mathematics

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Fuzzy Graph Structures with Application

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Received: 23 November 2018; Accepted: 3 January 2019; Published: 8 January 2019

Abstract: In this article, we introduce the notions of maximal products of fuzzy graph structures, regular fuzzy graph structures, and describe these notions with examples and properties. Further, we present the degree and total degree of a vertex in maximal product of fuzzy graph structures and explain some of their properties. Furthermore, we develop a flowchart to show general procedure of application of fuzzy graph structure, regarding identification of most controversial issues among countries.

Keywords: graph structure; maximal product; degree of vertex; total degree of vertex; controversial issues

1. Introduction

A fuzzy set, as a superset of a crisp set, owes its origin to the work of Zadeh [1] in 1965 that has been introduced to deal with the notion of partial truth between absolute true and absolute false. Zadeh's remarkable idea has found many applications in several fields, including chemical industry, telecommunication, decision making, networking, computer science, and discrete mathematics. Rosenfeld [2] used the concept of a fuzzy subset of a set to introduce the notion of a fuzzy subgroup of a group. Rosenfeld's paper spearheaded the development of fuzzy abstract algebra.

A graph is a mathematical representation of a network and it describes the relationship between vertices and edges. Graph theory is used to represent real-life phenomena, but sometimes graphs are not able to properly represent many phenomena because uncertainty of different attributes of the systems exists naturally. Many real-world phenomena provided motivation to define the fuzzy graphs. Kauffman [3] introduced fuzzy graphs using Zadeh's fuzzy relation [4]. Fuzzy-graph theory is growing rapidly, with numerous applications in many domains, including networking, communication, data mining, clustering, image capturing, image segmentation, planning, and scheduling.

Rosenfeld [5] described fuzzy analogue of some graph theoretical notions. Later on, Bhattacharya [6] provided remarks on fuzzy graphs. Sunitha and Vijayakumar [7,8] defined the complement of fuzzy graph and some characterization of fuzzy trees. Mordeson and Nair [9] defined some fuzzy graphs. Bhutani and Battou [10] introduced the concept of M-strong fuzzy graphs with some properties. Mathew and Sunitha defined types of arcs in a fuzzy graph [11]. Mordeson and Chang-Shyh [12] defined operations on fuzzy graphs. Nagoor Gani and Radha [13–16] described some properties of conjunction of fuzzy graphs, regular fuzzy graphs, some sequences in fuzzy graphs, and the degree of vertex in some fuzzy graphs. Akram et al. [17–20] defined certain notions of soft graphs, bipolar fuzzy graphs, and hypergraphs.

Sampathkumar [21] introduced the notion of graph structures. Graph structures are the generalization of graphs and widely useful in the study of some structures, like graphs, signed graphs, semigraphs, edge-colored graphs, and edge-labeled graphs. Graph structures are very useful in the

study of different domains of computer science and computational intelligence. Dinesh [22] introduced the concept of fuzzy-graph structures and described some related concepts. Fuzzy-graph structures are more useful than graph structures because they deal with the uncertainty and ambiguity of many real-world phenomena. Ramakrishnan and Dinesh [23–25] worked on generalized fuzzy-graph structures. Harinath and Lavanya discussed fuzzy graph structures for wheel, helm, and star graphs [26].

In this article, we present a new framework to handle fuzzy information by combining fuzzy sets with graph structures. We introduce many basic notions concerning fuzzy-graph structures and investigate a few related properties. We also consider an application of fuzzy-graph structures. In particular, a flowchart is developed to show general procedure of application, regarding identification of most controversial issues among countries.

2. Maximal Product of Fuzzy Graph Structures

Definition 1. *[21] A graph structure(GS) $G = (V, R_1, \ldots, R_n)$ consists of a nonempty set V with relations $R_1, R_2, \ldots, R_n$ on set V which are mutually disjoint such that each relation R_i, $1 \leq i \leq n$ is symmetric and irreflexive.*

Graph structure $G^* = (V, R_1, \ldots, R_n)$ can be represented just like a graph where each edge is labeled as R_i, $1 \leq i \leq n$.

Example 1. *Consider a set $V = \{v_1, v_2, v_3, v_4\}$, define two relations on set V, i.e., $R_1 = \{v_1v_3, v_1v_2, v_3v_4\}$ and $R_2 = \{v_1v_4, v_2v_3, v_2v_4\}$ such that these relations are disjoint. Moreover, they do not have elements of same vertices, like v_1v_1, v_2v_2, v_3v_3, v_4v_4; hence, these two relations R_1 and R_2 are irreflexive. Furthermore, these relations are symmetric, since graph structure is nondirected, so each element v_iv_j is also considered as v_jv_i. Relations R_1 and R_2 are disjoint, irreflexive and symmetric. Hence, vertex set V with relations R_1 and R_2 is a graph structure $G^* = (V, R_1, R_2)$ and is shown in Figure 1.*

In this graph structure, each edge is labeled as R_1 or R_2.

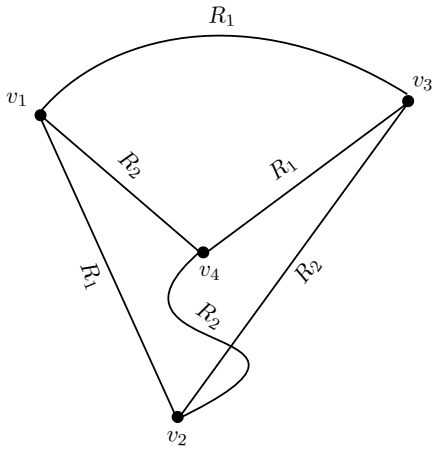


Figure 1. Graph Structure $G^* = (V, R_1, R_2)$.

Definition 2. *[22] Let σ be the fuzzy set on set V and $\mu_1, \mu_2, \ldots, \mu_n$ be fuzzy sets on $R_1, R_2, \ldots, R_n$, respectively. If $0 \leq \mu_i(vu) \leq \sigma(v) \wedge \sigma(u)$ $\forall u, v \in V$, $i = 1, 2, \ldots, n$, then $G = (\sigma, \mu_1, \mu_2, \ldots, \mu_n)$ is called fuzzy graph structure(FGS) of graph structure G^*.*

If $vu \in supp(\mu_i)$, then vu is named as μ_i-edge of FGS G.

Example 2. *Consider a graph structure $G^* = (V, R_1, R_2)$ as shown in Figure 1. We define fuzzy set $\sigma : V \to [0, 1]$ by*

$$\sigma(v_1) = 0.3, \sigma(v_2) = 0.3, \sigma(v_3) = 0.4, \sigma(v_4) = 0.4.$$

We define fuzzy sets μ_1, μ_2 on relations R_1, R_2, respectively, as follows:

$$\mu_1(v_1v_3) = 0.3, \mu_1(v_1v_2) = 0.3, \mu_1(v_3v_4) = 0.4,$$

$$\mu_2(v_1v_4) = 0.3, \mu_2(v_2v_3) = 0.3, \mu_2(v_2v_4) = 0.3.$$

By routine calculations, it is easy to see that $G = (\sigma, \mu_1, \mu_2)$ is a fuzzy-graph structure as shown in Figure 2.

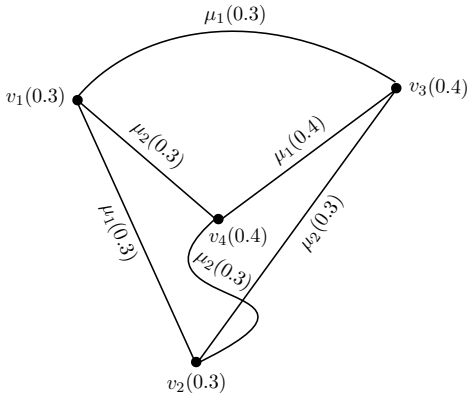


Figure 2. Fuzzy graph structure $G = (\sigma, \mu_1, \mu_2)$.

Definition 3. *Let $G_1 = (\sigma_1, \mu_1', \mu_2', \ldots, \mu_n')$ and $G_2 = (\sigma_2, \mu_1'', \mu_2'', \ldots, \mu_n'')$ be two fuzzy-graph structures with underlying crisp graph structures $G_1^* = (V_1, R_1', R_2', \ldots, R_n')$ and $G_2^* = (V_2, R_1'', R_2'', \ldots, R_n'')$, respectively. $G_1 * G_2 = (\sigma, \mu_1, \mu_2, \ldots \mu_n)$ is called maximal fuzzy graph structure with underlying crisp graph structure $G^* = (V, R_1, R_2, \ldots, R_n)$, where, $V = V_1 \times V_2$ and $R_i = \{(u_1, v_1)(u_2, v_2) / u_1 = u_2, v_1v_2 \in R_i''$ or $v_1 = v_2, u_1u_2 \in R_i'\}$. Fuzzy vertex set σ and fuzzy relations μ_i in maximal product $G_1 * G_2 = (\sigma, \mu_1, \mu_2, \ldots \mu_n)$ are defined as: $\sigma = \sigma_1 * \sigma_2$,*

$$\sigma(u, v) = \sigma_1(u) \vee \sigma_2(v), \text{for all } (u, v) \in V = V_1 \times V_2,$$

and $\mu_i = \mu_i' * \mu_i''$,

$$\mu_i((u_1, v_1)(u_2, v_2)) = \begin{cases} \sigma_1(u_1) \vee \mu_i''(v_1v_2), & u_1 = u_2, v_1v_2 \in R_i'', \\ \sigma_2(v_1) \vee \mu_i'(u_1u_2), & v_1 = v_2, u_1u_2 \in R_i', \end{cases}$$

$i = 1, 2, \ldots, n$.

Example 3. *Consider two fuzzy-graph structures $G_1 = (\sigma_1, \mu_1', \mu_2', \mu_3')$ and $G_2 = (\sigma_2, \mu_1'')$, which are shown in Figure 3.*

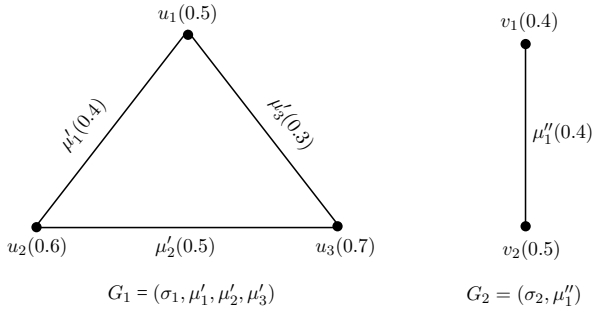


Figure 3. Two fuzzy-graph structures G_1 and G_2.

Maximal product of G_1 and G_2 is shown in Figure 4.

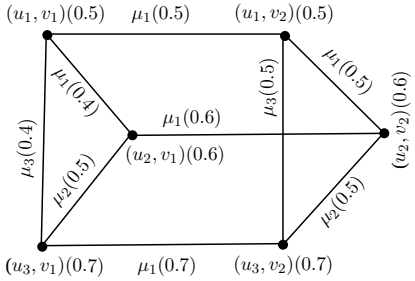


Figure 4. $G_1 * G_2 = (\sigma, \mu_1, \mu_2, \mu_3)$ (maximal product).

Definition 4. *A fuzzy-graph structure $G = (\sigma, \mu_1, \mu_2, \ldots \mu_n)$ is μ_i-strong if*

$$\mu_i(v_1v_2) = \sigma(v_1) \wedge \sigma(v_2), \text{for all } v_1v_2 \in R_i,\ i \in \{1, 2, \ldots, n\}.$$

If G is μ_i-strong $\forall\ i \in \{1, 2, \ldots, n\}$, then G is called strong fuzzy-graph structure.

Theorem 1. *Maximal product of two strong fuzzy-graph structures is also a strong fuzzy-graph structure.*

Proof. Let $G_1 = (\sigma_1, \mu_1', \mu_2', \ldots, \mu_n')$ and $G_2 = (\sigma_2, \mu_1'', \mu_2'', \ldots, \mu_n'')$ be two strong fuzzy-graph structures. Then, $\mu_i'(v_1v_2) = \sigma_1(v_1) \wedge \sigma_1(v_2)$ for any $v_1v_2 \in R_i'$ and $\mu_i''(u_1u_2) = \sigma_2(u_1) \wedge \sigma_2(u_2)$ for any $u_1u_2 \in R_i''$, $i = 1, 2, \ldots, n$. Then, proceeding according to the definition of maximal product,

Case (i): $u_1 = u_2$ and $v_1v_2 \in R_i''$. Then,

$$\begin{aligned} \mu_i((u_1, v_1)(u_2, v_2)) &= \sigma_1(u_1) \vee \mu_i''(v_1v_2) = \sigma_1(u_1) \vee [\sigma_2(v_1) \wedge \sigma_2(v_2)] \\ &= [\sigma_1(u_1) \vee \sigma_2(v_1)] \wedge [\sigma_1(u_1) \vee \sigma_2(v_2)] \\ &= [\sigma(u_1, v_1) \wedge \sigma(u_2, v_2)]. \end{aligned}$$

Case (ii): $v_1 = v_2$ and $u_1u_2 \in R_i'$. Then,

$$\begin{aligned} \mu_i((u_1, v_1)(u_2, v_2)) &= \sigma_2(v_1) \vee \mu_i'(u_1u_2) = \sigma_2(v_1) \vee [\sigma_1(u_1) \wedge \sigma_1(u_2)] \\ &= [\sigma_1(u_1) \vee \sigma_2(v_1)] \wedge [\sigma_1(u_2) \vee \sigma_2(v_1)] \\ &= [\sigma(u_1, v_1) \wedge \sigma(u_2, v_2)]. \end{aligned}$$

Thus, $\mu_i((u_1, v_1)(u_2, v_2)) = \sigma(u_1, v_1) \wedge \sigma(u_2, v_2)$ for all edges of maximal product.
Hence $G = G_1 * G_2 = (\sigma, \mu_1, \mu_2, \ldots \mu_n)$ is a strong fuzzy graph structure.

□

Remark 1. *Converse of Theorem 1 above may not be true. That is, maximal product $G = G_1 * G_2$ may be a strong fuzzy-graph structure, when G_1 and G_2 are not strong fuzzy-graph structures.*

Example 4. *Consider two fuzzy-graph structures $G_1 = (\sigma_1, \mu_1')$ and $G_2 = (\sigma_2, \mu_1'')$, which are not strong as shown in Figure 5.*

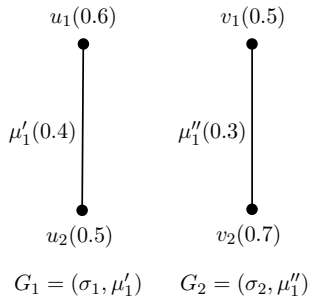


Figure 5. Not strong fuzzy-graph structures.

Maximal product $G_1 * G_2 = (\sigma, \mu_1)$ of G_1 and G_2 is shown in Figure 6.

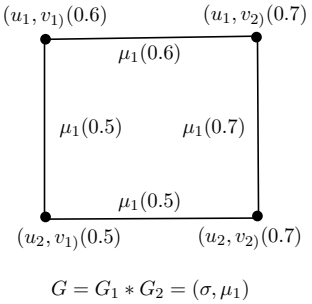


Figure 6. Strong fuzzy-graph structure.

In Figure 6, we can see that,

$\mu_1((u_1, v_1)(u_1, v_2)) = \sigma_1(u_1, v_1) \wedge \sigma_2(u_1, v_2)$, $\mu_1((u_1, v_1)(u_2, v_1)) = \sigma_1(u_1, v_1) \wedge \sigma_2(u_2, v_1)$,
$\mu_1((u_2, v_1)(u_2, v_2)) = \sigma_1(u_2, v_1) \wedge \sigma_2(u_2, v_2)$, $\mu_1((u_1, v_2)(u_2, v_2)) = \sigma_1(u_1, v_2) \wedge \sigma_2(u_2, v_2)$.

Hence $G_1 * G_2 = (\sigma, \mu_1)$ is a strong fuzzy-graph structure.

Theorem 2. *The maximal product of two connected fuzzy-graph structures is a connected fuzzy graph structure.*

Proof. Let $G_1 = (\sigma_1, \mu'_1, \mu'_2, \ldots, \mu'_n)$ and $G_2 = (\sigma_2, \mu''_1, \mu''_2, \ldots, \mu''_n)$ be two connected fuzzy-graph structures with underlying crisp graph structures $G_1^* = (V_1, R'_1, R'_2, \ldots, R'_n)$ and $G_2^* = (V_2, R''_1, R''_2, \ldots, R''_n)$, respectively. Let $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$. Then $\mu_i'^{\infty}(u_i u_j) > 0$ for all $u_i, u_j \in V_1$ and $\mu_i''^{\infty}(v_i v_j) > 0$ for all $v_i, v_j \in V_2$. The maximal product of $G_1 = (\sigma_1, \mu'_1, \mu'_2, \ldots, \mu'_n)$ and $G_2 = (\sigma_2, \mu''_1, \mu''_2, \ldots, \mu''_n)$ is written as $G = (\sigma, \mu_1, \mu_2, \ldots, \mu_n)$. Now consider 'm' subgraphs of G with the vertex sets $\{u_i v_1, u_i v_2, \ldots, u_i v_n\}$ for $i = 1, 2, \ldots, m$. Each of these subgraphs of G is connected since $u_i's$ are the same and G_2 is connected, each v_i is adjacent to at least one of the vertices in V_2. Since G_1 is connected, each u_i is also adjacent to at least one of the vertices in V_1. Therefore, there exists one edge between any pair of the above 'm' subgraphs. Thus, we have $\mu_i^{\infty}((u_i v_j)(u_k v_l)) > 0$ for all $(u_i, v_j)(u_k, v_l) \in R_i$. Hence, G is a connected fuzzy-graph structure. □

Remark 2. *The maximal product of two complete fuzzy-graph structures is not a complete fuzzy-graph structure due to the absence of case $u_1 u_2 \in R'_i$ and $v_1 v_2 \in R''_i$ in the maximal-product definition. Since every complete fuzzy-graph structure is strong, according to Theorem 1, the maximal product of two complete fuzzy-graph structures is a strong fuzzy-graph structure.*

Definition 5. *The degree of a vertex in maximal product $G_1 * G_2$ of two fuzzy-graph structures $G_1 = (\sigma_1, \mu'_1, \mu'_2, \ldots, \mu'_n)$ and $G_2 = (\sigma_2, \mu''_1, \mu''_2, \ldots, \mu''_n)$ is given by:*

$$d_{G_1 * G_2}(u_i, v_j) = \sum_{u_i u_k \in R'_i, v_j = v_l} \mu'_i(u_i u_k) \vee \sigma_2(v_j) + \sum_{v_j v_l \in R''_j, u_i = u_k} \mu''_j(v_j v_l) \vee \sigma_1(u_i).$$

μ_i−degree of a vertex of maximal product $G_1 * G_2$ is given by:

$$\mu_i - d_{G_1 * G_2}(u_i, v_j) = \sum_{u_i u_k \in R'_i, v_j = v_l} \mu'_i(u_i u_k) \vee \sigma_2(v_j) + \sum_{v_j v_l \in R''_i, u_i = u_k} \mu''_i(v_j v_l) \vee \sigma_1(u_i).$$

Example 5. *Consider two fuzzy-graph structures $G_1 = (\sigma_1, \mu'_1, \mu'_2, \mu'_3)$ and $G_2 = (\sigma_2, \mu''_1)$, which are shown in Figure 7.*

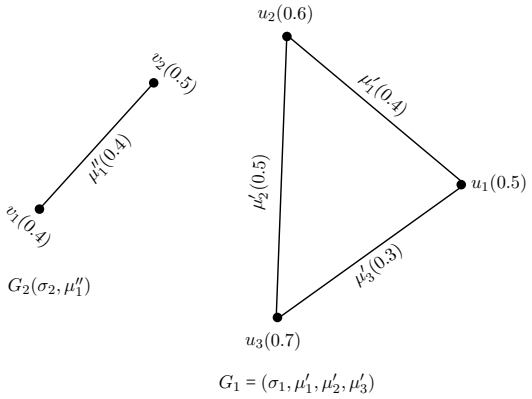


Figure 7. Two fuzzy-graph structures G_1 and G_2.

The maximal product of G_1 and G_2 is shown in Figure 8.

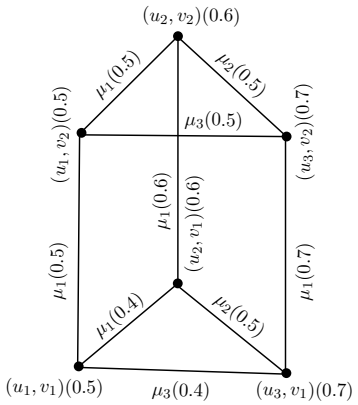


Figure 8. $G_1 * G_2 = (\sigma, \mu_1, \mu_2, \mu_3)$ (maximal product).

Degree of vertex in maximal product is given by:

$$d_{G_1*G_2}(u_i, v_j) = \sum_{u_iu_k \in R_i', v_j = v_l} \mu_i'(u_iu_k) \vee \sigma_2(v_j) + \sum_{v_jv_l \in R_j'', u_i = u_k} \mu_j''(v_jv_l) \vee \sigma_1(u_i).$$

Using this formula, we calculate degree of vertices in maximal product as:

$$\begin{aligned} d_{G_1*G_2}(u_1, v_1) &= (\mu_1'(u_1u_2) \vee \sigma_2(v_1)) + (\mu_3'(u_1u_3) \vee \sigma_2(v_1)) + (\mu_1''(v_1v_2) \vee \sigma_1(u_1)) \\ &= (0.4 \vee 0.4) + (0.3 \vee 0.4) + (0.4 \vee 0.5) \\ &= 0.4 + 0.4 + 0.5 \\ &= 1.3, \end{aligned}$$

$$\begin{aligned} d_{G_1*G_2}(u_1, v_2) &= (\mu_1'(u_1u_2) \vee \sigma_2(v_2)) + (\mu_3'(u_1u_3) \vee \sigma_2(v_2)) + (\mu_1''(v_1v_2) \vee \sigma_1(u_1)) \\ &= (0.4 \vee 0.5) + (0.3 \vee 0.5) + (0.4 \vee 0.5) \\ &= 0.5 + 0.5 + 0.5 \\ &= 1.5, \end{aligned}$$

$$\begin{aligned} d_{G_1*G_2}(u_2, v_1) &= (\mu'_1(u_2u_1) \vee \sigma_2(v_1)) + (\mu'_2(u_2u_3) \vee \sigma_2(v_1)) + (\mu''_1(v_1v_2) \vee \sigma_1(u_2)) \\ &= (0.4 \vee 0.4) + (0.5 \vee 0.4) + (0.4 \vee 0.6) \\ &= 0.4 + 0.5 + 0.6 \\ &= 1.5, \end{aligned}$$

$$\begin{aligned} d_{G_1*G_2}(u_2, v_2) &= (\mu'_1(u_2u_1) \vee \sigma_2(v_2)) + (\mu'_2(u_2u_3) \vee \sigma_2(v_2)) + (\mu''_1(v_2v_1) \vee \sigma_1(u_2)) \\ &= (0.4 \vee 0.5) + (0.5 \vee 0.5) + (0.4 \vee 0.6) \\ &= 0.5 + 0.5 + 0.6 \\ &= 1.6, \end{aligned}$$

$$\begin{aligned} d_{G_1*G_2}(u_3, v_1) &= (\mu'_3(u_3u_1) \vee \sigma_2(v_1)) + (\mu'_2(u_3u_2) \vee \sigma_2(v_1)) + (\mu''_1(v_1v_2) \vee \sigma_1(u_3)) \\ &= (0.3 \vee 0.4) + (0.5 \vee 0.4) + (0.4 \vee 0.7) \\ &= 0.4 + 0.5 + 0.7 \\ &= 1.6, \end{aligned}$$

$$\begin{aligned} d_{G_1*G_2}(u_3, v_2) &= (\mu'_2(u_3u_2) \vee \sigma_2(v_2)) + (\mu'_3(u_3u_1) \vee \sigma_2(v_2)) + (\mu''_1(v_1v_2) \vee \sigma_1(u_3)) \\ &= (0.5 \vee 0.5) + (0.3 \vee 0.5) + (0.4 \vee 0.7) \\ &= 0.5 + 0.5 + 0.7 \\ &= 1.7. \end{aligned}$$

μ_i—degree of a vertex of maximal product $G_1 * G_2$ is given by:

$$\mu_i - d_{G_1*G_2}(u_i, v_j) = \sum_{u_iu_k \in R'_i, v_j = v_l} \mu'_i(u_iu_k) \vee \sigma_2(v_j) + \sum_{v_jv_l \in R''_i, u_i = u_k} \mu''_i(v_jv_l) \vee \sigma_1(u_i).$$

Using this formula, we calculate μ_i—degree of vertices in maximal product as:

$$\begin{aligned} \mu_1 - d_{G_1*G_2}(u_1, v_1) &= (\mu'_1(u_1u_2) \vee \sigma_2(v_1)) + (\mu''_1(v_1v_2) \vee \sigma_1(u_1)) \\ &= (0.4 \vee 0.4) + (0.4 \vee 0.5) \\ &= 0.4 + 0.5 \\ &= 0.9, \end{aligned}$$

$$\begin{aligned} \mu_1 - d_{G_1*G_2}(u_1, v_2) &= (\mu'_1(u_1u_2) \vee \sigma_2(v_2)) + (\mu''_1(v_1v_2) \vee \sigma_1(u_1)) \\ &= (0.4 \vee 0.5) + (0.4 \vee 0.5) \\ &= 0.5 + 0.5 \\ &= 1.0, \end{aligned}$$

$$\begin{aligned} \mu_1 - d_{G_1*G_2}(u_2, v_1) &= (\mu'_1(u_2u_1) \vee \sigma_2(v_1)) + (\mu''_1(v_1v_2) \vee \sigma_1(u_2)) \\ &= (0.4 \vee 0.4) + (0.4 \vee 0.6) \\ &= 0.4 + 0.6 \\ &= 1.0, \end{aligned}$$

$$\begin{aligned}\mu_1 - d_{G_1*G_2}(u_2, v_2) &= (\mu_1'(u_2u_1) \vee \sigma_2(v_2)) + (\mu_1''(v_2v_1) \vee \sigma_1(u_2)) \\ &= (0.4 \vee 0.5) + (0.4 \vee 0.6) \\ &= 0.5 + 0.6 \\ &= 1.1,\end{aligned}$$

$$\begin{aligned}\mu_1 - d_{G_1*G_2}(u_3, v_1) &= \mu_1''(v_1v_2) \vee \sigma_1(u_3) = 0.4 \vee 0.7 = 0.7, \\ \mu_1 - d_{G_1*G_2}(u_3, v_2) &= \mu_1''(v_1v_2) \vee \sigma_1(u_3) = 0.4 \vee 0.7 = 0.7, \\ \mu_2 - d_{G_1*G_2}(u_2, v_1) &= \mu_2'(u_2u_3) \vee \sigma_2(v_1) = 0.5 \vee 0.4 = 0.5, \\ \mu_2 - d_{G_1*G_2}(u_2, v_2) &= \mu_2'(u_2u_3) \vee \sigma_2(v_2) = 0.5 \vee 0.5 = 0.5, \\ \mu_2 - d_{G_1*G_2}(u_3, v_1) &= \mu_2'(u_3u_2) \vee \sigma_2(v_1) = 0.5 \vee 0.4 = 0.5, \\ \mu_2 - d_{G_1*G_2}(u_3, v_2) &= \mu_2'(u_3u_2) \vee \sigma_2(v_2) = 0.5 \vee 0.5 = 0.5, \\ \mu_3 - d_{G_1*G_2}(u_1, v_1) &= \mu_3'(u_1u_3) \vee \sigma_2(v_1) = 0.3 \vee 0.4 = 0.4, \\ \mu_3 - d_{G_1*G_2}(u_1, v_2) &= \mu_3'(u_1u_3) \vee \sigma_2(v_2) = 0.3 \vee 0.5 = 0.5, \\ \mu_3 - d_{G_1*G_2}(u_3, v_1) &= \mu_3'(u_3u_1) \vee \sigma_2(v_1) = 0.3 \vee 0.4 = 0.4, \\ \mu_3 - d_{G_1*G_2}(u_3, v_2) &= \mu_3'(u_3u_1) \vee \sigma_2(v_2) = 0.3 \vee 0.5 = 0.5.\end{aligned}$$

Theorem 3. *If $G_1 = (\sigma_1, \mu_1', \mu_2', \ldots, \mu_n')$ and $G_2 = (\sigma_2, \mu_1'', \mu_2'', \ldots, \mu_n'')$ are two fuzzy-graph structures, such that $\sigma_1 \leq \mu_i''$, $i = 1, 2, \ldots, n$, then degree of any vertex in maximal product $G_1 * G_2$ is given by $d_{G_1*G_2}(u_i, v_j) = d_{G_1^*}(u_i)\sigma_2(v_j) + d_{G_2}(v_j)$.*

Proof. Let $G_1 = (\sigma_1, \mu_1', \mu_2', \ldots, \mu_n')$ and $G_2 = (\sigma_2, \mu_1'', \mu_2'', \ldots, \mu_n'')$ be two fuzzy-graph structures such that $\sigma_1 \leq \mu_i''$, then $\mu_i' \leq \sigma_2$, $i = 1, 2, \ldots, n$. Then, degree of any vertex in $G_1 * G_2$ (maximal product) is given by:

$$\begin{aligned}d_{G_1*G_2}(u_i, v_j) &= \sum_{u_iu_k \in R_i', v_j = v_l} \mu_i'(u_iu_k) \vee \sigma_2(v_j) + \sum_{v_jv_l \in R_j'', u_i = u_k} \mu_j''(v_jv_l) \vee \sigma_1(u_i) \\ &= \sum_{u_iu_k \in R_i', v_j = v_l} \sigma_2(v_j) + \sum_{v_jv_l \in R_j'', u_i = u_k} \mu_j''(v_jv_l) \\ &= d_{G_1^*}(u_i)\sigma_2(v_j) + d_{G_2}(v_j).\end{aligned}$$

□

Theorem 4. *If $G_1 = (\sigma_1, \mu_1', \mu_2', \ldots, \mu_n')$ and $G_2 = (\sigma_2, \mu_1'', \mu_2'', \ldots, \mu_n'')$ are two fuzzy graph structures such that $\sigma_1 \leq \mu_i''$, $i = 1, 2, \ldots, n$ and σ_2 is a constant function of value 'c', then degree of any vertex in $G_1 * G_2$ (maximal product) is given by: $d_{G_1*G_2}(u_i, v_j) = d_{G_1^*}(u_i)c + d_{G_2}(v_j)$.*

Proof. Let $G_1 = (\sigma_1, \mu_1', \mu_2', \ldots, \mu_n')$ and $G_2 = (\sigma_2, \mu_1'', \mu_2'', \ldots, \mu_n'')$ be two fuzzy-graph structures, such that $\sigma_1 \leq \mu_i''$, $i = 1, 2, \ldots, n$ and σ_2 is a constant function of value 'c'. Moreover, $\sigma_1 \leq \mu_i''$ implies $\mu_i' \leq \sigma_2$, $i = 1, 2, \ldots, n$. Then degree of any vertex in $G_1 * G_2$ (maximal product) is given by:

$$\begin{aligned}d_{G_1*G_2}(u_i, v_j) &= \sum_{u_iu_k \in R_i', v_j = v_l} \mu_i'(u_iu_k) \vee \sigma_2(v_j) + \sum_{v_jv_l \in R_j'', u_i = u_k} \mu_j''(v_jv_l) \vee \sigma_1(u_i) \\ &= \sum_{u_iu_k \in R_i', v_j = v_l} \sigma_2(v_j) + \sum_{v_jv_l \in R_j'', u_i = u_k} \mu_j''(v_jv_l) \\ &= d_{G_1^*}(u_i)c + d_{G_2}(v_j).\end{aligned}$$

□

Theorem 5. *If $G_1 = (\sigma_1, \mu'_1, \mu'_2, \ldots, \mu'_n)$ and $G_2 = (\sigma_2, \mu''_1, \mu''_2, \ldots, \mu''_n)$ are two fuzzy-graph structures such that $\sigma_2 \leq \mu'_i$, $i = 1, 2, \ldots, n$, then degree of any vertex in maximal product $G_1 * G_2$ is given by: $d_{G_1 * G_2}(u_i, v_j) = d_{G_2^*}(v_j)\sigma_1(u_i) + d_{G_1}(u_i)$.*

Proof. Let $G_1 = (\sigma_1, \mu'_1, \mu'_2, \ldots, \mu'_n)$ and $G_2 = (\sigma_2, \mu''_1, \mu''_2, \ldots, \mu''_n)$ be two fuzzy-graph structures such that $\sigma_2 \leq \mu'_i$, then $\mu''_i \leq \sigma_1$, $i = 1, 2, \ldots, n$. Then, degree of any vertex in $G_1 * G_2$ (maximal product) is given by:

$$\begin{aligned} d_{G_1 * G_2}(u_i, v_j) &= \sum_{u_i u_k \in R'_i, v_j = v_l} \mu'_i(u_i u_k) \vee \sigma_2(v_j) + \sum_{v_j v_l \in R''_j, u_i = u_k} \mu''_j(v_j v_l) \vee \sigma_1(u_i) \\ &= \sum_{u_i u_k \in R'_i, v_j = v_l} \mu'_i(u_i u_k) + \sum_{v_j v_l \in R''_j, u_i = u_k} \sigma_1(u_i) \\ &= d_{G_1}(u_i) + d_{G_2^*}(v_j)\sigma_1(u_i). \end{aligned}$$

□

Theorem 6. *If $G_1 = (\sigma_1, \mu'_1, \mu'_2, \ldots, \mu'_n)$ and $G_2 = (\sigma_2, \mu''_1, \mu''_2, \ldots, \mu''_n)$ are two fuzzy-graph structures such that $\sigma_2 \leq \mu'_i$, $i = 1, 2, \ldots, n$ and σ_1 is a constant function of value 'c', then degree of any vertex in $G_1 * G_2$ (maximal product) is given by: $d_{G_1 * G_2}(u_i, v_j) = d_{G_1}(u_i) + d_{G_2^*}(v_j)c$.*

Proof. Let $G_1 = (\sigma_1, \mu'_1, \mu'_2, \ldots, \mu'_n)$ and $G_2 = (\sigma_2, \mu''_1, \mu''_2, \ldots, \mu''_n)$ be two fuzzy-graph structures such that $\sigma_2 \leq \mu'_i, i = 1, 2, \ldots, n$ and σ_1 is a constant function of value 'c'. Moreover, $\sigma_2 \leq \mu'_i$ implies $\mu''_i \leq \sigma_1$, $i = 1, 2, \ldots, n$. Then, degree of any vertex in $G_1 * G_2$ (maximal product) is given by:

$$\begin{aligned} d_{G_1 * G_2}(u_i, v_j) &= \sum_{u_i u_k \in R'_i, v_j = v_l} \mu'_i(u_i u_k) \vee \sigma_2(v_j) + \sum_{v_j v_l \in R''_j, u_i = u_k} \mu''_j(v_j v_l) \vee \sigma_1(u_i) \\ &= \sum_{u_i u_k \in R'_i, v_j = v_l} \mu'_i(u_i u_k) + \sum_{v_j v_l \in R''_j, u_i = u_k} \sigma_1(u_i) \\ &= d_{G_1}(u_i) + d_{G_2^*}(v_j)c. \end{aligned}$$

□

Theorem 7. *If $G_1 = (\sigma_1, \mu'_1, \mu'_2, \ldots, \mu'_n)$ and $G_2 = (\sigma_2, \mu''_1, \mu''_2, \ldots, \mu''_n)$ are two fuzzy-graph structures such that $\mu''_i \leq \sigma_1$ and $\mu'_i \leq \sigma_2$, $i = 1, 2, \ldots, n$, then degree of any vertex in $G_1 * G_2$ (maximal product) is given by: $d_{G_1 * G_2}(u_i, v_j) = d_{G_1^*}(u_i)\sigma_2(v_j) + d_{G_2^*}(v_j)\sigma_1(u_i)$.*

Proof. Let $G_1 = (\sigma_1, \mu'_1, \mu'_2, \ldots, \mu'_n)$ and $G_2 = (\sigma_2, \mu''_1, \mu''_2, \ldots, \mu''_n)$ be two fuzzy-graph structures such that $\mu''_i \leq \sigma_1$ and $\mu'_i \leq \sigma_2$, $i = 1, 2, \ldots, n$, then degree of any vertex in $G_1 * G_2$ (maximal product) is given by:

$$\begin{aligned} d_{G_1 * G_2}(u_i, v_j) &= \sum_{u_i u_k \in R'_i, v_j = v_l} \mu'_i(u_i u_k) \vee \sigma_2(v_j) + \sum_{v_j v_l \in R''_j, u_i = u_k} \mu''_j(v_j v_l) \vee \sigma_1(u_i) \\ &= \sum_{u_i u_k \in R'_i, v_j = v_l} \sigma_2(v_j) + \sum_{v_j v_l \in R''_j, u_i = u_k} \sigma_1(u_i) \\ &= d_{G_1^*}(u_i)\sigma_2(v_j) + d_{G_2^*}(v_j)\sigma_1(u_i). \end{aligned}$$

□

Example 6. *Consider two fuzzy-graph structures $G_1 = (\sigma_1, \mu'_1, \mu'_2, \mu'_3)$ and $G_2 = (\sigma_2, \mu''_1)$, which are shown in Figure 9.*

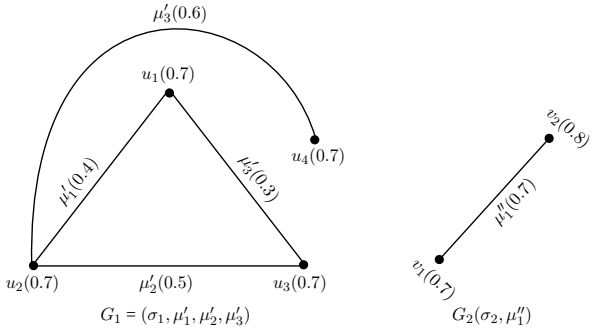


Figure 9. Two fuzzy-graph structures G_1 and G_2.

The maximal product of G_1 and G_2 is shown in Figure 10.

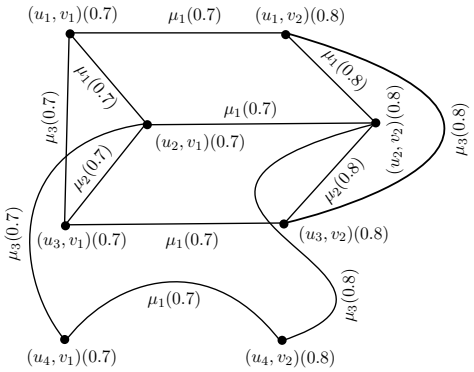


Figure 10. $G_1 * G_2 = (\sigma, \mu_1, \mu_2, \mu_3)$.

In Figure 9. $\mu''_1 = \sigma_1$ and $\mu'_i < \sigma_2$, $i = 1, 2, 3$. Then, the degree of vertex in maximal product is calculated by using following formula:

$$d_{G_1 * G_2}(u_i, v_j) = d_{G_1^*}(u_i)\sigma_2(v_j) + d_{G_2^*}(v_j)\sigma_1(u_i).$$

$$\begin{aligned} d_{G_1 * G_2}(u_1, v_1) &= d_{G_1^*}(u_1)\sigma_2(v_1) + d_{G_2^*}(v_1)\sigma_1(u_1) \\ &= (2)(0.7) + 1(0.7) \\ &= 1.4 + 0.7 \\ &= 2.1, \end{aligned}$$

$$\begin{aligned} d_{G_1 * G_2}(u_1, v_2) &= d_{G_1^*}(u_1)\sigma_2(v_2) + d_{G_2^*}(v_2)\sigma_1(u_1) \\ &= (2)(0.8) + 1(0.7) \\ &= 1.6 + 0.7 \\ &= 2.3, \end{aligned}$$

$$\begin{aligned} d_{G_1 * G_2}(u_2, v_1) &= d_{G_1^*}(u_2)\sigma_2(v_1) + d_{G_2^*}(v_1)\sigma_1(u_2) \\ &= (3)(0.7) + 1(0.7) \\ &= 2.1 + 0.7 \\ &= 2.8, \end{aligned}$$

$$\begin{aligned} d_{G_1*G_2}(u_2, v_2) &= d_{G_1^*}(u_2)\sigma_2(v_2) + d_{G_2^*}(v_2)\sigma_1(u_2) \\ &= (3)(0.8) + 1(0.7) \\ &= 2.4 + 0.7 \\ &= 3.1, \end{aligned}$$

$$\begin{aligned} d_{G_1*G_2}(u_3, v_1) &= d_{G_1^*}(u_3)\sigma_2(v_1) + d_{G_2^*}(v_1)\sigma_1(u_3) \\ &= (2)(0.7) + 1(0.7) \\ &= 1.4 + 0.7 \\ &= 2.1, \end{aligned}$$

$$\begin{aligned} d_{G_1*G_2}(u_3, v_2) &= d_{G_1^*}(u_3)\sigma_2(v_2) + d_{G_2^*}(v_2)\sigma_1(u_3) \\ &= (2)(0.8) + 1(0.7) \\ &= 1.6 + 0.7 \\ &= 2.3, \end{aligned}$$

$$\begin{aligned} d_{G_1*G_2}(u_4, v_1) &= d_{G_1^*}(u_4)\sigma_2(v_1) + d_{G_2^*}(v_1)\sigma_1(u_4) \\ &= (1)(0.7) + 1(0.7) \\ &= 0.7 + 0.7 \\ &= 1.4, \end{aligned}$$

$$\begin{aligned} d_{G_1*G_2}(u_4, v_2) &= d_{G_1^*}(u_4)\sigma_2(v_2) + d_{G_2^*}(v_2)\sigma_1(u_4) \\ &= (1)(0.8) + 1(0.7) \\ &= 0.8 + 0.7 \\ &= 1.5. \end{aligned}$$

By direct calculations:

$$d_{G_1*G_2}(u_4, v_1) = 0.7 + 0.7 = 1.4,$$

$$d_{G_1*G_2}(u_4, v_2) = 0.8 + 0.7 = 1.5,$$

$$d_{G_1*G_2}(u_1, v_1) = 0.7 + 0.7 + 0.7 = 2.1,$$

$$d_{G_1*G_2}(u_3, v_1) = 0.7 + 0.7 + 0.7 = 2.1,$$

$$d_{G_1*G_2}(u_1, v_2) = 0.8 + 0.8 + 0.7 = 2.3,$$

$$d_{G_1*G_2}(u_3, v_2) = 0.8 + 0.8 + 0.7 = 2.3,$$

$$d_{G_1*G_2}(u_2, v_1) = 0.7 + 0.7 + 0.7 + 0.7 = 2.8,$$

$$d_{G_1*G_2}(u_2, v_2) = 0.8 + 0.8 + 0.8 + 0.7 = 3.1.$$

It is clear from the above calculations that degrees of vertices calculated by using the formula of the above theorem and by direct method are same.

Theorem 8. *If $G_1 = (\sigma_1, \mu_1', \mu_2', \ldots, \mu_n')$ and $G_2 = (\sigma_2, \mu_1'', \mu_2'', \ldots, \mu_n'')$ are two fuzzy-graph structures, such that $\mu_i'' \geq \sigma_1$, $i = 1, 2, \ldots, n$, then total degree of any vertex in $G_1 * G_2$ (maximal product) is given by: $td_{G_1*G_2}(u_i, v_j) = d_{G_1^*}(u_i)\sigma_2(v_j) + td_{G_2}(v_j)$.*

Proof. Let $G_1 = (\sigma_1, \mu_1', \mu_2', \ldots, \mu_n')$ and $G_2 = (\sigma_2, \mu_1'', \mu_2'', \ldots, \mu_n'')$ be two fuzzy-graph structures such that $\mu_i'' \geq \sigma_1$, then $\mu_i' \leq \sigma_2$, $\sigma_1 \leq \sigma_2$, $i = 1, 2, \ldots, n$. Then, total degree of any vertex in $G_1 * G_2$ (maximal product) is given by:

$$\begin{aligned} td_{G_1*G_2}(u_i, v_j) &= \sum_{u_iu_k \in R_i', v_j = v_l} \mu_i'(u_iu_k) \vee \sigma_2(v_j) + \sum_{v_jv_l \in R_j'', u_i = u_k} \mu_j''(v_jv_l) \vee \sigma_1(u_i) + \sigma(u_i, v_j) \\ &= \sum_{u_iu_k \in R_i', v_j = v_l} \sigma_2(v_j) + \sum_{v_jv_l \in R_j'', u_i = u_k} \mu_j''(v_jv_l) + [\sigma_1(u_i) \vee \sigma_2(v_j)] \\ &= d_{G_1^*}(u_i)\sigma_2(v_j) + [d_{G_2}(v_j) + \sigma_2(v_j)] \\ &= d_{G_1^*}(u_i)\sigma_2(v_j) + td_{G_2}(v_j). \end{aligned}$$

□

Example 7. *Consider two fuzzy-graph structures $G_1 = (\sigma_1, \mu_1', \mu_2', \mu_3')$ and $G_2 = (\sigma_2, \mu_1'')$, which are shown in Figure 11.*

In Figure 11. $\mu_1'' \geq \sigma_1$. Then, the total degree of vertex in maximal product is calculated by using the following formula:

$$td_{G_1*G_2}(u_i, v_j) = d_{G_1^*}(u_i)\sigma_2(v_j) + td_{G_2}(v_j).$$

$$\begin{aligned} td_{G_1*G_2}(u_1, v_1) &= d_{G_1^*}(u_1)\sigma_2(v_1) + td_{G_2^*}(v_1) \\ &= 2(0.7) + (0.7 + 0.7) \\ &= 1.4 + 1.4 \\ &= 2.8, \end{aligned}$$

$$\begin{aligned} td_{G_1*G_2}(u_1, v_2) &= d_{G_1^*}(u_1)\sigma_2(v_2) + td_{G_2^*}(v_2) \\ &= (2)(0.8) + (0.7 + 0.8) \\ &= 1.6 + 1.5 \\ &= 3.1, \end{aligned}$$

$$\begin{aligned} td_{G_1*G_2}(u_2, v_1) &= d_{G_1^*}(u_2)\sigma_2(v_1) + td_{G_2^*}(v_1) \\ &= (3)(0.7) + (0.7 + 0.7) \\ &= 2.1 + 1.4 \\ &= 3.5, \end{aligned}$$

$$\begin{aligned} td_{G_1*G_2}(u_2, v_2) &= d_{G_1^*}(u_2)\sigma_2(v_2) + td_{G_2^*}(v_2) \\ &= (3)(0.8) + (0.7 + 0.8) \\ &= 2.4 + 1.5 \\ &= 3.9, \end{aligned}$$

$$\begin{aligned} td_{G_1*G_2}(u_3, v_1) &= d_{G_1^*}(u_3)\sigma_2(v_1) + td_{G_2^*}(v_1) \\ &= (2)(0.7) + (0.7 + 0.7) \\ &= 1.4 + 1.4 \\ &= 2.8, \end{aligned}$$

$$\begin{aligned} td_{G_1*G_2}(u_3, v_2) &= d_{G_1^*}(u_3)\sigma_2(v_2) + td_{G_2^*}(v_2) \\ &= (2)(0.8) + (0.7 + 0.8) \\ &= 1.6 + 1.5 \\ &= 3.1, \end{aligned}$$

$$\begin{aligned} td_{G_1*G_2}(u_4, v_1) &= d_{G_1^*}(u_4)\sigma_2(v_1) + td_{G_2^*}(v_1) \\ &= (1)(0.7) + (0.7 + 0.7) \\ &= 0.7 + 1.4 \\ &= 2.1, \end{aligned}$$

$$\begin{aligned} td_{G_1*G_2}(u_4, v_2) &= d_{G_1^*}(u_4)\sigma_2(v_2) + td_{G_2^*}(v_2) \\ &= (1)(0.8) + (0.7 + 0.8) \\ &= 0.8 + 1.5 \\ &= 2.3. \end{aligned}$$

By direct calculations:

$$td_{G_1*G_2}(u_4, v_1) = 0.7 + 0.7 + 0.7 = 2.1,$$

$$td_{G_1*G_2}(u_4, v_2) = 0.8 + 0.7 + 0.8 = 2.3,$$

$$td_{G_1*G_2}(u_1, v_1) = 0.7 + 0.7 + 0.7 + 0.7 = 2.8,$$

$$td_{G_1*G_2}(u_3, v_1) = 0.7 + 0.7 + 0.7 + 0.7 = 2.8,$$

$$td_{G_1*G_2}(u_3, v_2) = 0.8 + 0.8 + 0.7 + 0.8 = 3.1,$$

$$td_{G_1*G_2}(u_1, v_2) = 0.8 + 0.8 + 0.7 + 0.8 = 3.1,$$

$$td_{G_1*G_2}(u_2, v_1) = 0.7 + 0.7 + 0.7 + 0.7 + 0.7 = 3.5,$$

$$td_{G_1*G_2}(u_2, v_2) = 0.8 + 0.8 + 0.8 + 0.7 + 0.8 = 3.9.$$

It is clear from the above calculations that the total degrees of vertices calculated by using the formula of the above theorem and by direct method are the same.

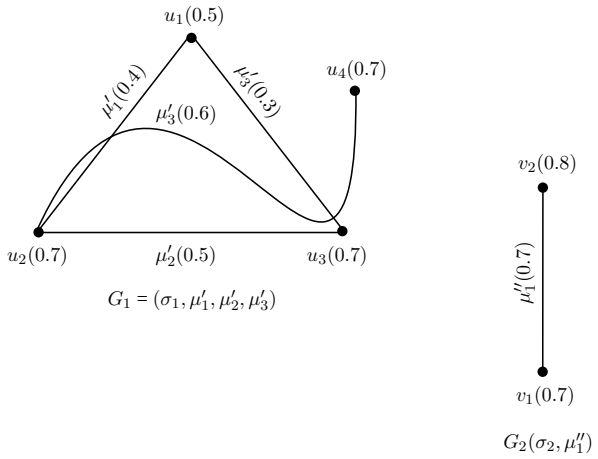


Figure 11. Two fuzzy-graph structures.

Theorem 9. *If $G_1 = (\sigma_1, \mu'_1, \mu'_2, \ldots, \mu'_n)$ and $G_2 = (\sigma_2, \mu''_1, \mu''_2, \ldots, \mu''_n)$ are two fuzzy-graph structures, such that $\mu''_i \geq \sigma_1$, $i = 1, 2, \ldots, n$ and σ_2 is a constant function of value 'c', then the total degree of any vertex in $G_1 * G_2$ (maximal product) (Figure 12) is given by: $td_{G_1 * G_2}(u_i, v_j) = td_{G_2}(v_j) + d_{G_1^*}(u_i)c$.*

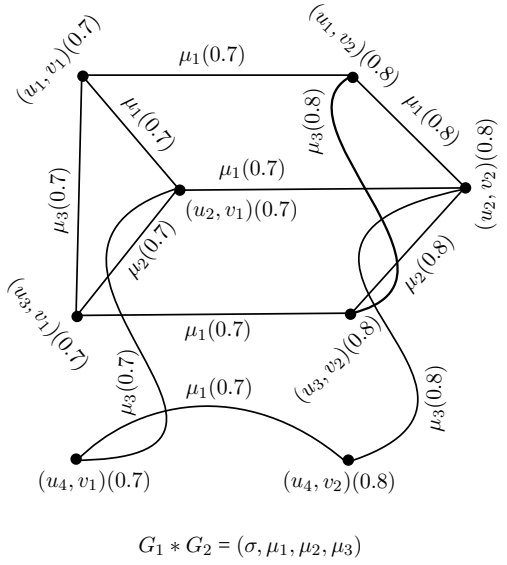


Figure 12. Maximal product of two fuzzy-graph structures (FGSs).

Proof. Let $G_1 = (\sigma_1, \mu'_1, \mu'_2, \ldots, \mu'_n)$ and $G_2 = (\sigma_2, \mu''_1, \mu''_2, \ldots, \mu''_n)$ be two fuzzy-graph structures such that $\mu''_i \geq \sigma_1, i = 1, 2, \ldots, n$ and σ_2 is a constant function of value 'c'. Moreover, $\mu''_i \geq \sigma_1$ implies $\sigma_2 \geq \sigma_1$, $\mu'_i \leq \sigma_2$, $i = 1, 2, \ldots, n$. Then, the total degree of any vertex in $G_1 * G_2$ (maximal product) is given by:

$$\begin{aligned} d_{G_1 * G_2}(u_i, v_j) &= \sum_{u_i u_k \in R'_i, v_j = v_l} \mu'_i(u_i u_k) \vee \sigma_2(v_j) + \sum_{v_j v_l \in R''_j, u_i = u_k} \mu''_j(v_j v_l) \vee \sigma_1(u_i) + \sigma(u_i, v_j) \\ &= \sum_{u_i u_k \in R'_i, v_j = v_l} \sigma_2(v_j) + \sum_{v_j v_l \in R''_j, u_i = u_k} \mu''_j(v_j v_l) + [\sigma_1(u_i) \vee \sigma_2(v_j)] \\ &= d_{G_2}(v_j) + d_{G_1^*}(u_i)c + \sigma_2(v_j) \\ &= td_{G_2}(v_j) + d_{G_1^*}(u_i)c. \end{aligned}$$

□

Theorem 10. *If $G_1 = (\sigma_1, \mu'_1, \mu'_2, \ldots, \mu'_n)$ and $G_2 = (\sigma_2, \mu''_1, \mu''_2, \ldots, \mu''_n)$ are two fuzzy-graph structures such that $\mu'_i \geq \sigma_2$, $i = 1, 2, \ldots, n$, then the total degree of any vertex in $G_1 * G_2$ (maximal product) is given by: $td_{G_1 * G_2}(u_i, v_j) = d_{G_2^*}(v_j)\sigma_1(u_i) + td_{G_1}(u_i)$.*

Proof. Let $G_1 = (\sigma_1, \mu'_1, \mu'_2, \ldots, \mu'_n)$ and $G_2 = (\sigma_2, \mu''_1, \mu''_2, \ldots, \mu''_n)$ be two fuzzy-graph structures, such that $\mu'_i \geq \sigma_2$, then $\sigma_1 \geq \sigma_2$, $\mu''_i \leq \sigma_1$, $i = 1, 2, \ldots, n$. The total degree of any vertex in $G_1 * G_2$ (maximal product) is given by:

$$\begin{aligned} d_{G_1 * G_2}(u_i, v_j) &= \sum_{u_i u_k \in R'_i, v_j = v_l} \mu'_i(u_i u_k) \vee \sigma_2(v_j) + \sum_{v_j v_l \in R''_j, u_i = u_k} \mu''_j(v_j v_l) \vee \sigma_1(u_i) + \sigma(u_i, v_j) \\ &= \sum_{u_i u_k \in R'_i, v_j = v_l} \mu'_i(u_i u_k) + \sum_{v_j v_l \in R''_j, u_i = u_k} \sigma_1(u_i) + [\sigma_1(u_i) \vee \sigma_2(v_j)] \\ &= d_{G_1}(u_i) + d_{G_2^*}(v_j)\sigma_1(u_i) + \sigma_1(u_i) \\ &= td_{G_1}(u_i) + d_{G_2^*}(v_j)\sigma_1(u_i). \end{aligned}$$

□

Theorem 11. *If $G_1 = (\sigma_1, \mu'_1, \mu'_2, \ldots, \mu'_n)$ and $G_2 = (\sigma_2, \mu''_1, \mu''_2, \ldots, \mu''_n)$ are two fuzzy-graph structures, such that $\mu'_i \geq \sigma_2$, $i = 1, 2, \ldots, n$ and σ_1 is a constant function of value 'c', then the total degree of any vertex in $G_1 * G_2$ (maximal product) is given by: $td_{G_1 * G_2}(u_i, v_j) = d_{G_2^*}(v_j)c + td_{G_1}(u_i)$.*

Proof. Let $G_1 = (\sigma_1, \mu'_1, \mu'_2, \ldots, \mu'_n)$ and $G_2 = (\sigma_2, \mu''_1, \mu''_2, \ldots, \mu''_n)$ be two fuzzy-graph structures, such that $\mu'_i \geq \sigma_2, i = 1, 2, \ldots, n$ and σ_1 is a constant function of value 'c'. Moreover, $\mu'_i \geq \sigma_2$ implies $\sigma_1 \geq \sigma_2$, $\mu''_i \leq \sigma_1$, $i = 1, 2, \ldots, n$. Then, the total degree of any vertex in $G_1 * G_2$ (maximal product) is given by:

$$\begin{aligned} td_{G_1 * G_2}(u_i, v_j) &= \sum_{u_i u_k \in R'_i, v_j = v_l} \mu'_i(u_i u_k) \vee \sigma_2(v_j) + \sum_{v_j v_l \in R''_j, u_i = u_k} \mu''_j(v_j v_l) \vee \sigma_1(u_i) + \sigma(u_i, v_j) \\ &= \sum_{u_i u_k \in R'_i, v_j = v_l} \mu'_i(u_i u_k) + \sum_{v_j v_l \in R''_j, u_i = u_k} \sigma_1(u_i) + [\sigma_1(u_i) \vee \sigma_2(v_j)] \\ &= d_{G_1}(u_i) + d_{G_2^*}(v_j)c + \sigma_1(u_i) \\ &= td_{G_1}(u_i) + d_{G_2^*}(v_j)c. \end{aligned}$$

□

Remark 3. *The maximal product of two regular fuzzy-graph structures may not be a regular fuzzy-graph structure.*

Example 8. *Consider two fuzzy-graph structures $G_1 = (\sigma_1, \mu'_1, \mu'_2)$ and $G_2 = (\sigma_2, \mu''_1, \mu''_2)$ as shown in Figure 13.*

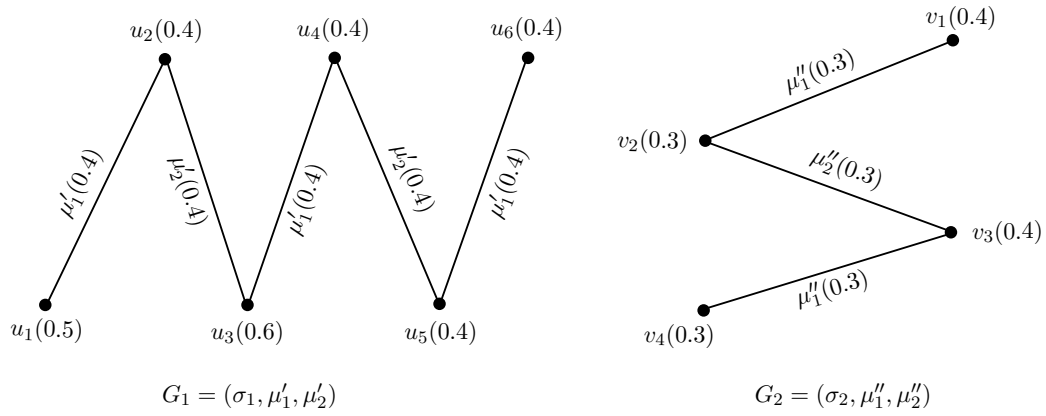


Figure 13. μ'_1-regular and μ''_1-regular FGSs.

It can easily be seen from Figure 13, that each vertex in $G_1 = (\sigma_1, \mu'_1, \mu'_2)$ has one μ'_1 edge with same membership value that is 0.4, hence G_1 is 0.4 μ'_1-regular FGS and each vertex in $G_2 = (\sigma_2, \mu''_1, \mu''_2)$ has one μ''_1 edge with same membership value that is 0.3; hence, G_2 is 0.3 μ''_1-regular FGS.

It can easily be seen from Figure 14, that each vertex in maximal product $G_1 * G_2$ does not have same number of edges of any type with same membership value. Hence, e $G_1 * G_2$ is not a regular FGS. This example shows that the maximal product of two regular FGSs is not a regular FGS.

Theorem 12. *If $G_1 = (\sigma_1, \mu'_1, \mu'_2, \ldots, \mu'_n)$ is partially regular fuzzy-graph structure and $G_2 = (\sigma_2, \mu''_1, \mu''_2, \ldots, \mu''_n)$ is fuzzy-graph structure such that $\mu''_i \geq \sigma_1$, $i = 1, 2, \ldots, n$ and σ_2 is a constant function of value 'c', then maximal product $G_1 * G_2$ is regular if and only if G_2 is regular.*

Proof. Let $G_1 = (\sigma_1, \mu'_1, \mu'_2, \ldots, \mu'_n)$ be partially regular fuzzy-graph structure such that G_1^* is r_1-regular and $G_2 = (\sigma_2, \mu''_1, \mu''_2, \ldots, \mu''_n)$ be fuzzy-graph structures such that $\mu''_i \geq \sigma_1, i = 1, 2, \ldots, n$ and σ_2 is a

constant function of value 'c'. Moreover, suppose that $G_2 = (\sigma_2, \mu_1'', \mu_2'', \ldots, \mu_n'')$ is k-regular fuzzy-graph structure. Then, $d_{G_1 * G_2}(u_i, v_j) = d_{G_2}(v_j) + d_{G_1^*}(u_i)\sigma_2(v_j) = k + r_1 c$. This holds for all vertices of $V_1 \times V_2$. Hence, maximal product $(G_1 * G_2)$ is regular fuzzy-graph structure. Conversely, suppose that maximal product $(G_1 * G_2)$ is a regular fuzzy-graph structure. Then, for any two vertices of $V_1 \times V_2$,

$$\begin{aligned} & d_{G_1 * G_2}(u_1, v_1) = d_{G_1 * G_2}(u_2, v_2) \\ \Rightarrow\ & d_{G_2}(v_1) + d_{G_1^*}(u_1)\sigma_2(v_1) = d_{G_2}(v_2) + d_{G_1^*}(u_2)\sigma_2(v_2) \\ \Rightarrow\ & d_{G_2}(v_1) + r_1 c = d_{G_2}(v_2) + r_1 c \\ \Rightarrow\ & d_{G_2}(v_1) = d_{G_2}(v_2). \end{aligned}$$

This holds for all vertices of G_2. Thus, G_2 is regular fuzzy-graph structure. □

Theorem 13. *If $G_2 = (\sigma_2, \mu_1'', \mu_2'', \ldots, \mu_n'')$ is partially regular fuzzy-graph structure and $G_1 = (\sigma_1, \mu_1', \mu_2', \ldots, \mu_n')$ is fuzzy-graph structure, such that $\mu_i' \geq \sigma_2$, $i = 1, 2, \ldots, n$ and σ_1 is a constant function of value 'c', then maximal product $G_1 * G_2$ is regular if and only if G_1 is regular.*

Proof. Let $G_2 = (\sigma_2, \mu_1'', \mu_2'', \ldots, \mu_n'')$ be partially regular fuzzy-graph structure such that G_2^* is r_2-regular and $G_1 = (\sigma_1, \mu_1', \mu_2', \ldots, \mu_n')$ be fuzzy-graph structure such that $\mu_i' \geq \sigma_2, i = 1, 2, \ldots, n$ and σ_1 is a constant function of value 'c'. Moreover, suppose that $G_1 = (\sigma_1, \mu_1', \mu_2', \ldots, \mu_n')$ is $k-$regular fuzzy-graph structure. Then, $d_{G_1 * G_2}(u_i, v_j) = d_{G_1}(u_i) + d_{G_2^*}(v_j)\sigma_1(u_i) = k + r_2 c$. This holds for all vertices of $V_1 \times V_2$. Hence, maximal product $(G_1 * G_2)$ is a regular fuzzy-graph structure. Conversely, suppose that maximal product $(G_1 * G_2)$ is a regular fuzzy-graph structure. Then, for any two vertices of $V_1 \times V_2$,

$$\begin{aligned} & d_{G_1 * G_2}(u_1, v_1) = d_{G_1 * G_2}(u_2, v_2) \\ \Rightarrow\ & d_{G_1}(u_1) + d_{G_2^*}(v_1)\sigma_1(u_1) = d_{G_1}(u_2) + d_{G_2^*}(v_2)\sigma_1(u_2) \\ \Rightarrow\ & d_{G_1}(u_1) + r_2 c = d_{G_1}(u_2) + r_2 c \\ \Rightarrow\ & d_{G_1}(u_1) = d_{G_1}(u_2). \end{aligned}$$

This holds for all vertices of G_1. Thus, G_1 is a regular fuzzy-graph structure. □

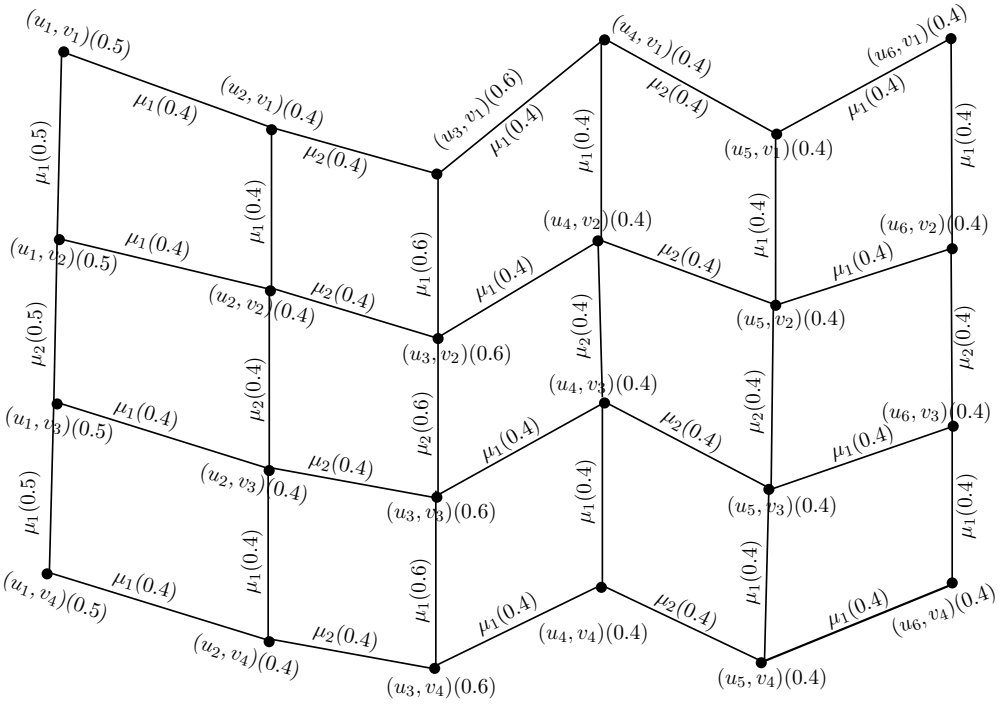


Figure 14. Maximal Product of G_1 and G_2.

Theorem 14. *If $G_1 = (\sigma_1, \mu'_1, \mu'_2, \ldots, \mu'_n)$ and $G_2 = (\sigma_2, \mu''_1, \mu''_2, \ldots, \mu''_n)$ are two partially regular fuzzy-graph structures such that $\mu''_i \leq \sigma_1$, $\mu''_i \leq \sigma_2$, $i = 1, 2, \ldots, n$ and σ_2 is a constant function of value 'c', then maximal product $G_1 * G_2$ is regular if and only if σ_1 is a constant function.*

Proof. Let $G_1 = (\sigma_1, \mu'_1, \mu'_2, \ldots, \mu'_n)$ and $G_2 = (\sigma_2, \mu''_1, \mu''_2, \ldots, \mu''_n)$ be partially regular fuzzy-graph structures such that $\mu''_i \leq \sigma_1$, $\mu''_i \leq \sigma_2$, $i = 1, 2, \ldots, n$ and σ_2 is a constant function of value 'c'. Moreover, G_1^* is r_1-regular and G_2^* is r_2-regular. Furthermore, suppose that σ_1 is a constant function of value k. Then, $d_{G_1 * G_2}(u_i, v_j) = d_{G_2^*}(v_j)\sigma_1(u_i) + d_{G_1^*}(u_i)\sigma_2(v_j) = r_2 k + r_1 c$. This holds for all vertices of $V_1 \times V_2$. Hence, maximal product $(G_1 * G_2)$ is a regular fuzzy-graph structure. Conversely, suppose that maximal product $(G_1 * G_2)$ is a regular fuzzy-graph structure. Then, for any two vertices of $V_1 \times V_2$,

$$
\begin{aligned}
& d_{G_1 * G_2}(u_1, v_1) = d_{G_1 * G_2}(u_2, v_2) \\
\Rightarrow\ & d_{G_2^*}(v_1)\sigma_1(u_1) + d_{G_1^*}(u_1)\sigma_2(v_1) = d_{G_2^*}(v_2)\sigma_1(u_2) + d_{G_1^*}(u_2)\sigma_2(v_2) \\
\Rightarrow\ & r_2\sigma_1(u_1) + r_1 c = r_2\sigma_1(u_2) + r_1 c \\
\Rightarrow\ & \sigma_1(u_1) = \sigma_1(u_2).
\end{aligned}
$$

This holds for all vertices of G_1. Thus, σ_1 is a constant function. □

Remark 4. *$G_1 * G_2$ (maximal product) of two full regular fuzzy-graph structures G_1 and G_2 is not always full regular. Moreover, if G_1 and G_2 are two regular fuzzy-graph structures on complete graph structures G_1^* and G_2^*, respectively, then $G_1 * G_2$ (maximal product) is a partially regular fuzzy-graph structure on a complete underlying graph structure $G_1^* * G_2^*$.*

3. Application

Identification of most controversial issues among countries: Nowadays , the world has become a global village, due to which all countries are related to each other. Relationships among different countries are not of the same nature. Some countries have good relationships with each other; for example, Pakistan and China have a very good relationship for many years. However, some countries do not have good relationships due to which global peace is at stake. The reasons for bad relationships are some controversial issues between those countries. There are many controversial issues that are permanently dangerous to global peace, including line of control issues, counterterrorism activities, proliferation of nuclear weapons, power struggles, religious issues, and occupations of other countries.

Different countries have many issues with each other, but, in particular time period, one issue is the most controversial that needs the attention of peace-loving organizations to be solved, so that warfare activities among those countries can be contained. For example, India and Pakistan have many issues with each other, including the issue of Kashmir, water issues, religious issues, terrorism, power struggles, and line of control issues. In different time periods, the India–Pakistan relationship has been disturbed due to different issues. Nowadays, the most controversial issue between Pakistan and India is the line of control issue, which needs to be promptly solved.

We can use a fuzzy-graph structure to highlight the most controversial issue between any two countries in a particular time period, and can also tell the severity level of the issue at that time with the help of the membership function. The fuzzy-graph structure of the most controversial issues can be very helpful for peace-keeping organizations and the United Nations to maintain global peace. Consider a set S of eight powerful countries:

$$S = \{NorthKorea, America, SouthKorea, Pakistan, Iran, Russia, India, Afghanistan\}.$$

Let σ be a fuzzy set on S, defined in Table 1.

Table 1. Fuzzy set σ of countries.

Country	Degree of Membership
North Korea	0.8
America	0.9
South Korea	0.7
Pakistan	0.8
Iran	0.7
Russia	0.9
India	0.8
Afghanistan	0.5

In Table 1, we describe the degrees of membership in a set of powerful countries. In Tables 2–10, we have mentioned the membership values of controversial issues between each pair of countries. Membership value of each pair of countries is according to $\mu(v_1v_2) \leq \sigma(v_1) \wedge \sigma(v_2)$, for all $v_1, v_2 \in S$. By using these membership values, we show the severity level of each controversial issue between each pair of countries.

Table 2. Fuzzy set of controversial issues between America and other countries.

Type of Controversial Issues	(America, North Korea)	(America, South Korea)	(America, Pakistan)	(America, Iran)
Proliferation of nuclear weapons	0.8	0.5	0.7	0.7
Counter terrorism activities	0.6	0.4	0.8	0.6
Line of control issues	0.1	0.1	0.3	0.2
To be more powerful	0.7	0.2	0.4	0.6
Religious issues	0.5	0.3	0.6	0.5
To occupy other country	0.3	0.5	0.6	0.5

Table 3. Fuzzy set of controversial issues between Russia and other countries.

Type of Controversial Issues	(Russia, North Korea)	(Russia, India)	(Russia, Afghanistan)	(Russia, America)
Proliferation of nuclear weapons	0.8	0.5	0.2	0.3
Counter terrorism activities	0.6	0.6	0.2	0.4
Line of control issues	0.2	0.6	0.1	0.5
To be more powerful	0.6	0.5	0.2	0.9
Religious issues	0.5	0.5	0.4	0.3
To occupy other country	0.6	0.5	0.5	0.4

Table 4. Fuzzy set of controversial issues between North Korea and other countries.

Type of Controversial Issues	(North Korea, America)	(North Korea, South Korea)	(North Korea, Pakistan)
Proliferation of nuclear weapons	0.8	0.5	0.8
Counter terrorism activities	0.6	0.2	0.4
Line of control issues	0.1	0.6	0.3
To be more powerful	0.7	0.7	0.4
Religious issues	0.5	0.5	0.6
To occupy other country	0.3	0.6	0.4

Table 5. Fuzzy set of controversial issues between India and other countries.

Type of Controversial Issues	(India, North Korea)	(India, America)	(India, South Korea)	(India, Russia)
Proliferation of nuclear weapons	0.8	0.2	0.5	0.7
Counter terrorism activities	0.6	0.3	0.4	0.6
Line of control issues	0.1	0.2	0.3	0.4
To be more powerful	0.6	0.8	0.6	0.6
Religious issues	0.2	0.5	0.3	0.5
To occupy other country	0.3	0.3	0.4	0.4

Table 6. Fuzzy set of controversial issues between Pakistan and other countries.

Type of Controversial Issues	(Pakistan, Iran)	(Pakistan, Russia)	(Pakistan, India)	(Pakistan, Afghanistan)
Proliferation of nuclear weapons	0.3	0.8	0.5	0.3
Counter terrorism activities	0.2	0.6	0.4	0.2
Line of control issues	0.5	0.5	0.8	0.5
To be more powerful	0.6	0.4	0.7	0.1
Religious issues	0.7	0.5	0.6	0.1
To occupy other country	0.3	0.3	0.6	0.1

Table 7. Fuzzy set of controversial issues between Afghanistan and other countries.

Type of Controversial Issues	(Afghanistan, South Korea)	(Afghanistan, Pakistan)	(Afghanistan, America)
Proliferation of nuclear weapons	0.1	0.1	0.1
Counter terrorism activities	0.1	0.2	0.1
Line of control issues	0.2	0.5	0.0
To be more powerful	0.1	0.2	0.0
Religious issues	0.5	0.2	0.4
To occupy other country	0.0	0.1	0.5

Table 8. Fuzzy set of controversial issues between Iran and other countries.

Type of Controversial Issues	(Iran, Russia)	(Iran, India)	(Iran, Afghanistan)	(Iran, North Korea)
Proliferation of nuclear weapons	0.7	0.6	0.4	0.1
Counter terrorism activities	0.2	0.1	0.0	0.1
Line of control issues	0.1	0.4	0.2	0.0
To be more powerful	0.1	0.1	0.0	0.0
Religious issues	0.2	0.7	0.3	0.5
To occupy other country	0.0	0.2	0.0	0.0

Table 9. Fuzzy set of controversial issues between South Korea and other countries.

Type of Controversial Issues	(South Korea, Iran)	(South Korea, Russia)	(South Korea, North Korea)
Proliferation of nuclear weapons	0.1	0.2	0.6
Counter terrorism activities	0.0	0.3	0.4
Line of control issues	0.0	0.2	0.6
To be more powerful	0.1	0.2	0.7
Religious issues	0.5	0.1	0.4
To occupy other country	0.3	0.5	0.6

Table 10. Fuzzy set of controversial issues between some countries.

Type of Controversial Issues	(North Korea, Iran)	(Afghanistan, India)	(South Korea, Pakistan)
Proliferation of nuclear weapons	0.1	0.1	0.5
Counter terrorism activities	0.0	0.2	0.4
Line of control issues	0.0	0.5	0.0
To be more powerful	0.4	0.1	0.4
Religious issues	0.5	0.3	0.6
To occupy other country	0.3	0.0	0.2

On set S, many relations can be defined. Let us define following relations on S: R_1 = Proliferation of nuclear weapons, R_2 = Counter terrorism activities, R_3 = Line of control issues, R_4 = To be more powerful , R_5 = Religious issues, R_6 = To occupy other country, such that $(S, R_1, R_2, R_3, R_4, R_5, R_6)$ is a graph structure. Each element in the relation depicts a particular kind of most controversial issues among those two countries. As $(S, R_1, R_2, R_3, R_4, R_5, R_6)$ is the graph structure, therefore a pair of countries can appear in just one relation. Hence, it would be considered an element of that relation, for which its membership value is comparatively high than those of other relations. Using the given data above, elements in relations are paired with their membership values, resulting sets are the fuzzy sets on R_1, R_2, R_3, R_4, R_5, R_6, respectively. These fuzzy sets are μ_1, μ_2, μ_3, μ_4, μ_5, μ_6, respectively.

Let

$R_1 = \{(America, NorthKorea), (America, Iran),$
$(NorthKorea, Russia), (Pakistan, NorthKorea)\},$
$R_2=\{(Pakistan, America)\},$
$R_3=\{(Pakistan, India), (Afghanistan, India)\},$
$R_4=\{(America, Russia), (SouthKorea, NorthKorea)\},$
$R_5 = \{(Iran, India), (Afghanistan, SouthKorea)\},$
$R_6 = \{(America, Afghanistan)\}.$

And the corresponding fuzzy sets are:

$\mu_1=\{((America, NorthKorea), 0.8), ((America, Iran), 0.7),$
$((NorthKorea, Russia), 0.8), ((Pakistan, NorthKorea), 0.8)\},$
$\mu_2=\{((Pakistan, America), 0.8)\}$
$\mu_3=\{((Pakistan, India), 0.8), ((Afghanistan, India), 0.5)\},$
$\mu_4=\{(America, Russia), 0.9), ((SouthKorea, NorthKorea), 0.7)\},$
$\mu_5=\{((Iran, India), 0.7), ((Afghanistan, SouthKorea), 0.5)\},$
$\mu_6 = \{((America, Afghanistan), 0.5)\}.$

Clearly, $(\sigma, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6)$ is a fuzzy-graph structure and is shown in Figure 15.

In the FGS shown in Figure 15, each edge depicts the most controversial issue in the corresponding countries. For example: the most controversial issue between America and North Korea is the proliferation of nuclear weapons, and its membership value is 0.8. It can be noted that vertex America has the highest vertex degree for the relation proliferation of nuclear weapons. This means that America has the proliferation of nuclear weapons as a controversial issue with other countries. Moreover, according to this fuzzy-graph structure, Pakistan and India have the line of control issue as the most controversial issue at this time with membership value 0.8. A FGS of all countries can be very helpful for United Nations and other organizations to maintain global peace. It would highlight those controversial issues that needed to be promptly solved.

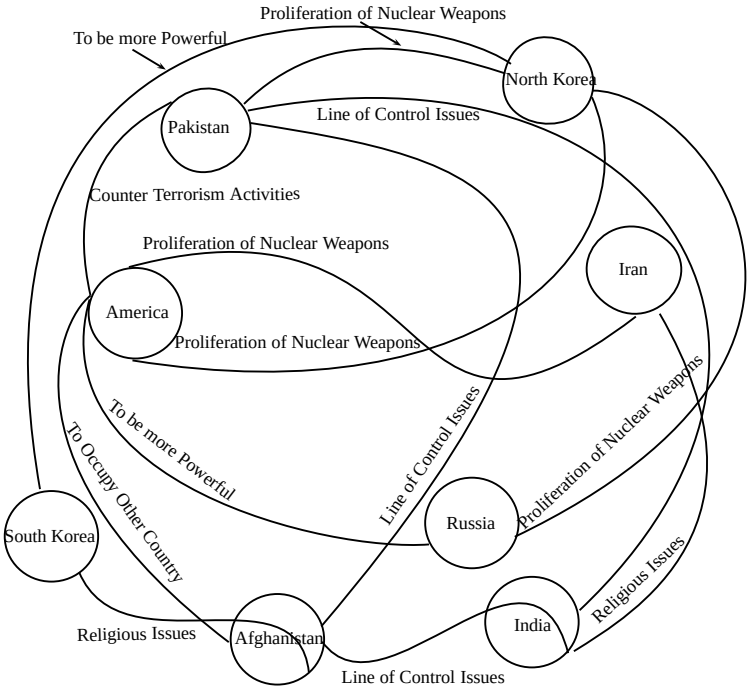


Figure 15. Fuzzy-graph structure depicting most controversial issues among different countries.

The general procedure used in this application is shown in the following flowchart (Figure 16).

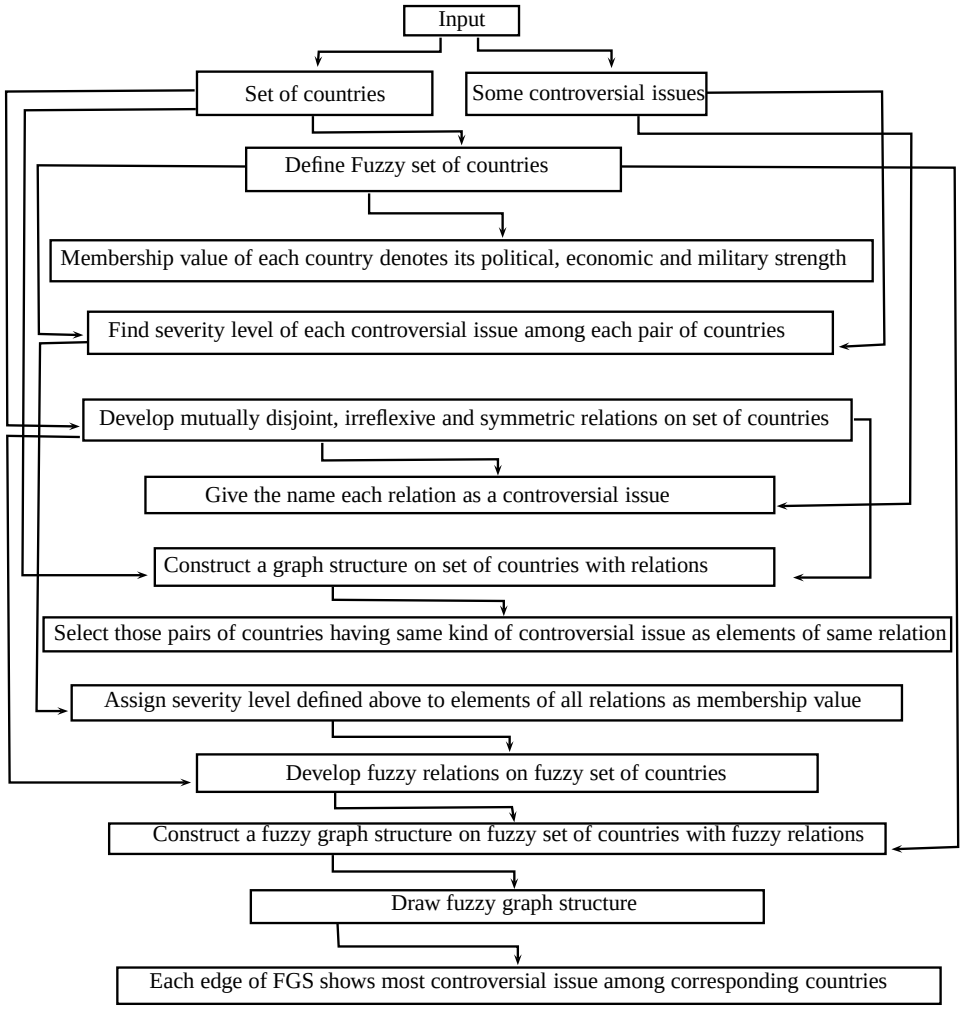


Figure 16. Flowchart.

4. Conclusions

Graph theory has many applications in solving various problems of several domains, including networking, communication, data mining, clustering, image capturing, image segmentation, planning, and scheduling. However, in some situations, certain aspects of a graph-theoretical system may be uncertain. Use of fuzzy-graphical methods in dealing with ambiguity and vague notions is very natural. Fuzzy-graph theory has a large number of applications in modeling various real-time systems where the level of information inherent in the system varies with different levels of precision. In this article, we have presented a new framework to handle fuzzy information by combining fuzzy sets with graph structures. We have introduced many basic notions concerning fuzzy-graph structures including maximal product of two fuzzy-graph structures and regular fuzzy-graph structures, and investigated a few related properties. We have also defined the degree and total degree of vertex in the maximal product of fuzzy-graph structures, and discussed some properties with examples. Furthermore, we have considered an application of fuzzy-graph structures. In particular, a flowchart was developed to show the general procedure of the application regarding the identification of the most controversial issues among countries. In the future, we will extend our fuzzification to the maximal product of (1) soft fuzzy-graph structures; (2) fuzzy soft graph structures; (3) rough fuzzy soft graph structures; and (4) rough fuzzy graph structures.

Author Contributions: M.S., M.A. and M.Y.B. conceived of the presented idea. M.S. and M.Y.B. developed the theory and performed the computations. M.A verified the analytical methods.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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