Article

Arithmetic Means for a Class of Functions and the Modified Bessel Functions of the First Kind

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Abstract: In the paper, by virtue of the residue theorem in the theory of complex functions, the authors establish several identities between arithmetic means for a class of functions and the modified Bessel functions of the first kind, present several identities between arithmetic means for a class of functions and infinite series, and find several series expressions for the modified Bessel functions of the first kind.

Keywords: arithmetic mean for function; modified Bessel function of the first kind; identity; series expression; residue theorem; Toader–Qi mean

MSC: 26E60; 30B10; 33C10

1. Preliminaries

Recall from ([1], Chapter 6) and ([2], Chapter 5) that the classical Euler gamma function can be defined by

$$\Gamma(w) = \int_0^\infty t^{w-1} e^{-t} dt, \quad \Re(w) > 0$$

or by

$$\Gamma(w) = \lim_{n \rightarrow \infty} \frac{n! n^w}{\prod_{k=0}^n (w+k)}, \quad w \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$$

and that the logarithmic derivative

$$\psi(w) = [\ln \Gamma(w)]' = \frac{\Gamma'(w)}{\Gamma(w)}$$

is called the psi or digamma function. One can also find these notions in the papers [3,4], the handbooks [1,2], and closely related references therein.

In most calculus texts, the quantity $\frac{1}{b-a} \int_a^b f(x) dx$ is called the arithmetic mean for $f(x)$ on the interval $[a, b]$. This is justified by noting that the approximating Riemann sums are the arithmetic means of values of $f(x)$ at points distributed across $[a, b]$. See ([5], p. 368).

Recall from ([1], Chapter 9) and ([2], Chapter 10) that the modified Bessel functions of the first kind $I_\nu(w)$ can be defined by

$$I_\nu(w) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(\nu + k + 1)} \left(\frac{w}{2}\right)^{2k+\nu}, \quad |\arg w| < \pi, \quad \nu \in \mathbb{C}.$$

One can find some inequalities for $I_\nu(w)$ in the papers [6–10] and closely related references therein.

2. Motivations

In ([11], Lemma 2.1), by virtue of two approaches, the identity

$$\frac{2}{\pi} \int_0^{\pi/2} \alpha^{\cos^2 \phi} \beta^{\sin^2 \phi} d\phi = \sqrt{\alpha\beta} I_0\left(\ln \sqrt{\frac{\alpha}{\beta}}\right) \quad (1)$$

for $\alpha, \beta > 0$ was derived. By virtue of the identity (1) and a double inequality for $I_0(w)$, the double inequality

$$L(\alpha, \beta) < \frac{2}{\pi} \int_0^{\pi/2} \alpha^{\cos^2 \phi} \beta^{\sin^2 \phi} d\phi < I(\alpha, \beta)$$

for $\alpha, \beta > 0$ and $\alpha \neq \beta$ was discovered in ([11], Theorem 1.1), where

$$I(\alpha, \beta) = \frac{1}{e} \left(\frac{\beta^\beta}{\alpha^\alpha} \right)^{1/(\beta-\alpha)} \quad \text{and} \quad L(\alpha, \beta) = \frac{\beta - \alpha}{\ln \beta - \ln \alpha}$$

for $\alpha, \beta > 0$ and $\alpha \neq \beta$ are respectively called the exponential and logarithmic means [5,12–14]. Subsequently, there have been several papers, such as [10,15–20], dedicated to estimating or bounding the so-called Toader–Qi mean

$$TQ(\alpha, \beta) = \frac{2}{\pi} \int_0^{\pi/2} \alpha^{\cos^2 \phi} \beta^{\sin^2 \phi} d\phi = \frac{1}{2\pi} \int_0^{2\pi} \alpha^{\cos^2 \phi} \beta^{\sin^2 \phi} d\phi, \quad \alpha, \beta > 0.$$

In this paper, we consider the following arithmetic means for a class of functions,

$$Q(\alpha, \beta; k, \ell) = \frac{1}{2\pi} \int_0^{2\pi} \alpha^{\cos^k \phi} \beta^{\sin^\ell \phi} d\phi, \quad \alpha, \beta > 0, \quad k, \ell \in \mathbb{N}.$$

It is clear that $Q(\alpha, \beta; 2, 2)$ is just the above Toader–Qi mean $TQ(\alpha, \beta)$.

3. A Definition and a Lemma

In this paper, one of our main tools is the residue theorem in the theory of complex functions.

Definition 1 ([21], p. 112) and ([2], p. 19)). Let $f(w)$ have an isolated singularity at $w = w_0$ and let

$$f(w) = \sum_{n=-\infty}^{\infty} c_n (w - w_0)^n$$

be its Laurent expansion about $w = w_0$. Then the residue of $f(w)$ at $w = w_0$ is defined to be the coefficient c_{-1} and is denoted by $\text{Res}[f(w), w_0] = c_{-1}$.

Lemma 1 (Residue theorem ([21], p. 112) and ([2], p. 19)). Let f be analytic in the domain D except for the isolated singularities $w_1, w_2, \dots, w_n$. If γ is a simple closed contour in D and does not pass through any of the points w_k for $1 \leq k \leq n$, then

$$\frac{1}{2\pi i} \int_{\gamma} f(w) dw = \sum_{k=1}^n \text{Res}[f(w), w_k].$$

4. Several Identities and Series for Arithmetic Means

We now state and prove the first identity for the arithmetic mean $Q(\alpha, \beta; 1, 1)$ and the modified Bessel function of the first kind $I_0(w)$ as follows.

Theorem 1. For $\alpha, \beta > 0$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} \alpha^{\cos \phi} \beta^{\sin \phi} d\phi = I_0\left(\sqrt{(\ln \alpha)^2 + (\ln \beta)^2}\right). \quad (2)$$

Proof. Let $i = \sqrt{-1}$ and $w = e^{i\phi}$. Then

$$\cos \phi = \frac{w + w^{-1}}{2} \quad \text{and} \quad \sin \phi = \frac{w - w^{-1}}{2i}. \quad (3)$$

Accordingly, by direct calculation and Lemma 1, we have

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \alpha^{\cos \phi} \beta^{\sin \phi} d\phi &= \frac{1}{2\pi i} \oint_{|w|=1} \frac{\alpha^{(w+w^{-1})/2} \beta^{(w-w^{-1})/(2i)}}{w} dw \\ &= \text{Res} \left[\frac{\alpha^{(w+w^{-1})/2} \beta^{(w-w^{-1})/(2i)}}{w}, 0 \right]. \end{aligned} \quad (4)$$

Straightforward computation yields

$$\begin{aligned} \alpha^{(w+w^{-1})/2} \beta^{(w-w^{-1})/(2i)} &= \exp \left(\frac{w + w^{-1}}{2} \ln \alpha + \frac{w - w^{-1}}{2i} \ln \beta \right) \\ &= \exp \left(\frac{\ln \alpha - i \ln \beta}{2} w + \frac{\ln \alpha + i \ln \beta}{2} \frac{1}{w} \right) = \sum_{p=0}^{\infty} \frac{1}{p!} \left(\frac{\ln \alpha - i \ln \beta}{2} w + \frac{\ln \alpha + i \ln \beta}{2} \frac{1}{w} \right)^p \\ &= \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{q=0}^p \binom{p}{q} \left(\frac{\ln \alpha - i \ln \beta}{2} w \right)^q \left(\frac{\ln \alpha + i \ln \beta}{2} \frac{1}{w} \right)^{p-q} \\ &= \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{q=0}^p \binom{p}{q} \left(\frac{\ln \alpha - i \ln \beta}{2} \right)^q \left(\frac{\ln \alpha + i \ln \beta}{2} \right)^{p-q} w^{2q-p} \\ &= \sum_{m=0}^{\infty} \frac{1}{(2m)!} \sum_{q=0}^{2m} \binom{2m}{q} \left(\frac{\ln \alpha - i \ln \beta}{2} \right)^q \left(\frac{\ln \alpha + i \ln \beta}{2} \right)^{2m-q} w^{2(q-m)} \\ &\quad + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \sum_{q=0}^{2m+1} \binom{2m+1}{q} \left(\frac{\ln \alpha - i \ln \beta}{2} \right)^q \left(\frac{\ln \alpha + i \ln \beta}{2} \right)^{2m+1-q} w^{2(q-m)-1} \\ &= \sum_{m=0}^{\infty} \frac{1}{(2m)!} \binom{2m}{m} \left(\frac{\ln \alpha - i \ln \beta}{2} \right)^m \left(\frac{\ln \alpha + i \ln \beta}{2} \right)^{2m-m} w^{2(m-m)} \\ &\quad + \sum_{m=0}^{\infty} \frac{1}{(2m)!} \left(\sum_{q=0}^{m-1} + \sum_{q=m+1}^{2m} \right) \binom{2m}{q} \left(\frac{\ln \alpha - i \ln \beta}{2} \right)^q \left(\frac{\ln \alpha + i \ln \beta}{2} \right)^{2m-q} w^{2(q-m)} \\ &\quad + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \sum_{q=0}^{2m+1} \binom{2m+1}{q} \left(\frac{\ln \alpha - i \ln \beta}{2} \right)^q \left(\frac{\ln \alpha + i \ln \beta}{2} \right)^{2m+1-q} w^{2(q-m)-1} \\ &= \sum_{m=0}^{\infty} \frac{1}{(2m)!} \binom{2m}{m} \left[\frac{(\ln \alpha)^2 + (\ln \beta)^2}{4} \right]^m \\ &\quad + \sum_{m=0}^{\infty} \frac{1}{(2m)!} \left(\sum_{q=0}^{m-1} + \sum_{q=m+1}^{2m} \right) \binom{2m}{q} \left(\frac{\ln \alpha - i \ln \beta}{2} \right)^q \left(\frac{\ln \alpha + i \ln \beta}{2} \right)^{2m-q} w^{2(q-m)} \end{aligned}$$

$$+ \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \sum_{q=0}^{2m+1} \binom{2m+1}{q} \left(\frac{\ln \alpha - i \ln \beta}{2} \right)^q \left(\frac{\ln \alpha + i \ln \beta}{2} \right)^{2m-q+1} w^{2(q-m)-1},$$

where any empty sum is taken to be zero. Therefore, by Definition 1, we obtain

$$\begin{aligned} \operatorname{Res} \left[\frac{\alpha^{(w+w^{-1})/2} \beta^{(w-w^{-1})/(2i)}}{w}, 0 \right] &= \sum_{m=0}^{\infty} \frac{1}{(2m)!} \binom{2m}{m} \left[\frac{(\ln \alpha)^2 + (\ln \beta)^2}{4} \right]^m \\ &= \sum_{m=0}^{\infty} \frac{1}{(m!)^2} \left[\frac{\sqrt{(\ln \alpha)^2 + (\ln \beta)^2}}{2} \right]^{2m} = I_0 \left(\sqrt{(\ln \alpha)^2 + (\ln \beta)^2} \right). \end{aligned}$$

Substituting this equation into (4) results in the identity (2). The proof of Theorem 1 is thus complete. $\square$

We now state and prove the second and third identities between the arithmetic means $Q(\alpha, \beta; 1, 2)$ and $Q(\alpha, \beta; 2, 1)$ and two infinite series as follows.

Theorem 2. For $\alpha, \beta > 0$, we have

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \alpha^{\cos \phi} \beta^{\sin^2 \phi} d\phi &= \sum_{m=0}^{\infty} \frac{(\ln \beta)^{2m}}{(2m)!} \sum_{n=0}^m \binom{2m}{2n} \binom{(2n-1)/2}{2m-n} \left(\frac{\ln \alpha}{\ln \beta} \right)^{2n} \\ &\quad - \sum_{m=0}^{\infty} \frac{(\ln \beta)^{2m+1}}{(2m+1)!} \sum_{n=0}^m \binom{2m+1}{2n} \binom{(2n-1)/2}{2m-n+1} \left(\frac{\ln \alpha}{\ln \beta} \right)^{2n} \end{aligned} \quad (5)$$

and

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \alpha^{\cos^2 \phi} \beta^{\sin \phi} d\phi &= \sum_{m=0}^{\infty} \frac{(\ln \beta)^{2m}}{(2m)!} \sum_{n=0}^m \frac{1}{2^{2(m+n)}} \binom{2m}{2n} \left(\frac{\ln \alpha}{\ln \beta} \right)^{2n} \left[\sum_{q=0}^{m+n} (-1)^q \binom{2n}{q} \binom{2(m-n)}{m+n-q} \right] \\ &\quad + \frac{1}{4} \sum_{m=0}^{\infty} \frac{(-1)^m (\ln \beta)^{2m+1}}{(2m+1)!} \sum_{n=0}^m (-1)^n \binom{2m+1}{2n+1} \binom{[2(2n+1)-1]/2}{m+n+1} \left(\frac{\ln \alpha}{\ln \beta} \right)^{2n+1}. \end{aligned} \quad (6)$$

Proof. As did in the proof of Theorem 1, we can obtain

$$\frac{1}{2\pi} \int_0^{2\pi} \alpha^{\cos \phi} \beta^{\sin^2 \phi} d\phi = \operatorname{Res} \left[\frac{\alpha^{(w+w^{-1})/2} \beta^{[(w-w^{-1})/(2i)]^2}}{w}, 0 \right].$$

Direct computation yields

$$\begin{aligned} \alpha^{(w+w^{-1})/2} \beta^{[(w-w^{-1})/(2i)]^2} &= \exp \left[\frac{w+w^{-1}}{2} \ln \alpha + \left(\frac{w-w^{-1}}{2i} \right)^2 \ln \beta \right] \\ &= \exp \left[\frac{\ln \alpha}{2} \left(w + \frac{1}{w} \right) - \frac{\ln \beta}{4} \left(w - \frac{1}{w} \right)^2 \right] = \sum_{m=0}^{\infty} \frac{1}{m!} \left[\frac{\ln \alpha}{2} \left(w + \frac{1}{w} \right) - \frac{\ln \beta}{4} \left(w - \frac{1}{w} \right)^2 \right]^m \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n=0}^m \binom{m}{n} \left[\frac{\ln \alpha}{2} \left(w + \frac{1}{w} \right) \right]^n (-1)^{m-n} \left[\frac{\ln \beta}{4} \left(w - \frac{1}{w} \right)^2 \right]^{m-n} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n=0}^m (-1)^{m-n} \binom{m}{n} \frac{(\ln \alpha)^n (\ln \beta)^{m-n}}{2^{2m-n}} \frac{(1+w^2)^n (w^2-1)^{2(m-n)}}{w^{2m-n}} \\ &= \sum_{m=0}^{\infty} \frac{1}{(2m)!} \sum_{n=0}^{2m} (-1)^{2m-n} \binom{2m}{n} \frac{(\ln \alpha)^n (\ln \beta)^{2m-n}}{2^{4m-n}} \frac{(1+w^2)^n (w^2-1)^{2(2m-n)}}{w^{4m-n}} \end{aligned}$$

$$\begin{aligned}
& + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \sum_{n=0}^{2m+1} (-1)^{2m+1-n} \binom{2m+1}{n} \frac{(\ln \alpha)^n (\ln \beta)^{2m+1-n}}{2^{2(2m+1)-n}} \frac{(1+w^2)^n (w^2-1)^{2(2m+1-n)}}{w^{2(2m+1)-n}} \\
& = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \sum_{n=0}^m \binom{2m}{2n} \frac{(\ln \alpha)^{2n} (\ln \beta)^{2m-2n}}{2^{2(2m-n)}} \frac{(1+w^2)^{2n} (w^2-1)^{2(2m-2n)}}{w^{2(2m-n)}} \\
& + \sum_{m=0}^{\infty} \frac{-1}{(2m)!} \sum_{n=0}^{m-1} \binom{2m}{2n+1} \frac{(\ln \alpha)^{2n+1} (\ln \beta)^{2m-(2n+1)}}{2^{4m-(2n+1)}} \frac{(1+w^2)^{2n+1} (w^2-1)^{2[2m-(2n+1)]}}{w^{4m-(2n+1)}} \\
& + \sum_{m=0}^{\infty} \frac{-1}{(2m+1)!} \sum_{n=0}^m \binom{2m+1}{2n} \frac{(\ln \alpha)^{2n} (\ln \beta)^{2m+1-2n}}{2^{2(2m+1)-2n}} \frac{(1+w^2)^{2n} (w^2-1)^{2(2m+1-2n)}}{w^{2(2m+1)-2n}} \\
& + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \sum_{n=0}^m \binom{2m+1}{2n+1} \frac{(\ln \alpha)^{2n+1} (\ln \beta)^{2m+1-(2n+1)}}{2^{2(2m+1)-(2n+1)}} \frac{(1+w^2)^{2n+1} (w^2-1)^{2[2m+1-(2n+1)]}}{w^{2(2m+1)-(2n+1)}}.
\end{aligned}$$

In the second and fourth sums above, all terms are not constant and dependent of the variable w . The first and third sums above can be rewritten as

$$\begin{aligned}
& \sum_{m=0}^{\infty} \frac{1}{(2m)!} \sum_{n=0}^m \binom{2m}{2n} \frac{(\ln \alpha)^{2n} (\ln \beta)^{2m-2n}}{2^{2(2m-n)}} \frac{(1+w^2)^{2n} (w^2-1)^{2(2m-2n)}}{w^{2(2m-n)}} \\
& = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \sum_{n=0}^m \binom{2m}{2n} \frac{(\ln \alpha)^{2n} (\ln \beta)^{2(m-n)}}{(2w)^{2(2m-n)}} \sum_{p=0}^{2n} \binom{2n}{p} w^{2p} \sum_{p=0}^{4(m-n)} \binom{4(m-n)}{p} (-1)^p w^{2p} \\
& = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \sum_{n=0}^m \binom{2m}{2n} \frac{(\ln \alpha)^{2n} (\ln \beta)^{2(m-n)}}{(2w)^{2(2m-n)}} \sum_{p=0}^{2m-n} \left[\sum_{q=0}^p \binom{2n}{q} \binom{4(m-n)}{p-q} (-1)^{p-q} \right] w^{2p} \\
& = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \sum_{n=0}^m \binom{2m}{2n} \frac{(\ln \alpha)^{2n} (\ln \beta)^{2(m-n)}}{2^{2(2m-n)}} \left[\sum_{q=0}^{2m-n} \binom{2n}{q} \binom{4(m-n)}{2m-n-q} (-1)^{2m-n-q} \right] \\
& \quad + \sum_{m=0}^{\infty} \frac{1}{(2m)!} \sum_{n=0}^m \binom{2m}{2n} \frac{(\ln \alpha)^{2n} (\ln \beta)^{2(m-n)}}{(2w)^{2(2m-n)}} \\
& \quad \times \left[\sum_{p=0}^{2m-n-1} + \sum_{p=2m-n+1}^{2(2m-n)} \right] \left[\sum_{q=0}^p \binom{2n}{q} \binom{4(m-n)}{p-q} (-1)^{p-q} \right] w^{2p}
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{m=0}^{\infty} \frac{-1}{(2m+1)!} \sum_{n=0}^m \binom{2m+1}{2n} \frac{(\ln \alpha)^{2n} (\ln \beta)^{2m+1-2n}}{2^{2(2m+1)-2n}} \frac{(1+w^2)^{2n} (w^2-1)^{2(2m+1-2n)}}{w^{2(2m+1)-2n}} \\
& = \sum_{m=0}^{\infty} \frac{-1}{(2m+1)!} \sum_{n=0}^m \binom{2m+1}{2n} \frac{(\ln \alpha)^{2n} (\ln \beta)^{2m-2n+1}}{(2w)^{2(2m-n+1)}} \\
& \quad \times \sum_{p=0}^{2n} \binom{2n}{p} w^{2p} \sum_{p=0}^{2(2m-2n+1)} \binom{2(2m-2n+1)}{p} w^{2p} (-1)^p \\
& = \sum_{m=0}^{\infty} \frac{-1}{(2m+1)!} \sum_{n=0}^m \binom{2m+1}{2n} \frac{(\ln \alpha)^{2n} (\ln \beta)^{2m-2n+1}}{(2w)^{2(2m-n+1)}} \\
& \quad \times \sum_{p=0}^{2(2m-n+1)} \left[\sum_{q=0}^p \binom{2n}{q} \binom{2(2m-2n+1)}{p-q} (-1)^{p-q} \right] w^{2p} \\
& = \sum_{m=0}^{\infty} \frac{-1}{(2m+1)!} \sum_{n=0}^m \binom{2m+1}{2n} \frac{(\ln \alpha)^{2n} (\ln \beta)^{2m-2n+1}}{2^{2(2m-n+1)}}
\end{aligned}$$

$$\begin{aligned} & \times \left[\sum_{q=0}^{2m-n+1} \binom{2n}{q} \binom{2(2m-2n+1)}{2m-n+1-q} (-1)^{2m-n+1-q} \right] \\ & + \sum_{m=0}^{\infty} \frac{-1}{(2m+1)!} \sum_{n=0}^m \binom{2m+1}{2n} \frac{(\ln \alpha)^{2n} (\ln \beta)^{2m-2n+1}}{(2w)^{2(2m-n+1)}} \\ & \times \left[\sum_{p=0}^{2m-n} + \sum_{p=2m-n+2}^{2(2m-n+1)} \right] \left[\sum_{q=0}^p \binom{2n}{q} \binom{2(2m-2n+1)}{p-q} (-1)^{p-q} \right] w^{2p}. \end{aligned}$$

Consequently, the constant term in the Laurent expansion of the function $\alpha^{(w+w^{-1})/2} \beta^{[(w-w^{-1})/(2i)]^2}$ at $w = 0$ is equal to

$$\begin{aligned} & \sum_{m=0}^{\infty} \frac{1}{(2m)!} \sum_{n=0}^m \binom{2m}{2n} \frac{(\ln \alpha)^{2n} (\ln \beta)^{2(m-n)}}{2^{2(2m-n)}} \left[\sum_{q=0}^{2m-n} \binom{2n}{q} \binom{4(m-n)}{2m-n-q} (-1)^{2m-n-q} \right] \\ & + \sum_{m=0}^{\infty} \frac{-1}{(2m+1)!} \sum_{n=0}^m \binom{2m+1}{2n} \frac{(\ln \alpha)^{2n} (\ln \beta)^{2m-2n+1}}{2^{2(2m-n+1)}} \\ & \times \left[\sum_{q=0}^{2m-n+1} \binom{2n}{q} \binom{2(2m-2n+1)}{2m-n+1-q} (-1)^{2m-n+1-q} \right] \\ & = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \sum_{n=0}^m (-1)^n \binom{2m}{2n} \frac{(\ln \alpha)^{2n} (\ln \beta)^{2(m-n)}}{2^{2(2m-n)}} \sum_{q=0}^{2m-n} (-1)^q \binom{2n}{q} \binom{4(m-n)}{2m-n-q} \\ & + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \sum_{n=0}^m (-1)^n \binom{2m+1}{2n} \frac{(\ln \alpha)^{2n} (\ln \beta)^{2m-2n+1}}{2^{2(2m-n+1)}} \sum_{q=0}^{2m-n+1} (-1)^q \binom{2n}{q} \binom{2(2m-2n+1)}{2m-n+1-q} \\ & = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \sum_{n=0}^m (-1)^n \binom{2m}{2n} \frac{(\ln \alpha)^{2n} (\ln \beta)^{2(m-n)}}{2^{2(2m-n)}} (-4)^{2m-n} \binom{(2n-1)/2}{2m-n} \\ & + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \sum_{n=0}^m (-1)^n \binom{2m+1}{2n} \frac{(\ln \alpha)^{2n} (\ln \beta)^{2m-2n+1}}{2^{2(2m-n+1)}} (-4)^{2m-n+1} \binom{(2n-1)/2}{2m-n+1} \\ & = \sum_{m=0}^{\infty} \frac{(\ln \beta)^{2m}}{(2m)!} \sum_{n=0}^m \binom{2m}{2n} \binom{(2n-1)/2}{2m-n} \left(\frac{\ln \alpha}{\ln \beta} \right)^{2n} \\ & - \sum_{m=0}^{\infty} \frac{(\ln \beta)^{2m+1}}{(2m+1)!} \sum_{n=0}^m \binom{2m+1}{2n} \binom{(2n-1)/2}{2m-n+1} \left(\frac{\ln \alpha}{\ln \beta} \right)^{2n}, \end{aligned}$$

where we used the formula

$$\sum_{r=0}^j (-1)^r \binom{q}{r} \binom{2j-q}{j-r} = (-4)^j \binom{(q-1)/2}{j} \quad (7)$$

in ([22], p. 6, Equation (1.40)) and ([23], p. 63, Equation (3.42)). In a word, it follows that

$$\begin{aligned} \text{Res} \left[\frac{\alpha^{(w+w^{-1})/2} \beta^{[(w-w^{-1})/(2i)]^2}}{w}, 0 \right] &= \sum_{m=0}^{\infty} \frac{(\ln \beta)^{2m}}{(2m)!} \sum_{n=0}^m \binom{2m}{2n} \binom{(2n-1)/2}{2m-n} \left(\frac{\ln \alpha}{\ln \beta} \right)^{2n} \\ &- \sum_{m=0}^{\infty} \frac{(\ln \beta)^{2m+1}}{(2m+1)!} \sum_{n=0}^m \binom{2m+1}{2n} \binom{(2n-1)/2}{2m-n+1} \left(\frac{\ln \alpha}{\ln \beta} \right)^{2n}. \end{aligned}$$

The Equation (5) is thus proved.

As did in the above or in the proof of Theorem 1, we can also obtain

$$\frac{1}{2\pi} \int_0^{2\pi} \alpha^{\cos^2 \phi} \beta^{\sin^2 \phi} d\phi = \text{Res} \left[\frac{\alpha^{[(w+w^{-1})/2]^2} \beta^{(w-w^{-1})/(2i)}}{w}, 0 \right].$$

Straightforward calculation shows

$$\begin{aligned}
 & \alpha^{[(w+w^{-1})/2]^2} \beta^{(w-w^{-1})/(2i)} = \exp \left[\left(\frac{w+w^{-1}}{2} \right)^2 \ln \alpha + \frac{w-w^{-1}}{2i} \ln \beta \right] \\
 & = \exp \left[\frac{\ln \alpha}{4} \left(w + \frac{1}{w} \right)^2 - \frac{i \ln \beta}{2} \left(w - \frac{1}{w} \right) \right] = \sum_{m=0}^{\infty} \frac{1}{m!} \left[\frac{\ln \alpha}{4} \left(w + \frac{1}{w} \right)^2 - \frac{i \ln \beta}{2} \left(w - \frac{1}{w} \right) \right]^m \\
 & = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n=0}^m \binom{m}{n} \left[\frac{\ln \alpha}{4} \left(w + \frac{1}{w} \right)^2 \right]^n (-1)^{m-n} \left[\frac{i \ln \beta}{2} \left(w - \frac{1}{w} \right) \right]^{m-n} \\
 & = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n=0}^m (-1)^{m-n} \binom{m}{n} \frac{(\ln \alpha)^n (i \ln \beta)^{m-n}}{2^{m+n}} \frac{(1+w^2)^{2n} (w^2-1)^{m-n}}{w^{m+n}} \\
 & = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \sum_{n=0}^{2m} (-1)^{2m-n} \binom{2m}{n} \frac{(\ln \alpha)^n (i \ln \beta)^{2m-n}}{2^{2m+n}} \frac{(1+w^2)^{2n} (w^2-1)^{2m-n}}{w^{2m+n}} \\
 & + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \sum_{n=0}^{2m+1} (-1)^{2m+1-n} \binom{2m+1}{n} \frac{(\ln \alpha)^n (i \ln \beta)^{2m+1-n}}{2^{2m+1+n}} \frac{(1+w^2)^{2n} (w^2-1)^{2m+1-n}}{w^{2m+1+n}} \\
 & = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \sum_{n=0}^m (-1)^{2m-2n} \binom{2m}{2n} \frac{(\ln \alpha)^{2n} (i \ln \beta)^{2m-2n}}{2^{2m+2n}} \frac{(1+w^2)^{4n} (w^2-1)^{2m-2n}}{w^{2m+2n}} \\
 & + \sum_{m=0}^{\infty} \frac{-1}{(2m+1)!} \sum_{n=0}^m \binom{2m+1}{2n} \frac{(\ln \alpha)^{2n} (i \ln \beta)^{2m+1-2n}}{2^{2m+1+2n}} \frac{(1+w^2)^{4n} (w^2-1)^{2m+1-2n}}{w^{2m+1+2n}} \\
 & + \sum_{m=0}^{\infty} \frac{-1}{(2m)!} \sum_{n=0}^{m-1} \binom{2m}{2n+1} \frac{(\ln \alpha)^{2n+1} (i \ln \beta)^{2m-(2n+1)}}{2^{2m+2n+1}} \frac{(1+w^2)^{2(2n+1)} (w^2-1)^{2m-(2n+1)}}{w^{2m+2n+1}} \\
 & + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \sum_{n=0}^m \binom{2m+1}{2n+1} \frac{(\ln \alpha)^{2n+1} (i \ln \beta)^{2(m-n)}}{2^{2(m+n+1)}} \frac{(1+w^2)^{2(2n+1)} (w^2-1)^{2(m-n)}}{w^{2(m+n+1)}}.
 \end{aligned}$$

In the second and third sums above, all terms are not constant and dependent of the variable w . The first and fourth sums above can be formulated as

$$\begin{aligned}
 & \sum_{m=0}^{\infty} \frac{1}{(2m)!} \sum_{n=0}^m (-1)^{2m-2n} \binom{2m}{2n} \frac{(\ln \alpha)^{2n} (i \ln \beta)^{2m-2n}}{2^{2m+2n}} \frac{(1+w^2)^{4n} (w^2-1)^{2m-2n}}{w^{2m+2n}} \\
 & = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \sum_{n=0}^m \binom{2m}{2n} \frac{(\ln \alpha)^{2n} (i \ln \beta)^{2(m-n)}}{(2w)^{2(m+n)}} \sum_{p=0}^{4n} \binom{4n}{p} w^{2p} \sum_{p=0}^{2(m-n)} \binom{2(m-n)}{p} (-1)^p w^{2p} \\
 & = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \sum_{n=0}^m \binom{2m}{2n} \frac{(\ln \alpha)^{2n} (i \ln \beta)^{2(m-n)}}{(2w)^{2(m+n)}} \sum_{p=0}^{2(m+n)} \left[\sum_{q=0}^p \binom{4n}{q} \binom{2(m-n)}{p-q} (-1)^{p-q} \right] w^{2p} \\
 & = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \sum_{n=0}^m \binom{2m}{2n} \frac{(\ln \alpha)^{2n} (i \ln \beta)^{2(m-n)}}{2^{2(m+n)}} \left[\sum_{q=0}^{m+n} \binom{2n}{q} \binom{2(m-n)}{m+n-q} (-1)^{m+n-q} \right] \\
 & \quad + \sum_{m=0}^{\infty} \frac{1}{(2m)!} \sum_{n=0}^m \binom{2m}{2n} \frac{(\ln \alpha)^{2n} (i \ln \beta)^{2(m-n)}}{(2w)^{2(m+n)}} \\
 & \quad \times \left[\sum_{p=0}^{m+n-1} + \sum_{p=m+n+1}^{2(m+n)} \right] \left[\sum_{q=0}^p \binom{4n}{q} \binom{2(m-n)}{p-q} (-1)^{p-q} \right] w^{2p}
 \end{aligned}$$

and

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \sum_{n=0}^m \binom{2m+1}{2n+1} \frac{(\ln \alpha)^{2n+1} (i \ln \beta)^{2(m-n)}}{2^{2(m+n+1)}} \frac{(1+w^2)^{2(2n+1)} (w^2-1)^{2(m-n)}}{w^{2(m+n+1)}}$$

$$\begin{aligned}
&= \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \sum_{n=0}^m \binom{2m+1}{2n+1} \frac{(\ln \alpha)^{2n+1} (i \ln \beta)^{2(m-n)}}{(2w)^{2(m+n+1)}} \\
&\times \sum_{p=0}^{2(2n+1)} \binom{2(2n+1)}{p} w^{2p} \sum_{p=0}^{2(m-n)} \binom{2(m-n)}{p} w^{2p} (-1)^{2(m-n)-p} \\
&= \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \sum_{n=0}^m \binom{2m+1}{2n+1} \frac{(\ln \alpha)^{2n+1} (i \ln \beta)^{2(m-n)}}{(2w)^{2(m+n+1)}} \\
&\times \sum_{p=0}^{2(m+n+1)} \left[\sum_{q=0}^p \binom{2(2n+1)}{q} \binom{2(m-n)}{p-q} (-1)^{p-q} \right] w^{2p} \\
&= \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \sum_{n=0}^m \binom{2m+1}{2n+1} \frac{(\ln \alpha)^{2n+1} (i \ln \beta)^{2(m-n)}}{2^{2(m+n+1)}} \\
&\times \left[\sum_{q=0}^{m+n+1} \binom{2(2n+1)}{q} \binom{2(m-n)}{m+n+1-q} (-1)^{m+n+1-q} \right] \\
&+ \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \sum_{n=0}^m \binom{2m+1}{2n+1} \frac{(\ln \alpha)^{2n+1} (i \ln \beta)^{2(m-n)}}{(2w)^{2(m+n+1)}} \\
&\times \left[\sum_{p=0}^{m+n} + \sum_{p=m+n+2}^{2(m+n+1)} \right] \left[\sum_{q=0}^p \binom{2(2n+1)}{q} \binom{2(m-n)}{p-q} (-1)^{p-q} \right] w^{2p}.
\end{aligned}$$

Consequently, the constant term in the Laurent expansion of the function $\alpha^{[(w+w^{-1})/2]^2} \beta^{(w-w^{-1})/(2i)}$ at $w = 0$ is equal to

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{1}{(2m)!} \sum_{n=0}^m \binom{2m}{2n} \frac{(\ln \alpha)^{2n} (i \ln \beta)^{2(m-n)}}{2^{2(m+n)}} \left[\sum_{q=0}^{m+n} \binom{2n}{q} \binom{2(m-n)}{m+n-q} (-1)^{m+n-q} \right] \\
&+ \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \sum_{n=0}^m \binom{2m+1}{2n+1} \frac{(\ln \alpha)^{2n+1} (i \ln \beta)^{2(m-n)}}{2^{2(m+n+1)}} \\
&\times \left[\sum_{q=0}^{m+n+1} \binom{2(2n+1)}{q} \binom{2(m-n)}{m+n+1-q} (-1)^{m+n+1-q} \right] \\
&= \sum_{m=0}^{\infty} \frac{1}{(2m)!} \sum_{n=0}^m \binom{2m}{2n} \frac{(\ln \alpha)^{2n} (\ln \beta)^{2(m-n)}}{2^{2(m+n)}} \left[\sum_{q=0}^{m+n} \binom{2n}{q} \binom{2(m-n)}{m+n-q} (-1)^q \right] \\
&+ \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \sum_{n=0}^m \binom{2m+1}{2n+1} \frac{(\ln \alpha)^{2n+1} (\ln \beta)^{2(m-n)}}{2^{2(m+n+1)}} \\
&\times \left[\sum_{q=0}^{m+n+1} \binom{2(2n+1)}{q} \binom{2(m-n)}{m+n+1-q} (-1)^{q-1} \right] \\
&= \sum_{m=0}^{\infty} \frac{(\ln \beta)^{2m}}{(2m)!} \sum_{n=0}^m \binom{2m}{2n} \left(\frac{\ln \alpha}{\ln \beta} \right)^{2n} \frac{1}{2^{2(m+n)}} \left[\sum_{q=0}^{m+n} \binom{2n}{q} \binom{2(m-n)}{m+n-q} (-1)^q \right] \\
&+ \frac{1}{4} \sum_{m=0}^{\infty} \frac{(-1)^m (\ln \beta)^{2m+1}}{(2m+1)!} \sum_{n=0}^m (-1)^n \binom{2m+1}{2n+1} \left(\frac{\ln \alpha}{\ln \beta} \right)^{2n+1} \binom{[2(2n+1)-1]/2}{m+n+1},
\end{aligned}$$

where we used the formula (7). In conclusion, it follows that

$$\begin{aligned}
& \text{Res} \left[\frac{\alpha^{(w+w^{-1})/2} \beta^{[(w-w^{-1})/(2i)]^2}}{w}, 0 \right] \\
&= \sum_{m=0}^{\infty} \frac{(\ln \beta)^{2m}}{(2m)!} \sum_{n=0}^m \binom{2m}{2n} \left(\frac{\ln \alpha}{\ln \beta} \right)^{2n} \frac{1}{2^{2(m+n)}} \left[\sum_{q=0}^{m+n} \binom{2n}{q} \binom{2(m-n)}{m+n-q} (-1)^q \right] \\
&\quad + \frac{1}{4} \sum_{m=0}^{\infty} \frac{(-1)^m (\ln \beta)^{2m+1}}{(2m+1)!} \sum_{n=0}^m (-1)^n \binom{2m+1}{2n+1} \left(\frac{\ln \alpha}{\ln \beta} \right)^{2n+1} \binom{[2(2n+1)-1]/2}{m+n+1}.
\end{aligned}$$

The Equation (6) is thus proved. The proof of Theorem 2 is complete. $\square$

We now state and prove an identity between the arithmetic mean $Q(\alpha, \beta; k, k)$ and an infinite series as follows.

Theorem 3. For $\alpha, \beta > 0$ and $k \in \mathbb{N}$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} \alpha^{\cos^k \phi} \beta^{\sin^k \phi} d\phi = \sum_{j=0}^{\infty} (-1)^{kj} \frac{(\ln \beta)^{2j}}{(2j)!} \sum_{q=0}^{2j} (-i)^{kq} \binom{2j}{q} \binom{(kq-1)/2}{kj} \left(\frac{\ln \alpha}{\ln \beta} \right)^q. \quad (8)$$

Proof. Employing the transforms in (3) and Lemma 1 reveals

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} \alpha^{\cos^k \phi} \beta^{\sin^k \phi} d\phi &= \frac{1}{2\pi i} \oint_{|w|=1} \frac{\alpha^{[(w+w^{-1})/2]^k} \beta^{[(w-w^{-1})/(2i)]^k}}{w} dw \\
&= \text{Res} \left[\frac{\alpha^{[(w+w^{-1})/2]^k} \beta^{[(w-w^{-1})/(2i)]^k}}{w}, 0 \right].
\end{aligned} \quad (9)$$

Easy computation gives

$$\begin{aligned}
& \alpha^{[(w+w^{-1})/2]^k} \beta^{[(w-w^{-1})/(2i)]^k} = \exp \left[\left(\frac{w+w^{-1}}{2} \right)^k \ln \alpha + \left(\frac{w-w^{-1}}{2i} \right)^k \ln \beta \right] \\
&= \exp \left[\frac{(1+w^2)^k \ln \alpha + (1-w^2)^k i^k \ln \beta}{(2w)^k} \right] = \sum_{j=0}^{\infty} \frac{1}{j!} \left[\frac{(1+w^2)^k \ln \alpha + (1-w^2)^k i^k \ln \beta}{(2w)^k} \right]^j \\
&= \sum_{j=0}^{\infty} \frac{1}{j!} \frac{1}{(2w)^{kj}} \sum_{q=0}^j \binom{j}{q} [(1+w^2)^k \ln \alpha]^q [(1-w^2)^k i^k \ln \beta]^{j-q} \\
&= \sum_{j=0}^{\infty} \frac{1}{j!} \frac{1}{(2w)^{kj}} \sum_{q=0}^j \binom{j}{q} (\ln \alpha)^q (\ln \beta)^{j-q} i^{(j-q)k} [(1+w^2)^q (1-w^2)^{j-q}]^k \\
&= \sum_{j=0}^{\infty} \sum_{q=0}^{2j} \binom{2j}{q} \frac{(\ln \alpha)^q (\ln \beta)^{2j-q} i^{(2j-q)k}}{(2j)!} \left[\frac{(1+w^2)^q (1-w^2)^{2j-q}}{(4w^2)^j} \right]^k \\
&\quad + \sum_{j=0}^{\infty} \sum_{q=0}^{2j+1} \binom{2j+1}{q} \frac{(\ln \alpha)^q (\ln \beta)^{2j+1-q} i^{(2j+1-q)k}}{(2j+1)!} \left[\frac{(1+w^2)^q (1-w^2)^{2j+1-q}}{(2w)^{2j+1}} \right]^k.
\end{aligned}$$

It is easy to see that, in the above sum $\sum_{j=0}^{\infty} \sum_{q=0}^{2j+1}$, there is no any constant term independent of w . On the other hand, because

$$\begin{aligned}(1+w)^q(1-w)^{2j-q} &= \sum_{r=0}^q \binom{q}{r} w^r \sum_{s=0}^{2j-q} \binom{2j-q}{s} (-1)^s w^s = \sum_{p=0}^{2j} \left[\sum_{r=0}^p \binom{q}{r} \binom{2j-q}{p-r} (-1)^{p-r} \right] w^p \\ &= \left[\sum_{r=0}^j \binom{q}{r} \binom{2j-q}{j-r} (-1)^{j-r} \right] w^j + \left(\sum_{p=0}^{j-1} + \sum_{p=j+1}^{2j} \right) \left[\sum_{r=0}^p \binom{q}{r} \binom{2j-q}{p-r} (-1)^{p-r} \right] w^p,\end{aligned}$$

where any empty sum is taken to be zero and $\binom{r}{s} = 0$ for $s > r \geq 0$ and $r, s \in \mathbb{N}$, we have

$$\begin{aligned}& \sum_{j=0}^{\infty} \sum_{q=0}^{2j} \binom{2j}{q} \frac{(\ln \alpha)^q (\ln \beta)^{2j-q} i^{(2j-q)k}}{(2j)!} \left[\frac{(1+w^2)^q (1-w^2)^{2j-q}}{(4w^2)^j} \right]^k \\ &= \sum_{j=0}^{\infty} \sum_{q=0}^{2j} \binom{2j}{q} \frac{(\ln \alpha)^q (\ln \beta)^{2j-q} i^{(2j-q)k}}{(2j)!} \frac{(1+w^2)^{kq} (1-w^2)^{k(2j-q)}}{(4w^2)^{kj}} \\ &= \sum_{j=0}^{\infty} \sum_{q=0}^{2j} \binom{2j}{q} \frac{(\ln \alpha)^q (\ln \beta)^{2j-q} i^{(2j-q)k}}{(2j)! 4^{kj}} \sum_{r=0}^{kj} (-1)^{kj-r} \binom{kq}{r} \binom{k(2j-q)}{kj-r} \\ &+ \sum_{j=0}^{\infty} \sum_{q=0}^{2j} \binom{2j}{q} \frac{(\ln \alpha)^q (\ln \beta)^{2j-q} i^{(2j-q)k}}{(2j)!} \frac{1}{(4w^2)^{kj}} \left(\sum_{p=0}^{kj-1} + \sum_{p=kj+1}^{2kj} \right) \left[\sum_{r=0}^p \binom{kq}{r} \binom{k(2j-q)}{p-r} (-1)^{p-r} \right] w^{2p}.\end{aligned}$$

It is not difficult to see that, in the sum $\sum_{p=0}^{kj-1} + \sum_{p=kj+1}^{2kj}$, there is no any constant term independent of w . In a word, the constant term independent of w in the Laurent expansion of the function

$$\alpha^{[(w+w^{-1})/2]^k} \beta^{[(w-w^{-1})/(2i)]^k}$$

at $z = 0$ is equal to

$$\sum_{j=0}^{\infty} \sum_{q=0}^{2j} \binom{2j}{q} \frac{(\ln \alpha)^q (\ln \beta)^{2j-q} i^{(2j-q)k}}{(2j)! 4^{kj}} \sum_{r=0}^{kj} (-1)^{kj-r} \binom{kq}{r} \binom{k(2j-q)}{kj-r}.$$

Consequently, by Definition 1, we deduce

$$\begin{aligned}\text{Res} \left[\frac{\alpha^{[(w+w^{-1})/2]^k} \beta^{[(w-w^{-1})/(2i)]^k}}{w}, 0 \right] \\ = \sum_{j=0}^{\infty} \frac{1}{4^{kj}} \frac{(\ln \beta)^{2j}}{(2j)!} \sum_{q=0}^{2j} \binom{2j}{q} \left(\frac{\ln \alpha}{\ln \beta} \right)^q (-i)^{kq} \sum_{r=0}^{kj} (-1)^r \binom{kq}{r} \binom{k(2j-q)}{kj-r}.\end{aligned}\quad (10)$$

Substituting the identity (7) into (10) and simplifying leads to

$$\text{Res} \left[\frac{\alpha^{[(w+w^{-1})/2]^k} \beta^{[(w-w^{-1})/(2i)]^k}}{w}, 0 \right] = \sum_{j=0}^{\infty} (-1)^{kj} \frac{(\ln \beta)^{2j}}{(2j)!} \sum_{q=0}^{2j} (-i)^{kq} \binom{2j}{q} \binom{(kq-1)/2}{kj} \left(\frac{\ln \alpha}{\ln \beta} \right)^q.$$

Substituting this equation into (9) leads to (8). The proof of Theorem 3 is thus complete. $\square$

We now state and prove an identity between the arithmetic mean $Q(\alpha, \beta; k, \ell)$ and an infinite series as follows.

Theorem 4. For $\alpha, \beta > 0$ and $k, \ell \in \mathbb{N}$, we have

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \alpha^{\cos^k \phi} \beta^{\sin^\ell \phi} d\phi &= \sum_{m=0}^{\infty} (-1)^{[kn+\ell(m-n)]/2} \frac{(-1)^{\ell(m-n)} + (-1)^{kn}}{2} \frac{1}{m!} \\ &\times \sum_{n=0}^m \binom{m}{n} \left(\frac{\ln \alpha}{2^k}\right)^n \left[\frac{\ln \beta}{(2i)^\ell}\right]^{m-n} \sum_{r=0}^{[kn+\ell(m-n)]/2} (-1)^r \binom{kn}{r} \binom{\ell(m-n)}{[kn+\ell(m-n)]/2-r}. \end{aligned} \quad (11)$$

Proof. Utilizing the transforms in (3) and Lemma 1 results in

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \alpha^{\cos^k \phi} \beta^{\sin^\ell \phi} d\phi &= \frac{1}{2\pi i} \oint_{|w|=1} \frac{\alpha^{[(w+w^{-1})/2]^k} \beta^{[(w-w^{-1})/(2i)]^\ell}}{w} dw \\ &= \text{Res} \left[\frac{\alpha^{[(w+w^{-1})/2]^k} \beta^{[(w-w^{-1})/(2i)]^\ell}}{w}, 0 \right]. \end{aligned} \quad (12)$$

Direct computation gives

$$\begin{aligned} \alpha^{[(w+w^{-1})/2]^k} \beta^{[(w-w^{-1})/(2i)]^\ell} &= \exp \left[\left(\frac{w+w^{-1}}{2} \right)^k \ln \alpha + \left(\frac{w-w^{-1}}{2i} \right)^\ell \ln \beta \right] \\ &= \exp \left[\left(w + \frac{1}{w} \right)^k \frac{\ln \alpha}{2^k} + \left(w - \frac{1}{w} \right)^\ell \frac{\ln \beta}{(2i)^\ell} \right] = \sum_{m=0}^{\infty} \frac{1}{m!} \left[\left(w + \frac{1}{w} \right)^k \frac{\ln \alpha}{2^k} + \left(w - \frac{1}{w} \right)^\ell \frac{\ln \beta}{(2i)^\ell} \right]^m \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n=0}^m \binom{m}{n} \left[\left(w + \frac{1}{w} \right)^k \frac{\ln \alpha}{2^k} \right]^n \left[\left(w - \frac{1}{w} \right)^\ell \frac{\ln \beta}{(2i)^\ell} \right]^{m-n} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n=0}^m \binom{m}{n} \left(\frac{\ln \alpha}{2^k} \right)^n \left[\frac{\ln \beta}{(2i)^\ell} \right]^{m-n} \left(w + \frac{1}{w} \right)^{kn} \left(w - \frac{1}{w} \right)^{\ell(m-n)} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n=0}^m \binom{m}{n} \left(\frac{\ln \alpha}{2^k} \right)^n \left[\frac{\ln \beta}{(2i)^\ell} \right]^{m-n} \sum_{p=0}^{kn} \binom{kn}{p} w^p \left(\frac{1}{w} \right)^{kn-p} \sum_{q=0}^{\ell(m-n)} \binom{\ell(m-n)}{q} w^q \left(-\frac{1}{w} \right)^{\ell(m-n)-q} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n=0}^m \binom{m}{n} \left(\frac{\ln \alpha}{2^k} \right)^n \left[\frac{\ln \beta}{(2i)^\ell} \right]^{m-n} \sum_{p=0}^{kn} \binom{kn}{p} w^{2p-kn} \sum_{q=0}^{\ell(m-n)} \binom{\ell(m-n)}{q} (-1)^{\ell(m-n)-q} w^{2q-\ell(m-n)} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n=0}^m \binom{m}{n} \left(\frac{\ln \alpha}{2^k} \right)^n \left[\frac{\ln \beta}{(2i)^\ell} \right]^{m-n} \frac{(-1)^{\ell(m-n)}}{w^{kn+\ell(m-n)}} \sum_{p=0}^{kn} \binom{kn}{p} w^{2p} \sum_{q=0}^{\ell(m-n)} \binom{\ell(m-n)}{q} (-1)^q w^{2q} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n=0}^m \binom{m}{n} \left(\frac{\ln \alpha}{2^k} \right)^n \left[\frac{\ln \beta}{(2i)^\ell} \right]^{m-n} \frac{(-1)^{\ell(m-n)}}{w^{kn+\ell(m-n)}} \sum_{j=0}^{kn+\ell(m-n)} \left[\sum_{r=0}^j (-1)^{j-r} \binom{kn}{r} \binom{\ell(m-n)}{j-r} \right] w^{2j}. \end{aligned}$$

Therefore, by Definition 1, we deduce

$$\begin{aligned} \text{Res} \left[\frac{\alpha^{[(w+w^{-1})/2]^k} \beta^{[(w-w^{-1})/(2i)]^\ell}}{w}, 0 \right] &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n=0}^m \binom{m}{n} \left(\frac{\ln \alpha}{2^k} \right)^n \left[\frac{\ln \beta}{(2i)^\ell} \right]^{m-n} \\ &\times (-1)^{\ell(m-n)} \frac{1 + (-1)^{kn+\ell(m-n)}}{2} \sum_{r=0}^{[kn+\ell(m-n)]/2} (-1)^{[kn+\ell(m-n)]/2-r} \binom{kn}{r} \binom{\ell(m-n)}{[kn+\ell(m-n)]/2-r}. \end{aligned}$$

Substituting this result into (12) leads to the identity (11). The proof of Theorem 4 is thus complete. $\square$

5. Two New Series Expressions for I_0

Combining the Equation (8) in Theorem 3 with the identities (2) and (1) in sequence, we can find two series expressions for I_0 as follows.

Theorem 5. For $x \in \mathbb{R}$, we have

$$I_0(x) = \sum_{j=0}^{\infty} (-1)^j \binom{-1/2}{j} \frac{x^{2j}}{(2j)!}. \quad (13)$$

Proof. When taking $k = 1$, the right hand side of the Equation (8) in Theorem 3 becomes

$$\begin{aligned} & \sum_{j=0}^{\infty} (-1)^j \frac{(\ln \beta)^{2j}}{(2j)!} \sum_{q=0}^{2j} (-i)^q \binom{2j}{q} \binom{(q-1)/2}{j} \left(\frac{\ln \alpha}{\ln \beta} \right)^q \\ &= \sum_{j=0}^{\infty} (-1)^j \frac{(\ln \beta)^{2j}}{(2j)!} \sum_{q=0}^{2j} (-1)^q e^{q\pi i/2} \binom{2j}{q} \binom{(q-1)/2}{j} \left(\frac{\ln \alpha}{\ln \beta} \right)^q \\ &= \sum_{j=0}^{\infty} (-1)^j \frac{(\ln \beta)^{2j}}{(2j)!} \sum_{q=0}^{2j} (-1)^q \left(\cos \frac{q\pi}{2} + i \sin \frac{q\pi}{2} \right) \binom{2j}{q} \binom{(q-1)/2}{j} \left(\frac{\ln \alpha}{\ln \beta} \right)^q \\ &= \sum_{j=0}^{\infty} (-1)^j \frac{(\ln \beta)^{2j}}{(2j)!} \sum_{q=0}^j (-1)^q \binom{2j}{2q} \binom{(2q-1)/2}{j} \left(\frac{\ln \alpha}{\ln \beta} \right)^{2q} \\ &\quad + i \sum_{j=0}^{\infty} (-1)^j \frac{(\ln \beta)^{2j}}{(2j)!} \sum_{q=0}^{j-1} (-1)^{q+1} \binom{2j}{2q+1} \binom{q}{j} \left(\frac{\ln \alpha}{\ln \beta} \right)^{2q+1} \\ &= \sum_{q=0}^{\infty} (-1)^q \left[\sum_{j=q}^{\infty} (-1)^j \frac{(\ln \beta)^{2j}}{(2j)!} \binom{2j}{2q} \binom{(2q-1)/2}{j} \right] \left(\frac{\ln \alpha}{\ln \beta} \right)^{2q} \\ &\quad - i \sum_{q=0}^{\infty} (-1)^q \left[\sum_{j=q+1}^{\infty} \frac{(-1)^j}{(2j)!} \binom{2j}{2q+1} \binom{q}{j} (\ln \beta)^{2j} \right] \left(\frac{\ln \alpha}{\ln \beta} \right)^{2q+1} \\ &= \sum_{q=0}^{\infty} (-1)^q \left[\sum_{j=q}^{\infty} \frac{(-1)^j}{(2j)!} \binom{2j}{2q} \binom{(2q-1)/2}{j} (\ln \beta)^{2j} \right] \left(\frac{\ln \alpha}{\ln \beta} \right)^{2q}, \end{aligned}$$

where any empty sum is taken to be zero and $\binom{r}{s} = 0$ for $s > r \geq 0$ and $r, s \in \mathbb{N}$. This means that

$$I_0\left(\sqrt{(\ln \alpha)^2 + (\ln \beta)^2}\right) = \sum_{q=0}^{\infty} (-1)^q \left[\sum_{j=q}^{\infty} \frac{(-1)^j}{(2j)!} \binom{2j}{2q} \binom{(2q-1)/2}{j} (\ln \beta)^{2j} \right] \left(\frac{\ln \alpha}{\ln \beta} \right)^{2q}.$$

Further letting $\alpha \rightarrow 1$ and $\beta = e^x$ for $x \in \mathbb{R}$ reduces to (13). The proof of Theorem 5 is complete. $\square$

Theorem 6. For $x \in \mathbb{R}$, we have

$$I_0(x) = \frac{1}{\cosh x} \sum_{j=0}^{\infty} \binom{-1/2}{2j} \frac{(2x)^{2j}}{(2j)!}. \quad (14)$$

Proof. When taking $k = 2$, the right hand side of the Equation (8) in Theorem 3 becomes

$$\begin{aligned} & \sum_{j=0}^{\infty} (-1)^{2j} \frac{(\ln \beta)^{2j}}{(2j)!} \sum_{q=0}^{2j} (-i)^{2q} \binom{2j}{q} \binom{(2q-1)/2}{2j} \left(\frac{\ln \alpha}{\ln \beta} \right)^q \\ &= \sum_{j=0}^{\infty} \frac{(\ln \beta)^{2j}}{(2j)!} \sum_{q=0}^{2j} (-1)^q \binom{2j}{q} \binom{(2q-1)/2}{2j} \left(\frac{\ln \alpha}{\ln \beta} \right)^q. \end{aligned}$$

This implies that

$$\sqrt{\alpha\beta} I_0\left(\ln \sqrt{\frac{\alpha}{\beta}}\right) = \sum_{j=0}^{\infty} \frac{(\ln \beta)^{2j}}{(2j)!} \sum_{q=0}^{2j} (-1)^q \binom{2j}{q} \binom{(2q-1)/2}{2j} \left(\frac{\ln \alpha}{\ln \beta}\right)^q.$$

Further taking $\alpha \rightarrow 1$ reveals

$$\sqrt{\beta} I_0\left(\ln \sqrt{\frac{1}{\beta}}\right) = \sum_{j=0}^{\infty} \frac{(\ln \beta)^{2j}}{(2j)!} \binom{-1/2}{2j}.$$

Again letting $\beta = e^{\pm 2x}$ results in

$$e^{\pm x} I_0(x) = \sum_{j=0}^{\infty} \frac{(\pm 2x)^{2j}}{(2j)!} \binom{-1/2}{2j}$$

which can be rewritten as (14). The proof of Theorem 6 is complete. $\square$

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