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# Extending the Applicability of Two-Step Solvers for Solving Equations

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**Abstract:** We present a local convergence of two-step solvers for solving nonlinear operator equations under the generalized Lipschitz conditions for the first- and second-order derivatives and for the first order divided differences. In contrast to earlier works, we use our new idea of center average Lipschitz conditions, through which, we define a subset of the original domain that also contains the iterates. Then, the remaining average Lipschitz conditions are at least as tight as the corresponding ones in earlier works. This way, we obtain weaker sufficient convergence criteria, larger radius of convergence, tighter error estimates, and better information on the solution. These extensions require the same effort, since the new Lipschitz functions are special cases of the ones in earlier works. Finally, we give a numerical example that confirms the theoretical results, and compares favorably to the results from previous works.

**Keywords:** nonlinear equation; iterative process; the convergence order; radius of convergence; non-differentiable operator; generalized Lipschitz condition

## 1. Introduction

Let  $X, Y$  stand for Banach spaces,  $D \subset X$  denote an open and convex set. Consider the nonlinear equation

$$H(x) \equiv F(x) + G(x) = 0 \quad (1)$$

where  $F : D \rightarrow Y$ ,  $G : D \rightarrow Y$ ;  $F$  is a differentiable operator in the sense of Fréchet, and  $G$  is a continuous operator.

The difference one-step methods (Secant method, Kurchatov method) [1–7] and two-step method [8–10]:

$$\begin{aligned} x_{k+1} &= x_k - [H(x_k, y_k)]^{-1} H(x_k) \\ y_{k+1} &= x_{k+1} - [H(x_k, y_k)]^{-1} H(x_{k+1}), \quad k = 0, 1, 2, \dots, \end{aligned} \quad (2)$$

where  $H(x_k, y_k)$  is the first-order divided difference, which can be applied for solving nonlinear equations with nondifferentiable operator (Equation (1)). However, it is desirable to build iterative methods that take into account properties of the problem (Equation (1)). In particular, we can use only derivative  $F'(x_k)$  of differentiable part of operator instead of full derivative  $H'(x_k)$ , which in fact, does not exist. The method [11,12]:

$$x_{k+1} = x_k - F'(x_k)^{-1}(F(x_k) + G(x_k)), k = 0, 1, 2, \dots \quad (3)$$

obtained using this approach converges slowly. More efficient are methods that use sum of the derivatives of the differentiable part and divided difference of the nondifferentiable part of the operator instead of the full derivative  $H'(x_k)$ . In particular, the one-step Newton-Secant method [13–15]:

$$x_{k+1} = x_k - (F'(x_k) + G(x_k, x_{k-1}))^{-1}(F(x_k) + G(x_k)), \quad k = 0, 1, 2, \dots \tag{4}$$

and Newton-Kurchatov method [16,17]:

$$x_{k+1} = x_k - (F'(x_k) + G(2x_k - x_{k-1}, x_{k-1}))^{-1}(F(x_k) + G(x_k)), \quad k = 0, 1, 2, \dots \tag{5}$$

The method (3) has linear convergence order, Newton-Secant method (4) has order of convergence not higher than  $(1 + \sqrt{5})/2$  and Newton-Kurchatov method (5) has a quadratic convergence order.

In works [18,19], we first proposed two-step solvers for solving Equation (1) defined as:

$$\begin{aligned} x_{k+1} &= x_k - \left[ F' \left( \frac{x_k + y_k}{2} \right) + G(x_k, y_k) \right]^{-1} (F(x_k) + G(x_k)) \\ y_{k+1} &= x_{k+1} - \left[ F' \left( \frac{x_k + y_k}{2} \right) + G(x_k, y_k) \right]^{-1} (F(x_{k+1}) + G(x_{k+1})), \quad k = 0, 1, 2, \dots, \end{aligned} \tag{6}$$

where  $x_0, y_0$  are given.

Just like Newton-Secant method, which was studied by many authors, the new method requires at each iteration the computation of one operator  $A_k$  and its inverse. The operator  $A_k = F' \left( \frac{x_k + y_k}{2} \right) + G(x_k, y_k)$  consists of a combination of a Fréchet derivative of one part of the operator (the differentiable part) and the divided difference of the second part (which is, generally speaking, nondifferentiable). The number of computations of the function value increases by one at each step. Therefore, the number of computations at one iteration is almost identical in both methods. However, the convergence order of the new method is higher ( $1 + \sqrt{2} \approx 2.41$  vs.  $\approx 1.62$ ).

The investigations of method (6) have been conducted under Hölder’s conditions for the second order derivatives from  $F$  and usual Lipschitz conditions for the first order divided differences of  $G$  [18] of Equation (1) and under generalized Lipschitz conditions for the second order derivatives and divided differences of the first order [19].

There is a plethora of problems from computational sciences, that through mathematical modelling, can look like Equation (1) with a nonzero  $G$ . As an example, boundary value problems of first and second order appear in all branches of experimental sciences, e.g., Newton’s second law, in population dynamic modes, evaluating concentrations of different reagents during a reaction, etc. [20–22]. Let us consider nonlinear integral equation of Hammerstein-type defined as:

$$x(s) = v(s) + \int_0^t Q(s, t)(\lambda x(t)^{1+p} + \mu x(t)^q) dt$$

where  $v(s) \in C[0, 1], s \in [a, b], p, q \in [0, 1]$ , and  $Q$  is a continuous kernel. Clearly, this equation can be written in two parts with the second part involving the form  $x(t)^q$  and related to a nontrivial  $G$ . In particular, we can define  $F : C[0, 1] \rightarrow C[0, 1]$  and  $G : C[0, 1] \rightarrow C[0, 1]$  as:

$$\begin{aligned} F(x)(s) &= x(s) - v(s) - \lambda \int_a^b Q(s, t)x(t)^{1+p} dt, \\ G(x)(s) &= -\mu \int_a^b Q(s, t)x(t)^q dt, \quad \mu \neq 0 \end{aligned}$$

and

$$H(x)(s) = F(x)(s) + G(x)(s)$$

The discretized form of equation  $H(x)(s) = 0$  can also be written in a similar form where the discretized  $G$  is also nontrivial and nondifferentiable [22]. Therefore, there is a strong motivation for studying two-step solver (6) for a nonzero  $G$ .

We also like to bring to the attention of the motivated reader the works in References [23–25], which constitute subdivision-based solvers seeking all roots in the domain of interest using Newton’s method provided the domain is sufficiently small. These solvers can be used to include a non-differentiable part. Moreover, in References [26,27], the continuation method has also been used as another powerful tool for finding roots. However, these methods involve systems of equations on  $\mathbf{R}^i$  and cannot provide information about solutions of Equation (1) in a Banach space setting. In practice, one may use our results for the original equation, and the aforementioned references for the discretized versions, which are systems in  $\mathbf{R}^i$ .

In this work we continue to study a combined method (6) for solving nonlinear Equation (1). Using more precise estimates on the distances involved, under weaker hypotheses, and under the same computational cost, we provide an analysis of method (6) with the following advantages over the corresponding results in Reference [19]: larger convergence domain, finer error estimates on the distances involved, and an at least as precise information on the location of the solution.

The rest of the paper is structured as follows: Section 2 contains the definitions and auxiliary lemmas, in Sections 3 and 4, we present the local convergence analysis of method (6) and the uniqueness ball for solution of equation, respectively. In Section 5, we provide the numerical example. The article ends with some conclusions.

## 2. Definitions and Auxiliary Lemmas

**Definition 1.** Let  $G : D \rightarrow Y$  be a nonlinear operator and  $x, y \in D$ . If a linear operator  $G(x, y)$  satisfies the condition:

$$G(x, y)(x - y) = G(x) - G(y) \tag{7}$$

then it is called a divided difference of  $G$  at the points  $x$  and  $y$ .

Define  $B(x_0, r) = \{x : \|x - x_0\| < r\}$  and denote by  $\bar{B}(x_0, r)$  its closure.

**Lemma 1 [28].** Let  $\varphi(t) = \frac{1}{t} \int_0^t \lambda(u)du$ ,  $0 \leq t \leq r$ , where  $\lambda(u)$  is a positive integrable function and nondecreasing on  $[0, r]$ . Then  $\varphi(t)$  is nondecreasing monotonically.

**Lemma 2 [19].** Let  $\psi(t) = \frac{1}{t^3} \int_0^t \mu(u)(t - u)^2 du$ ,  $0 \leq t \leq r$ , where  $\mu(u)$  is a positive integrable function and nondecreasing monotonically on  $[0, r]$ . Then  $\psi(t)$  is nondecreasing monotonically.

**Lemma 3 [28].** Let  $p(t) = \frac{1}{t} \int_0^t (t - u)M(u)du$ ,  $0 \leq t \leq r$ , where  $M$  is a positive integrable function and a nondecreasing monotonically function on  $[0, r]$ . Then  $p(t)$  is increasing monotonically.

## 3. Convergence

Let  $B(x^*, r) \subseteq D$ , and  $r_0, r_1$  be the solutions of the systems of equations:

$$\begin{aligned} J\tau_0^2 + C\tau_1 &= 1 \\ [J\tau_0^2 + E(2\tau_0 + \tau_1)]\tau_0 &= \tau_1 \end{aligned} \tag{8}$$

where parameters  $J, r, E$ , and  $C$  are given in Theorem 1.

**Definition 2.** We say that the divided difference  $G(\cdot, \cdot)$  satisfies the center-Lipschitz condition on  $\Omega$  with  $M_0$  average if:

$$\|H'(x_*)^{-1}(G(x, y) - G'(x_*))\| \leq \int_0^{\|x-x_*\| + \|y-x_*\|} M_0(z) dz \tag{9}$$

**Definition 3.** We say that the operator  $F'$  satisfies the center-Lipschitz condition on  $\Omega$  with  $L_0$  average if:

$$\|H'(x_*)^{-1}(F'(x) - F'(x_*))\| \leq \int_0^{\rho(x)} L_0(u) du \text{ for each } x \in \Omega. \tag{10}$$

for each  $x, y \in \Omega \in S(x_*, \frac{R}{2}) \subset \Omega$ , and some  $R > 0$ .

Define function  $\zeta$  on the interval  $[0, \infty)$  as:

$$\zeta(t) = \int_0^t L_0(u) du + \int_0^{2t} M_0(z) dz \tag{11}$$

Suppose that the equation

$$\zeta(t) = 1 \tag{12}$$

has positive solutions. Denote by  $s$  the smallest such solution. Set  $\Omega_0 = \Omega \cap S(x_*, s)$ .

**Definition 4.** We say that the operator  $F'$  satisfies the restricted Lipschitz condition on  $\Omega_0$  with  $L$  average if:

$$\|H'(x_*)^{-1}(F'(y) - F'(x))\| \leq \int_0^{\|y-x\|} L(u) du \text{ for each } x, y \in \Omega_0 \tag{13}$$

**Definition 5.** We say that the divided difference  $G(\cdot, \cdot)$  satisfies the restricted Lipschitz condition on  $\Omega_0 \times \Omega_0$  with  $M$  average if:

$$\|H'(x_*)^{-1}(G(x, y) - G(u, v))\| \leq \int_0^{\|x-u\| + \|y-v\|} M(z) dz \tag{14}$$

for each  $x, y, u, v \in \Omega \in S(x_*, \frac{R}{4}) \subset \Omega$  and some  $R > 0$ .

**Definition 6.** We say that the operator  $F''$  satisfies the restricted Lipschitz condition on  $\Omega_0$  with  $N$  average if:

$$\|H'(x_*)^{-1}(F''(y) - F''(x))\| \leq \int_0^{\|y-x\|} N(u) du \text{ for each } x, y \in \Omega_0 \tag{15}$$

**Definition 7 [19].** We say that the operator  $F'$  satisfies the Lipschitz condition on  $\Omega$  with  $L_1$  average if:

$$\|H'(x_*)^{-1}(F'(y) - F'(x))\| \leq \int_0^{\|y-x\|} L_1(u) du \text{ for each } x, y \in \Omega \tag{16}$$

**Definition 8** [19]. We say that the divided difference  $G(\cdot, \cdot)$  satisfies the Lipschitz condition on  $\Omega \times \Omega$  with  $M_1$  average if:

$$\|H'(x_*)^{-1}(G(x, y) - G(u, v))\| \leq \int_0^{\|x-u\| + \|y-v\|} M_1(z) dz \tag{17}$$

for each  $x, y, u, v \in \Omega \in S(x_*, \frac{R}{4}) \subset \Omega$  and some  $R > 0$ .

**Definition 9.** We say that the operator  $F''$  satisfies the Lipschitz condition on  $\Omega$  with  $N_1$  average if:

$$\|H'(x_*)^{-1}(F''(y) - F''(x))\| \leq \int_0^{\|y-x\|} N_1(u) du \text{ for each } x, y \in \Omega. \tag{18}$$

**Remark 1.** We have that

$$\Omega_0 \subseteq \Omega. \tag{19}$$

Hence, it follows from these definitions that for each  $t \in [0, \lambda]$  for some  $\lambda > 0$

$$L_0(t) \leq L_1(t) \tag{20}$$

$$L(t) \leq L_1(t) \tag{21}$$

and

$$M(t) \leq M_1(t) \tag{22}$$

Then,  $(L_0, L)$ ,  $M$  can replace  $L_1, M_1$  in Reference [19] to obtain improved results as already noted in the introduction of this study.

The radius of the convergence ball and the order of convergence of the method (6) are defined by the following theorem.

**Theorem 1.** Let  $H(x) \equiv F(x) + G(x)$  be a nonlinear operator, defined in an open convex subset  $D$  of space  $X$  with values in the space  $Y$ . Suppose that:

- (i) Let  $H(x^*) \equiv 0$  for some  $x^*$  for which  $H(x^*)^{-1}$  exists;
- (ii) Conditions (9), (10), (13), and (15) are satisfied,  $s$  exists and is given in Reference [12].
- (iii)  $r > 0$  satisfies the inequality

$$\frac{\frac{1}{8} \int_0^r N(u)(r-u)^2 du + r \int_0^{3r/2} L(u) du + r \int_0^{3r} M(u) du}{r \left( 1 - \int_0^r L_0(u) du - \int_0^{2r} M_0(u) du \right)} \leq 1 \tag{23}$$

Then, for all  $x_0 \in B(x^*, r_0)$  and  $y_0 \in B(x^*, r_1)$  sequences  $\{x_k\}_{k \in \mathbb{N}}$  and  $\{y_k\}_{k \in \mathbb{N}}$ , as defined according to formulas (6), converge to the solution  $x^*$ ,  $x_k \in B(x^*, r_0)$  and  $y_k \in B(x^*, r_1)$  for  $k = 0, 1, 2, \dots$  and the estimations are being fulfilled:

$$\begin{aligned} \rho(x_{k+1}) &= \|x_{k+1} - x^*\| \leq J\rho(x_k)^3 + C\rho(x_k)\rho(y_k), k = 0, 1, 2, \dots, \\ \rho(y_{k+1}) &= \|y_{k+1} - x^*\| \leq J\rho(x_{k+1})^3 + E(\rho(x_k) + \rho(x_{k+1}) + \rho(y_k))\rho(x_{k+1}), k = 0, 1, 2, \dots, \end{aligned} \tag{24}$$

where

$$\begin{aligned}
 J &= \frac{Q_0}{8\rho(x_0)^3} \int_0^{\rho(x_0)} N(u)(\rho(x_0) - u)^2 du; E = \frac{Q_0}{z_0} \left( \int_0^{z_0/2} L(u)du + \int_0^{z_0} M(u)du \right); \\
 C &= \frac{Q_0}{\rho(y_0)} \left( \int_0^{\rho(y_0)/2} L(u)du + \int_0^{\rho(y_0)} M(u)du \right) \\
 Q_0 &= \left( 1 - \int_0^{\rho(x_0+y_0)/2} L_0(u)du - \int_0^{\rho(x_0)+\rho(y_0)} M_0(u)du \right)^{-1} \\
 z_0 &= \rho(x_0) + \rho(y_0) + \rho(x_1).
 \end{aligned}
 \tag{25}$$

Sequences  $\{x_k\}_{k \in \mathbb{N}}$  and  $\{y_k\}_{k \in \mathbb{N}}$  converge to  $x^*$  with order  $1 + \sqrt{2}$ .

**Proof.** Pick  $x_0 \in B(x^*, r_0)$  and  $y_0 \in B(x^*, r_1)$  arbitrarily.

By Lemmas 1 and 2 we get, that  $\frac{1}{t} \int_0^t L_0(u)du$ ,  $\frac{1}{t} \int_0^t L(u)du$ ,  $\frac{1}{t} \int_0^t M(u)du$ , and  $\frac{1}{t^3} \int_0^t N(u)(t - u)^2 du$  are nondecreasing with respect to  $t$ .

Set  $A = A(x, y)$  the linear operator  $A = F'\left(\frac{x+y}{2}\right) + G(x, y)$  with  $x, y \in B(x, r)$ . Then, by Inequalities (9) and (10), we get:

$$\begin{aligned}
 \|I - H'(x^*)^{-1}A\| &= \|H'(x^*)^{-1}(H'(x^*) - A)\| \\
 &= \|H'(x^*)^{-1}\left[F'(x^*) + G(x^*, x^*) - F'\left(\frac{x+y}{2}\right) - G(x, y)\right]\| \\
 &\leq \int_0^{\rho\left(\frac{x+y}{2}\right)} L_0(u)du + \int_0^{\rho(x)+\rho(y)} M_0(u)du
 \end{aligned}$$

It follows that from the definition of  $r$

$$\int_0^r L_0(u)du + \int_0^{2r} M_0(u)du = 1 - \frac{1}{8r} \int_0^r N(u)(r - u)^2 du - \int_0^{3r/2} L(u)du - \int_0^{3r} M(u)du < 1.$$

In view of the identity

$$\|A^{-1}H'(x^*)\| = \|I - \left(I - [H'(x^*)^{-1}A]\right)^{-1}\|$$

and the Banach perturbation lemma [13],  $A^{-1}$  exists, and:

$$\|A^{-1}H'(x^*)\| \leq \left( 1 - \int_0^{\rho\left(\frac{x+y}{2}\right)} L_0(u)du - \int_0^{\rho(x)+\rho(y)} M_0(u)du \right)^{-1}$$

Let us now suppose that  $x_k \in B(x^*, r_1)$ . Set  $A_k = A(x_k, y_k)$ . Then  $A_k$  is invertible and, according to (6), we can write:

$$\begin{aligned}
 x_{k+1} - x^* &= x_k - x^* - A_k^{-1}(F(x_k) + G(x_k) - F(x^*) - G(x^*)) \\
 &= -A_k^{-1}\left[F(x_k) + G(x_k) - F(x^*) - G(x^*) - \left(F'\left(\frac{x_k+y_k}{2}\right) + G(x_k, y_k)\right)(x_k - x^*)\right] \\
 &= A_k^{-1}H'(x^*)\left\{H'(x^*)^{-1}\left[F'\left(\frac{x_k+y_k}{2}\right)(x_k - x^*) - F(x_k) + F(x^*)\right] \right. \\
 &\quad \left. + H'(x^*)^{-1}[G(x_k, x^*) - G(x_k, y_k)](x_k - x^*)\right\} \\
 &= A_k^{-1}H'(x^*)\left\{H'(x^*)^{-1}\left[F'\left(\frac{x_k+x^*}{2}\right)(x_k - x^*) - F(x_k) + F(x^*)\right] \right. \\
 &\quad \left. + H'(x^*)^{-1}\left[F'\left(\frac{x_k+y_k}{2}\right) - F'\left(\frac{x_k+x^*}{2}\right)\right](x_k - x^*) \right. \\
 &\quad \left. + H'(x^*)^{-1}[G(x_k, x^*) - G(x_k, y_k)](x_k - x^*)\right\}.
 \end{aligned}
 \tag{26}$$

Let us write down the identity [29] (p. 336) (lemma 1) for the value  $\omega = \frac{1}{2}$ :

$$\begin{aligned}
 & F(x) - F(y) - F'\left(\frac{x+y}{2}\right)(x-y) \\
 &= \frac{1}{4} \int_0^1 (1-t) \left[ F''\left(\frac{x+y}{2} + \frac{t}{2}(x-y)\right) - F''\left(\frac{x+y}{2} + \frac{t}{2}(y-x)\right) \right] (x-y)(x-y) dt.
 \end{aligned}$$

Letting in this equality  $x = x^*$ ,  $y = x_k$ , we have in turn that:

$$\begin{aligned}
 & \|H'(x^*)^{-1} [F(x^*) - F(x_k) - F'\left(\frac{x_k+x^*}{2}\right)(x^* - x_k)]\| \\
 &= \frac{1}{4} \left\| \int_0^1 (1-t) H'(x^*)^{-1} \left[ F''\left(\frac{x_k+x^*}{2} + \frac{t}{2}(x^* - x_k)\right) - F''\left(\frac{x_k+x^*}{2} + \frac{t}{2}(x_k - x^*)\right) \right] \right. \\
 &\quad \times (x^* - x_k)(x^* - x_k) dt \left. \right\| \leq \frac{1}{4} \int_0^1 (1-t) \int_0^{t\|x_k-x^*\|} N(u) du \|x_k - x^*\|^2 dt \\
 &= \frac{1}{8} \int_0^{\|x_k-x^*\|} \left(1 - \frac{u}{\|x_k-x^*\|}\right) N(u) \left(1 - \frac{u}{\|x_k-x^*\|}\right) du \|x_k - x^*\|^2 \\
 &= \frac{1}{8} \int_0^{\rho(x_k)} N(u) (\rho(x_k) - u)^2 du
 \end{aligned}$$

Moreover, we have

$$\begin{aligned}
 & \|H'(x^*)^{-1} \left[ F'\left(\frac{x_k+y_k}{2}\right) - F'\left(\frac{x_k+x^*}{2}\right) \right]\| \leq \int_0^{\rho(y_k)/2} L(u) du \\
 & \|H'(x^*)^{-1} [G(x_k, x^*) - G(x_k, y_k)](x_k - x^*)\| \leq \int_0^{\rho(y_k)} M(u) du \|x_k - x^*\|
 \end{aligned}$$

Then, according to Lemmas 1–3, taking into account the latter inequalities, we obtain:

$$\begin{aligned}
 & \|x_{k+1} - x^*\| \leq \|A_k^{-1} H'(x^*)\| \left\{ \|H'(x^*)^{-1} \left[ F'\left(\frac{x_k+x^*}{2}\right)(x_k - x^*) - F(x_k) + F(x^*) \right]\| \right. \\
 & \quad + \|H'(x^*)^{-1} \left[ F'\left(\frac{x_k+y_k}{2}\right) - F'\left(\frac{x_k+x^*}{2}\right) \right](x_k - x^*)\| \\
 & \quad \left. + \|H'(x^*)^{-1} [G(x_k, x^*) - G(x_k, y_k)](x_k - x^*)\| \right\} \\
 & \leq \|A_k^{-1} H'(x^*)\| \left\{ \frac{1}{4} \int_0^1 (1-t) \|H'(x^*)^{-1} \left[ F''\left(\frac{x_k+x^*}{2} + \frac{t}{2}(x_k - x^*)\right) \right. \right. \\
 & \quad \left. \left. - F''\left(\frac{x_k+x^*}{2} + \frac{t}{2}(x^* - x_k)\right) \right] (x_k - x^*)(x_k - x^*) dt \right. \\
 & \quad + \|H'(x^*)^{-1} \left[ F'\left(\frac{x_k+y_k}{2}\right) - F'\left(\frac{x_k+x^*}{2}\right) \right](x_k - x^*)\| \\
 & \quad \left. + \|H'(x^*)^{-1} [G(x_k, x^*) - G(x_k, y_k)](x_k - x^*)\| \right\} \\
 & \leq \|A_k^{-1} H'(x^*)\| \left\{ \frac{1}{8} \int_0^{\rho(x_k)} N(u) (\rho(x_k) - u)^2 du \right. \\
 & \quad \left. + \int_0^{\rho(y_k)/2} L(u) du \rho(x_k) + \int_0^{\rho(y_k)} M(u) du \rho(x_k) \right\}.
 \end{aligned}$$

Therefore

$$\begin{aligned} \|x_{k+1} - x^*\| &\leq Q_k \left\{ \frac{1}{8\rho(x_k)^3} \int_0^{\rho(x_k)} N(u)(\rho(x_k) - u)^2 du \rho(x_k)^3 \right. \\ &\quad \left. + \frac{2}{\rho(y_k)} \int_0^{\rho(y_k)/2} L(u) du \rho(x_k)\rho(y_k)/2 + \frac{1}{\rho(y_k)} \int_0^{\rho(y_k)} M(u) du \rho(x_k)\rho(y_k) \right\} \\ &\leq Q_0 \left\{ \frac{1}{8\rho(x_0)^3} \int_0^{\rho(x_0)} N(u)(\rho(x_0) - u)^2 du \rho(x_0)^3 \right. \\ &\quad \left. + \frac{2}{\rho(y_0)} \int_0^{\rho(y_0)/2} L(u) du \rho(x_k)\rho(y_k)/2 + \frac{1}{\rho(y_0)} \int_0^{\rho(y_0)} M(u) du \rho(x_k)\rho(y_k) \right\} \\ &\leq J\rho(x_k)^3 + C\rho(x_k)\rho(y_k) < (Jr_0^2 + Cr_1)\rho(x_k) = \rho(x_k) < r_0 \end{aligned} \tag{27}$$

where  $Q_k = \left( 1 - \int_0^{\frac{\rho(x_k+y_k)}{2}} L_0(u) du - \int_0^{\rho(x_k)+\rho(y_k)} M_0(u) du \right)^{-1}$ .

Analogously [19]:

$$\begin{aligned} \|y_{k+1} - x^*\| &\leq Q_k \left\{ \frac{1}{8} \int_0^{\rho(x_{k+1})} N(u)(\rho(x_{k+1}) - u)^2 du + \int_0^{z_k/2} L(u) du \rho(x_{k+1}) + \int_0^{z_k} M(u) du \rho(x_{k+1}) \right\} \\ &\leq J\rho(x_{k+1})^3 + E(\rho(x_k) + \rho(y_k) + \rho(x_{k+1}))\rho(x_{k+1}) \\ &< [Jr_0^2 + E(2r_0 + r_1)]\rho(x_{k+1}) = \frac{r_1}{r_0}\rho(x_{k+1}) < r_1 \end{aligned} \tag{28}$$

where  $z_k = \rho(x_k) + \rho(y_k) + \rho(x_{k+1})$ .

By Inequality (27), sequence  $\{\|x_k - x^*\|\}$  is convergent monotonically to some  $a$ ,  $0 \leq a < r_0$ . From Inequality (27) comes  $a \leq Ja^3 + Car_1$ . We arrive at a contradiction for  $a \neq 0$ :

$$1 \leq Ja^2 + Cr_1 < Jr_0^2 + Cr_1 = 1$$

Hence, we get:  $\lim_{k \rightarrow \infty} x_k = x^* = \lim_{k \rightarrow \infty} y_k$ .

We set  $a_k = \rho(x_k)$ ,  $b_k = \rho(y_k)$ ,  $k = 0, 1, 2, \dots$ . From Inequalities (27) and (28), we get:

$$\begin{aligned} a_{k+1} &\leq Ja_k^3 + Ca_k b_k, \quad k = 0, 1, 2, \dots, \\ b_{k+1} &\leq a_{k+1} \min \left\{ \frac{r_1}{r_0}, Ja_{k+1}^2 + E(a_k + a_{k+1} + b_k) \right\} \\ &\leq a_{k+1} \min \left\{ \frac{r_1}{r_0}, (2E + Ja_k)a_k + Eb_k \right\} \\ &\leq a_{k+1} \min \left\{ \frac{r_1}{r_0}, (2E + Jr_0 + E\frac{r_1}{r_0})a_k \right\} \leq \left( Jr_0 + E \left( 2 + \frac{r_1}{r_0} \right) \right) a_{k+1} a_k \end{aligned} \tag{29}$$

$$\leq a_{k+1} \min \left\{ \frac{r_1}{r_0}, \left( 2E + Jr_0 + E\frac{r_1}{r_0} \right) a_k \right\} \leq \left( Jr_0 + E \left( 2 + \frac{r_1}{r_0} \right) \right) a_{k+1} a_k \tag{30}$$

From the Inequalities (29) and (30) for large enough  $k$  and some positive constant  $C_1$ , it follows that  $a_{k+1} \leq C_1 a_k^2$ , which leads to the  $\rho^2 - 2\rho - 1 = 0$ , with positive root  $\rho^* = 1 + \sqrt{2}$ .  $\square$

#### 4. The Uniqueness Ball for the Solution of Equations

The uniqueness ball for the solution is defined in the following theorem.

**Theorem 2.** Suppose that  $H(x^*) \equiv F(x^*) + G(x^*) = 0$ ,  $F'$  is continuous in  $B(x^*, r)$ ,  $H'(x^*)^{-1}$  exists,  $F'$  satisfies the center-Lipschitz condition with  $L_0$  average:

$$\|H'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq \int_0^{\|x-x^*\|} L_0(u)du, \forall x, y \in B(x^*, r) \tag{31}$$

the divided difference  $G(x, y)$  satisfies the center-Lipschitz condition with  $M_0$  average:

$$\|H'(x^*)^{-1}(G(x, y) - G(x_*, x_*))\| \leq \int_0^{\|x-x_*\| + \|y-x_*\|} M_0(u)du, \forall x, y \in B(x^*, r) \tag{32}$$

Suppose  $r$  satisfies

$$\frac{1}{r} \int_0^r (r-u)L_0(u)du + \int_0^r M_0(u)du \leq 1. \tag{33}$$

Then the equation  $H(x) = 0$  has a unique solution  $x^*$  in  $B(x^*, r)$ .

**Proof.** Pick  $x_0 \in B(x^*, r)$  arbitrarily. Then, by the iteration:

$$x_{k+1} = x_k - H'(x^*)^{-1}(F(x_k) + G(x_k)), k = 0, 1, 2, \dots, \tag{34}$$

we can write

$$\begin{aligned} x_1 - x^* &= x_0 - x^* - H'(x^*)^{-1}(F(x_0) + G(x_0)) \\ &= H'(x^*)^{-1}[H'(x^*)(x_0 - x^*) - F(x_0) + F(x^*) - G(x_0) + G(x^*)] = \\ &= H'(x^*)^{-1}\{F'(x^*)(x_0 - x^*) - F(x_0) + F(x^*) + G(x^*, x^*)(x_0 - x^*) - G(x_0) + G(x^*)\} \\ &= H'(x^*)^{-1}\{F'(x^*)(x_0 - x^*) - F(x_0) + F(x^*) + [G(x^*, x^*) - G(x_0, x^*)](x_0 - x^*)\} \end{aligned}$$

Let us estimate the rate of expression as  $H'(x^*)^{-1}\{F'(x^*)(x_0 - x^*) - F(x_0) + F(x^*)\}$ ,

$$\begin{aligned} &\|H'(x^*)^{-1}\{F'(x^*)(x_0 - x^*) - F(x_0) + F(x^*)\}\| \\ &\leq \left\| \int_0^1 H'(x^*)^{-1}\{F'(x^* + t(x_0 - x^*)) - F'(x^*)\}(x_0 - x^*)dt \right\| \\ &\leq \int_0^1 \int_0^{\|x_0-x^*\|} L_0(u)du \|x_0 - x^*\| dt \\ &\leq \int_0^{\|x_0-x^*\|} \left(1 - \frac{u}{\|x_0-x^*\|}\right) L_0(u)du \|x_0 - x^*\| \\ &= \int_0^{\rho(x_0)} (\rho(x_0) - u)L_0(u)du. \end{aligned}$$

Then:

$$\|x_1 - x^*\| \leq \left( \frac{1}{\rho(x_0)} \int_0^{\rho(x_0)} (\rho(x_0) - u)L_0(u)du + \int_0^{\rho(x_0)} M_0(u)du \right) \|x_0 - x^*\| = q_0 \|x_0 - x^*\|, \tag{35}$$

where:

$$\begin{aligned} q_0 &= \frac{1}{\rho(x_0)} \int_0^{\rho(x_0)} (\rho(x_0) - u)L_0(u)du + \int_0^{\rho(x_0)} M_0(u)du \\ &< \frac{1}{r} \int_0^r (r-u)L_0(u)du + \int_0^r M_0(u)du \leq 1. \end{aligned}$$

According to Inequality (35):

$$\|x_1 - x^*\| \leq q_0 \|x_0 - x^*\|.$$

Thus, the iteration (34) can be continued infinitely, and:

$$\|x_k - x^*\| \leq q_0^k \|x_0 - x^*\|, k = 1, 2, \dots$$

Therefore,  $\lim_{n \rightarrow \infty} x_k = x^*$ . However, if  $H(x_0) = 0$ , then from Equality (34),  $x_k = x_0$ . Therefore, from this follows  $x_0 = x^*$ .  $\square$

Let us denote that having set in Equation (1)  $G(x) \equiv 0$ , from Theorem 1 we obtain Theorem 1 from Reference [29], and from Theorems 2–4.1 with Reference [28] for Newton’s method and having set in Equation (1)  $F(x) \equiv 0$ , we obtain from Theorem 1 a corresponding Theorem 1 from Reference [10] for method (2), but with more precise and obvious estimations. With Lipschitz constants, we obtain the corresponding theorems from References [18,29–31].

**Remark 2.** If  $L_0 = L = L_1$ ,  $M_0 = M = M_1$ , and  $N = N_1$ , then our results reduce to the ones in Reference [19]. Otherwise, i.e., if the inequalities in (20) or (21) or (22) are strict, then the following advantages are available: larger convergence ball, tighter estimates on  $\|x_k - x^*\|$ , and better information regarding the location of the solution since all new constants and functions are more precise (see also the numerical example). Let us look for the corresponding Inequality (33) given in Reference [19], Inequality (23):

$$\frac{1}{r} \int_0^r (r - u)L_1(u)du + \int_0^r M_1(u)du \leq 1. \tag{36}$$

Then, it follows from Inequalities (33) and (36) that the new radius  $r$  is larger than the one in Reference [19] provided (20) or (22) hold as strict inequalities. The advantages are obtained under the same effort as in Reference [19]. This is because finding  $L_1, M_1, N_1$  requires finding  $L_0, L, M$ , and  $N$  as special cases.

**Remark 3.** In our earlier works [17,18], we reported the results of numerical experiments for solving Equation (1).

### 5. Numerical Examples

In first example we test the local convergence hypotheses and show that the new radius of convergence the is larger than the one in Reference [19]. The second example is used to show the superiority of our method (6) over other methods using the similar information.

**Example 1.** Let  $X = Y = \mathbf{R}^3$ ,  $\Omega = S(v_*, 1)$ ,  $v_* = (0, 0, 0)^T$ , and  $G = 0$  on  $\Omega$ .

Define  $F$  on  $\Omega$  by  $F(v) = (e^{v_1} - 1, \frac{e-1}{2}v_2^2 + v_2, v_3)^T$  for  $v = (v_1, v_2, v_3)^T$ . Then we have:

$$F'(v) = \begin{pmatrix} e^{v_1} & 0 & 0 \\ 0 & (e - 1)v_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so  $L_0(u) = (e - 1)u$ ,  $s = \frac{1}{e-1}$ ,  $M_0(u) = M(u) = M_1(u) = 0$ ,  $N(u) = L(u) = e^{\frac{1}{e-1}}u$ , and  $N_1(u) = L_1(u) = eu$ . Then, using Inequality (23),  $r_{new} = 0.614$ , and compared to  $r_{old} = 0.475$  using Inequality (23) again but for  $N \rightarrow N_1$ ,  $L \rightarrow L_1$ ,  $M \rightarrow M_1$ ,  $L_0 \rightarrow L_1$ , and  $M_0 \rightarrow M_1$ , we conclude that:

$$r_{old} < r_{new}$$

**Example 2.** Let  $X = Y = \mathbf{R}^3$ :

$$\begin{aligned} H_1(v) &= v_3^2(1 - v_2) - v_1v_2 + |v_2 - v_3^2| = 0, \\ H_2(v) &= v_3^2(v_1^3 - v_1) - v_2^2 + |3v_2^2 - v_3^2 + 1| = 0, \\ H_3(v) &= 6v_1v_2^3 + v_2^2v_3^2 - v_1v_2^2v_3 + |v_1 + v_3 - v_2| = 0, \\ v^* &= (-1; 2; 3). \end{aligned} \tag{37}$$

Define

$$\begin{aligned} G_1(v) &= |v_2 - v_3^2|, \\ G_2(v) &= |3v_2^2 - v_3^2 + 1|, \\ G_3(v) &= |v_1 + v_3 - v_2|. \end{aligned}$$

Let us denote  $x = v$ . We choose the initial approximation  $x_0 = (-2; 3; 5)p$ . An additional approximation is  $x_{-1} = y_0 = x_0 + 0.0001$ . We performed the calculations to fulfill the conditions  $\|x_{k+1} - x_k\|_\infty \leq \varepsilon$ ;  $\|H(x_{k+1})\| \leq \varepsilon$ .

To solve the System (37) we compared methods (3), (4), (6), and the secant method:

$$x_{k+1} = x_k - (F(x_k, x_{k-1}) + G(x_k, x_{k-1}))^{-1}(F(x_k) + G(x_k)), k = 0, 1, 2, \dots \tag{38}$$

The computations were performed using Matlab R2010a. The results of calculations are shown in Table 1.

**Table 1.** Numbers of iterations for solving system of Equation (37).

$p$	$\varepsilon$	Method			
		(3)	(4)	(38)	(6)
1	$10^{-5}$	85	7	10	7
	$10^{-15}$	266	10	12	8
10	$10^{-5}$	102	110	25	14
	$10^{-15}$	284	20	27	16
100	$10^{-5}$	110	28	39	23
	$10^{-15}$	292	30	41	24

As can be seen from column (6) of Table 1, the number of iterations for  $p = 1, 10, 100$  and  $\varepsilon = 10^{-5}, 10^{-15}$  was smaller than the corresponding one of the other methods. The presented results show the high efficiency of the proposed method (6).

### 6. Conclusions

We conducted an improved local convergence analysis of two-step solvers for solving nonlinear equations. To study the method we used a center and a restricted radius Lipschitz conditions to obtain a larger radius of convergence and tighter error estimates, and under the same computational effort. The uniqueness ball of the solution was established. Our results have the following advantages over earlier works: (1) larger convergence region, leading to more initial points; (2) tighter upper bound estimates on  $\|x_{k+1} - x^*\|$ , as well as  $\|x_k - x^*\|$ , which means that fewer iterations are needed

to arrive at a desired error tolerance; and (3) the information on the location of the solution is at least as precise. In partial cases, the obtained results of local convergence contain the results obtained in the works of other authors. Conducted numerical calculations confirm the theoretical results.

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## References

1. Argyros, I.K.; Magreñan, A.A. *A Contemporary Study of Iterative Methods*; Elsevier (Academic Press): New York, NY, USA, 2018.
2. Argyros, I.K.; Magreñan, A.A. *Iterative Methods and Their Dynamics with Applications*; CRC Press: New York, NY, USA, 2017.
3. Hernandez, M.A.; Rubio, M.J. The Secant method for nondifferentiable operators. *Appl. Math. Lett.* **2002**, *15*, 395–399. [[CrossRef](#)]
4. Ortega, J.M.; Rheinboldt, W.C. *Iterative Solution of Nonlinear Equations in Several Variables*; Academic Press: New York, NY, USA, 1970.
5. Ren, H.; Argyros, I.K. A new semilocal convergence theorem for a fast iterative method with nondifferentiable operators. *J. Appl. Math. Comp.* **2010**, *34*, 39–46. [[CrossRef](#)]
6. Shakhno, S.M. About the difference method with quadratic convergence for solving nonlinear operator equations. *Matematychni Studii* **2006**, *26*, 105–110. (In Ukrainian)
7. Shakhno, S.M. On the secant method under generalized Lipschitz conditions for the divided difference operator. *Proc. Appl. Math. Mech.* **2007**, *7*, 2060083–2060084. [[CrossRef](#)]
8. Bartish, M.Y. About one iterative method of solving functional equations. *Dopov. Acad. Nauk Ukr. RSR. Ser. A* **1968**, *5*, 387–391. (In Ukrainian)
9. Bartish, M.Y.; Shcherbyna, Y.M. About one difference method of solving operator equations. *Dopov. Acad. Nauk Ukr. RSR. Ser. A* **1972**, *7*, 579–582. (In Ukrainian)
10. Shakhno, S.M. On a two-step iterative process under generalized Lipschitz conditions for first-order divided differences. *J. Math. Sci.* **2010**, *168*, 576–584. [[CrossRef](#)]
11. Argyros, I.K.; Hilout, S. Newton-Kantorovich approximations under weak continuity conditions. *J. Appl. Math. Comput.* **2011**, *37*, 361–375. [[CrossRef](#)]
12. Zabrejko, P.P.; Nguen, D.F. The majorant method in the theory of Newton-Kantorovich approximations and the Pták error estimates. *Numer. Funct. Anal. Optim.* **1987**, *9*, 671–684. [[CrossRef](#)]
13. Argyros, I.K. *Convergence and Applications of Newton-Type Iterations*; Springer: New York, NY, USA, 2008.
14. Cătinăș, E. On some iterative methods for solving nonlinear equations. *Revue d'Analyse Numérique et de Théorie de l'Approximation.* **1994**, *23*, 47–53.
15. Shakhno, S.M.; Mel'nyk, I.V.; Yarmola, H.P. Analysis of the convergence of a combined method for the solution of nonlinear equations. *J. Math. Sci.* **2014**, *201*, 32–43. [[CrossRef](#)]
16. Shakhno, S.M. Combined Newton-Kurchatov method under the generalized Lipschitz conditions for the derivatives and divided differences. *J. Comput. Appl. Math. Kiev.* **2015**, *2*, 78–89.
17. Shakhno, S.M.; Yarmola, H.P. Two-point method for solving nonlinear equation with nondifferentiable operator. *Matematychni Studii* **2011**, *36*, 213–222. (In Ukrainian)
18. Shakhno, S.; Yarmola, H. Two-step method for solving nonlinear equations with nondifferentiable operator. *J. Comput. Appl. Math. Kiev.* **2012**, *3*, 105–115. [[CrossRef](#)]
19. Shakhno, S.M. Convergence of the two-step combined method and the uniqueness of solution of the nonlinear operator equations. *J. Comput. Appl. Math.* **2014**, *261*, 378–386. [[CrossRef](#)]
20. Chen, X.; Nashed, Z.; Qi, L. Convergence of Newton's method for singular smooth and nonsmooth equations using adaptive outer inverses. *SIAM J. Optim.* **1997**, *7*, 445–462. [[CrossRef](#)]
21. Chen, X.; Yamamoto, T. Convergence domains of certain iterative methods for solving nonlinear equations. *Numer. Funct. Anal. Optim.* **1989**, *10*, 37–48. [[CrossRef](#)]
22. Esquerro, J.A.; Hernández, M.A. *Newton's Method: An Updated Approach of Kantorovich's Theory*; Frontiers in Mathematics; Birkhäuser: Basel, Switzerland, 2017.

23. Elber, G.; Kim, M.S. Geometric constraint solver using multivariate rational spline functions. In Proceedings of the 6th ACM Symposium on Solid Modeling and Applications, Ann Arbor, MI, USA, 4–8 June 2001; pp. 1–10.
24. Aizenshtein, M.; Bartoň, M.; Elber, G. Global solutions of well-constrained transcendental systems using expression trees and a single solution test. *Comput. Aided Geom. Des.* **2012**, *29*, 265–279. [[CrossRef](#)]
25. Bartoň, M. Solving polynomial systems using no-root elimination blending schemes. *Comput.-Aided Des.* **2011**, *43*, 1870–1878. [[CrossRef](#)]
26. Morgan, A.P. A homotopy for solving polynomial systems. *Appl. Math. Comput.* **1986**, *18*, 87–92. [[CrossRef](#)]
27. Morgan, A.; Sommese, A. Computing all solutions to polynomial systems using homotopy continuation. *Appl. Math. Comput.* **1987**, *24*, 115–138. [[CrossRef](#)]
28. Wang, X. Convergence of Newton's method and uniqueness of the solution of equations in Banach space. *IMA J. Numer. Anal.* **2000**, *20*, 123–134. [[CrossRef](#)]
29. Werner, W. Über ein Verfahren der Ordnung zur Nullstellenbestimmung. *Numer. Math.* **1979**, *32*, 333–342. [[CrossRef](#)]
30. Shakhno, S.M. Convergence of the two-step Newton type method for solving of nonlinear equations under the generalized Lipschitz conditions. *Phys.-Math. Model. Inf. Technol.* **2012**, *16*, 163–172. (In Ukrainian)
31. Shakhno, S.M. On an iterative algorithm with superquadratic convergence for solving nonlinear operator equations. *J. Comp. App. Math.* **2009**, *231*, 222–235. [[CrossRef](#)]



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