforementioned work, we introduce a new subclass $\Phi_c^\delta(\lambda, \beta)$ of A involving Al-Shaqsi operator [6] as below:

Definition 1. *For $0 \le \lambda < 1$, $0 \le \beta < 1$, $c > 0$, $\delta > 0$, we say $f(z) \in A$ is in the class $\Phi_c^\delta(\lambda, \beta)$ if it satisfies the condition*

$$Re \left(\frac{z \left(\mathscr{I}_c^\delta f(z) \right)' + \lambda z^2 \left(\mathscr{I}_c^\delta f(z) \right)''}{\mathscr{I}_c^\delta f(z)} \right) > \beta, \quad (z \in U). \tag{5}$$

Also we denote by $T\Phi_c^\delta(\lambda, \beta) = \Phi_c^\delta(\lambda, \beta) \cap T$.

In this section we obtain a sufficient condition for a function f given by (1) to be in $\Phi_c^\delta(\lambda,\beta)$ and we prove that it is also a necessary condition for a function belonging to the class $T\Phi_c^\delta(\lambda,\beta)$. Also, distortion results and linear combinations for the class $T\Phi_c^\delta(\lambda,\beta)$ are obtained. We also investigate the results on partial sums for the functions in the class $\Phi_c^\delta(\lambda,\beta)$.

2. Main Results

2.1. Conditions for Functions to Be in the Class $\Phi_c^\delta(\lambda,\beta)$

Theorem 1. *A function $f \in A$ belongs to the class $\Phi_c^\delta(\lambda,\beta)$ if*

$$\sum_{k=2}^{\infty}\{k+\lambda k(k-1)-\beta\}\left(\frac{1+c}{k+c}\right)^{\delta}|a_k| \le 1-\beta\,. \tag{6}$$

Proof. Since $0 \le \beta < 1$ and $\lambda \ge 0$, now if we put

$$P(z) = \frac{z\left(\mathscr{I}_c^\delta f(z)\right)' + \lambda\, z^2\left(\mathscr{I}_c^\delta f(z)\right)''}{\mathscr{I}_c^\delta f(z)}, \quad (z \in U)$$

then it is sufficient to prove that $|P(z)-1| < 1-\beta,\ (z \in U)$. Indeed if $f(z) \equiv z\ (z \in U)$, then we have $P(z) \equiv z\ (z \in U)$. This implies that the desired in equality (6) holds. If $f(z) \neq z\ (|z| = r < 1)$, then there exist a coefficient $\left(\frac{1+c}{k+c}\right)^\delta a_k \neq 0$ for some $k \ge 2$. It follows that $\sum_{k=2}^{\infty}\left(\frac{1+c}{k+c}\right)^\delta |a_k| > 0$. Further note that

$$\sum_{k=2}^{\infty}\{k+\lambda k(k-1)-\beta\}\left(\frac{1+c}{k+c}\right)^{\delta}|a_k| > (1-\beta)\sum_{k=2}^{\infty}\left(\frac{1+c}{k+c}\right)^{\delta}|a_k|\,,$$

which implies that

$$\sum_{k=2}^{\infty}\left(\frac{1+c}{k+c}\right)^{\delta}|a_k| < 1\,.$$

By coefficeint inequality (6), we obtain

$$\begin{aligned}|P(z)-1| &= \left|\frac{\displaystyle\sum_{k=2}^{\infty}\{k+\lambda k(k-1)-1\}\left(\frac{1+c}{k+c}\right)^{\delta}a_k\, z^{k-1}}{\displaystyle 1+\sum_{k=2}^{\infty}\left(\frac{1+c}{k+c}\right)^{\delta}a_k\, z^{k-1}}\right| \\ &\le \frac{\displaystyle\sum_{k=2}^{\infty}\{k+\lambda k(k-1)-1\}\left(\frac{1+c}{k+c}\right)^{\delta}|a_k|}{\displaystyle 1-\sum_{k=2}^{\infty}\left(\frac{1+c}{k+c}\right)^{\delta}|a_k|}\end{aligned}$$

$$\le \frac{\displaystyle\sum_{k=2}^{\infty}\{k+\lambda k(k-1)-\beta\}\left(\frac{1+c}{k+c}\right)^{\delta}|a_k| - (1-\beta)\sum_{k=2}^{\infty}\left(\frac{1+c}{k+c}\right)^{\delta}|a_k|}{\displaystyle 1-\sum_{k=2}^{\infty}\left(\frac{1+c}{k+c}\right)^{\delta}|a_k|}$$

$$\leq \frac{(1-\beta)-(1-\beta)\sum_{k=2}^{\infty}\left(\frac{1+c}{k+c}\right)^{\delta}|a_k|}{1-\sum_{k=2}^{\infty}\left(\frac{1+c}{k+c}\right)^{\delta}|a_k|} = (1-\beta)\,,\ \ (z\in U).$$

Hence we obtain

$$Re\left(\frac{z\left(\mathscr{I}_c^{\delta}f(z)\right)'+\lambda\, z^2\left(\mathscr{I}_c^{\delta}f(z)\right)''}{\mathscr{I}_c^{\delta}f(z)}\right) = Re(P(z)) > 1-(1-\beta) = \beta,\ \ (z\in U).$$

Then $f\in\Phi_c^{\delta}(\lambda,\beta)$. This completes the proof. □

In the next theorem, we prove that the condition (6) is also necessary for a function $f\in T\Phi_c^{\delta}(\lambda,\beta)$.

Theorem 2. *Let f be given by (2) then the function $f\in T\Phi_c^{\delta}(\lambda,\beta)$ if and only if*

$$\sum_{k=2}^{\infty}\{k+\lambda k(k-1)-\beta\}\left(\frac{1+c}{k+c}\right)^{\delta}|a_k|\leq 1-\beta\,. \tag{7}$$

Proof. In view of Theorem 1 we need only to prove that $f\in T\Phi_c^{\delta}(\lambda,\beta)$ satisfies the coefficient inequality (6). If $f\in T\Phi_c^{\delta}(\lambda,\beta)$ then the function

$$P(z)=\frac{z\left(\mathscr{I}_c^{\delta}f(z)\right)'+\lambda\, z^2\left(\mathscr{I}_c^{\delta}f(z)\right)''}{\mathscr{I}_c^{\delta}f(z)},\ \ (z\in U)$$

satisfies $Re(P(z))>\beta\ \ (z\in U)$.

This implies that

$$\mathscr{I}_c^{\delta}f(z)=z-\sum_{k=2}^{\infty}\left(\frac{1+c}{k+c}\right)^{\delta}|a_k|\,z^k\neq 0;\ \ (z\in U\backslash\{0\}).$$

Noting that $\frac{\mathscr{I}_c^{\delta}f(r)}{r}$ is the real continuous function in the open interval $(0,1)$ with $f(0)=1$, we have

$$\frac{\mathscr{I}_c^{\delta}f(r)}{r}=1-\sum_{k=2}^{\infty}\left(\frac{1+c}{k+c}\right)^{\delta}|a_k|\,r^{k-1}>0,\ \ (0<r<1). \tag{8}$$

Now

$$\beta<P(r)=\frac{1-\sum_{k=2}^{\infty}\{k+\lambda k(k-1)\}\left(\frac{1+c}{k+c}\right)^{\delta}|a_k|\,r^{k-1}}{1-\sum_{k=2}^{\infty}\left(\frac{1+c}{k+c}\right)^{\delta}|a_k|\,r^{k-1}},$$

and consequently by (8) we get

$$\sum_{k=2}^{\infty}\{k+\lambda k(k-1)-\beta\}\left(\frac{1+c}{k+c}\right)^{\delta}|a_k|r^{k-1}\leq 1-\beta\,.$$

Letting $r\to 1$, we get

$$\sum_{k=2}^{\infty}\{k+\lambda k(k-1)-\beta\}\left(\frac{1+c}{k+c}\right)^{\delta}|a_k|\leq 1-\beta\,.$$

This proves the converse part. □

Remark 1. *If a function f of the form (2) belongs to the class $T\Phi_c^{\delta}(\lambda,\beta)$ then*

$$|a_k| \leq \frac{1-\beta}{\{k+\lambda k(k-1)-\beta\}\left(\frac{1+c}{k+c}\right)^{\delta}}, \quad k \geq 2.$$

The equality holds for the functions

$$f_k(z) = z - \frac{1-\beta}{\{k+\lambda k(k-1)-\beta\}\left(\frac{1+c}{k+c}\right)^{\delta}} z^n, \quad (z \in U), k \geq 2. \tag{9}$$

Next we obtain the distortion bounds for functions belonging to the class $T\Phi_c^{\delta}(\lambda,\beta)$.

Corollary 1. *Let f be in the class $T\Phi_c^{\delta}(\lambda,\beta)$, $|z| = r < 1$. If the sequence $\left\{\{k+\lambda k(k-1)-\beta\}\left(\frac{1+c}{k+c}\right)^{\delta}\right\}_{n=2}^{\infty}$ is nondecreasing, then*

$$r - \frac{1-\beta}{(2k-\beta+2)\left(\frac{1+c}{k+c}\right)^{\delta}} r^2 \leq |f(z)| \leq r - \frac{1-\beta}{(2k-\beta+2)\left(\frac{1+c}{k+c}\right)^{\delta}} r^2 . \tag{10}$$

If the sequence $\left\{\frac{\{k+\lambda k(k-1)-\beta\}}{k}\left(\frac{1+c}{k+c}\right)^{\delta}\right\}_{n=2}^{\infty}$ is nondecreasing, then

$$1 - \frac{2(1-\beta)}{(2k-\beta+2)\left(\frac{1+c}{k+c}\right)^{\delta}} r \leq |f(z)| \leq 1 - \frac{2(1-\beta)}{(2k-\beta+2)\left(\frac{1+c}{k+c}\right)^{\delta}} r . \tag{11}$$

The result is sharp. The extremal function is the function f_2 of the form (9).

Proof. Since $f \in T\Phi_c^{\delta}(\lambda,\beta)$, we apply Theorem 2 to obtain

$$\begin{aligned}(2\lambda-\beta+2)\left(\frac{1+c}{k+c}\right)^{\delta}\sum_{k=2}^{\infty}|a_k| &\leq \sum_{k=2}^{\infty}\{k+\lambda k(k-1)-\beta\}\left(\frac{1+c}{k+c}\right)^{\delta}|a_k| \\ &\leq 1-\beta .\end{aligned}$$

Thus

$$\begin{aligned}|f(z)| &\leq |z| + |z|^2 \sum_{k=2}^{\infty}|a_k| \\ &\leq r + \frac{1-\beta}{(2k-\beta+2)\left(\frac{1+c}{k+c}\right)^{\delta}} r^2.\end{aligned}$$

Also we have,

$$\begin{aligned}|f(z)| &\geq |z| - |z|^2 \sum_{k=2}^{\infty}|a_k| \\ &\geq r - \frac{1-\beta}{(2k-\beta+2)\left(\frac{1+c}{k+c}\right)^{\delta}} r^2 ,\end{aligned}$$

and (10) follows. In similar manner for f', the inequalities

$$\begin{aligned} |f'(z)| &\leq 1+\sum_{k=2}^{\infty} k|a_k||z^{k-1}|\,, \\ &\leq 1+|z|\sum_{k=2}^{\infty} k|a_k| \end{aligned}$$

and

$$\sum_{k=2}^{\infty} k|a_k| \leq \frac{2(1-\beta)}{(2k-\beta+2)\left(\frac{1+c}{k+c}\right)^{\delta}}$$

are satisfied, which leads to (11). This completes the proof. □

Theorem 3. *Let,* $f_1(z)=z$ *and*

$$f_k(z)=z-\frac{1-\beta}{\{k+\lambda k(k-1)-\beta\}\left(\frac{1+c}{k+c}\right)^{\delta}}z^k,\ \ (z\in U), k\geq 2. \tag{12}$$

Then $f\in T\Phi_c^{\delta}(\lambda,\beta)$ *if and only if* f *can be expressed in the form*

$$f(z)=\sum_{k=1}^{\infty}\mu_k f_k(z),\quad \mu_k\geq 0\,, \tag{13}$$

and $\sum_{k=1}^{\infty}\mu_k=1$.

Proof. If a function f is of the form $f(z)=\sum_{k=1}^{\infty}\mu_k f_k(z)$, $\mu_k\geq 0$ and $\sum_{k=1}^{\infty}\mu_k=1$ then

$$\begin{aligned} &\sum_{k=2}^{\infty}\{k+\lambda k(k-1)-\beta\}\left(\frac{1+c}{k+c}\right)^{\delta}|a_k| \\ =\ &\sum_{k=2}^{\infty}\{k+\lambda k(k-1)-\beta\}\left(\frac{1+c}{k+c}\right)^{\delta}\frac{(1-\beta)\mu_k}{\{k+\lambda k(k-1)-\beta\}\left(\frac{1+c}{k+c}\right)^{\delta}} \\ =\ &\sum_{k=2}^{\infty}\mu_k(1-\beta)=(1-\mu_1)(1-\beta) \\ \leq\ &(1-\beta) \end{aligned}\ .$$

which provides (7), hence $f\in T\Phi_c^{\delta}(\lambda,\beta)$ by Theorem 2.

Conversely, if f is in the class $f\in T\Phi_c^{\delta}(\lambda,\beta)$, then we may set

$$\mu_k=\frac{\{k+\lambda k(k-1)-\beta\}\left(\frac{1+c}{k+c}\right)^{\delta}}{1-\beta}|a_k|\,,\ \ k\geq 2\,, \tag{14}$$

and $\mu_1=1-\sum_{k=2}^{\infty}\mu_k$.

Then the function f is of the form (13) and this completes the proof. □

2.2. Partial Sums of Functions in the Class $\Phi_c^\delta(\lambda,\beta)$

For a function $f \in A$ given by (1), Silverman [12] investigated the partial sums f_1 and f_m defined by

$$f_1(z) = z; \text{ and } f_m(z) = z + \sum_{k=2}^{m} a_k\, z^k \quad m = 2,3,4,\cdots . \tag{15}$$

In [12], Silverman examined sharp lower bounds on the real part of the quotients between the normalized convex or starlike functions and their sequences of partial sums. Also, Srivastava et al. [13] and Owa et al. [14] have investigated an interesting results on the partial sums. In this section, we consider partial sums of functions in the class $\Phi_c^\delta(\lambda,\beta)$ and obtain sharp lower bounds for the ratios of real part of f to f_m and f' to f_m'.

Theorem 4. *Let a function f of the form (1) belong to the class $\Phi_c^\delta(\lambda,\beta)$ and satisfy (6). Then*

$$Re\left(\frac{f(z)}{f_m(z)}\right) \geq 1 - \frac{1}{d_{m+1}}, \quad z \in U,\ m \in \mathbb{N}, \tag{16}$$

where

$$d_k = \frac{\{k + \lambda k(k-1) - \beta\}}{1-\beta}. \tag{17}$$

Proof. Clearly, $d_{k+1} > d_k > 1, \quad k = 2,3,4,\cdots$.

Thus by Theorem 1 we get,

$$\sum_{k=2}^{m} |a_k| + d_{m+1} \sum_{k=m+1}^{\infty} |a_k| \leq \sum_{k=2}^{\infty} d_k\, |a_k| \leq 1 \tag{18}$$

Setting

$$g(z) = d_{m+1}\left\{\frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{d_{m+1}}\right)\right\}$$

$$g(z) = 1 + \frac{d_{m+1}\sum_{k=m+1}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^{m} a_k z^{k-1}} \tag{19}$$

it suffices to show that $Re(g(z)) > 0$, $z \in U$. Applying (18) we find that

$$\begin{aligned}\left|\frac{g(z)-1}{g(z)+1}\right| &\leq \frac{d_{m+1}\sum_{k=2}^{\infty}|a_k|}{2 - 2\sum_{k=2}^{m}|a_k| - d_{m+1}\sum_{m+1}^{\infty}|a_k|} \\ &\leq 1\,,\end{aligned}$$

which gives,

$$Re\left(\frac{f(z)}{f_m(z)}\right) \geq 1 - \frac{1}{d_{m+1}}\,,$$

and the proof is complete. □

Theorem 5. *Let a function f of the form (1) belong to the class $\Phi_c^\delta(\lambda,\beta)$ and satisfy (6). Then*

$$Re\left(\frac{f_m(z)}{f(z)}\right) \geq \frac{d_{m+1}}{1+d_{m+1}}, \quad z \in U,\ m \in \mathbb{N}, \tag{20}$$

where

$$d_k = \frac{\{k + \lambda k(k-1) - \beta\}}{1-\beta}. \tag{21}$$

Proof. Clearly, $d_{k+1} > d_k > 1, \quad k = 2, 3, 4, \ldots$.

Thus by Theorem 1 we get,

$$\sum_{k=2}^{m} |a_k| + d_{m+1} \sum_{k=m+1}^{\infty} |a_k| \leq \sum_{k=2}^{\infty} d_k \, |a_k| \leq 1 \, . \tag{22}$$

Setting

$$h(z) = (1 + d_{m+1}) \left\{ \frac{f_m(z)}{f(z)} - \left(\frac{d_{m+1}}{1 + d_{m+1}} \right) \right\}$$

$$h(z) = 1 - \frac{(1 + d_{m+1}) \sum_{k=m+1}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^{\infty} a_k z^{k-1}} \tag{23}$$

it suffices to show that $Re(h(z)) > 0, \; z \in U$. Applying (22) we can deduce that

$$\begin{aligned} \left| \frac{h(z) - 1}{h(z) + 1} \right| &\leq \frac{(1 + d_{m+1}) \sum_{k=2}^{\infty} |a_k|}{2 - 2 \sum_{k=2}^{m} |a_k| - (1 + d_{m+1}) \sum_{m+1}^{\infty} |a_k|} \\ &\leq 1 \, , \end{aligned}$$

which gives,

$$Re \left(\frac{f_m(z)}{f(z)} \right) \geq \frac{d_{m+1}}{1 + d_{m+1}} \, ,$$

and the proof is complete. □

Theorem 6. *Let a function f of the form (1) belong to the class $\Phi_c^{\delta}(\lambda, \beta)$ and satisfy (6). Then*

$$Re \left(\frac{f'(z)}{f'_m(z)} \right) \geq 1 - \frac{m+1}{d_{m+1}}, \quad z \in U, \; m \in \mathbb{N}, \tag{24}$$

and

$$Re \left(\frac{f'_m(z)}{f'(z)} \right) \geq \frac{d_{m+1}}{m + 1 + d_{m+1}}, \quad z \in U, \; m \in \mathbb{N}, \tag{25}$$

where

$$d_k = \frac{\{k + \lambda k(k-1) - \beta\}}{1-\beta}. \tag{26}$$

Proof. By Setting

$$g(z) = d_{m+1} \left\{ \frac{f'(z)}{f'_m(z)} - \left(1 - \frac{m+1}{d_{m+1}} \right) \right\} \, , \; z \in U,$$

and

$$h(z) = (m + 1 + d_{m+1}) \left\{ \frac{f'_m(z)}{f'(z)} - \left(\frac{d_{m+1}}{m + 1 + d_{m+1}} \right) \right\} \, , \; z \in U,$$

the proof is similar to that of Theorems 4 and 5, and so we omit the details. □

3. Conclusions

The theory of analytic function is an old subject, yet it remains an active field of current research. As a preferential topic concerning inequalities in complex analysis, there have been lots of studies based on the classes of analytic functions. The interplay of geometry and analysis is the most fascinating aspect of complex function theory. This rapid progress has been concerned primarily with such relations between analytic structure and geometric behavior. Motivated by this approach, in the present study, we have introduced a new subclass of analytic functions involving integral operator defined by polylogarithm function. Necessary and sufficient conditions are obtained for this class. Further distortion theorem, linear combination and results on partial sums are investigated in this study, and therefore it may be considered as a useful tool for those who are interested in the above-mentioned topics for further research.

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