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Comparison of the Orthogonal Polynomial Solutions for Fractional Integral Equations

Ayşegül Daşcıoğlu * and Serpil Salınan

Pamukkale University, Faculty of Art and Science, Department of Mathematics, Denizli 20160, Turkey; ssalinan@pau.edu.tr

* Correspondence: aakyuz@pau.edu.tr; Tel.: +90-258-296-3621

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Abstract: In this paper, a collocation method based on the orthogonal polynomials is presented to solve the fractional integral equations. Six numerical examples are given to illustrate the method. The results are compared with the other methods in the literature, and the results obtained by different kinds of polynomials are compared.

Keywords: fractional integral equations; Abel integral equations; Volterra integral equations; collocation method; singular integral equation

1. Introduction

Fractional analysis is used in many fields such as fluid flow of science and engineering, rheology, electromagnetic theory, and probability [1–3]. Fractional derivatives and fractional integrals are the generalization of these to non-integer arbitrary order. These terms entered into the literature with the letter that Leibniz wrote to L'Hôpital in 1695 [4]. There are different definitions of fractional integral [5], some of which—defined in $(0,\infty)$ and sometimes called left-sided integrals—are given below:

Definition 1 [6]. Let $f(x)$ be piecewise continuous on $(0,\infty)$ and an integrable function on any finite subinterval of $[0,\infty)$ and let $\mathrm{Re}(\alpha) > 0$. Then, for $x > a \geq 0$, the Riemann–Liouville fractional integral of order α is defined by

$$ {}_aD_x^{-\alpha} f(x) = I^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt. $$

Definition 2 [7]. Let $\alpha > 0$, $\mu \in \mathbb{C}$ and $x > a \geq 0$, then

$$ I^{\alpha}_{a+,\mu} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\frac{t}{x}\right)^{\mu} \left(\log\frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t}\, dt\,, $$

is called a Hadamard-type fractional integral.

Definition 3 [4]. Let (a,b) $(-\infty \leq a < b \leq \infty)$ be a finite or infinite interval of the half-axis $\mathbb{R}^+$. Also, let $\rho > 0$, $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, and $\mu \in \mathbb{C}$. Then Erdélyi-Kober type fractional integrals of order α are defined by

$$ I^{\alpha}_{a+;\rho,\mu} f(x) := \frac{\rho x^{-\rho(\alpha+\mu)}}{\Gamma(\alpha)} \int_a^x (x^{\rho} - t^{\rho})^{\alpha-1} t^{\rho\mu+\rho-1} f(t)\, dt, a \geq 0. $$

Definition 4 [8]. The fractional version which generalizes both Hadamard and Riemann–Liouville fractional integrals into a single form is called Katugampola fractional integral, and is defined by

$$ {}^{\rho}_{a}I^{\alpha}_{x}f(x) := \frac{(\rho+1)^{1-\alpha}}{\Gamma(\alpha)} \int\limits_{a}^{x} \left(x^{\rho+1} - t^{\rho+1}\right)^{\alpha-1} t^{\rho} f(t)dt,\ \alpha \in \mathbb{R},\ \rho \in \mathbb{R}\backslash\{-1\}. $$

A fractional integral equation is an integral equation which contains fractional integrals. The thought of a fractional integration is related to Abel's integral equation which is defined as follows:

$$ g(x) = \int\limits_{a}^{x} \frac{1}{(x-t)^{\alpha}} f(t)dt, 0 < \alpha < 1. $$

Generally, when $\alpha = 1/2$, this equation is called Abel's integral equation; in other cases, the equation is called generalized Abel's integral equation. These equations belong to the class of Volterra equations of the first kind. The second kind Abel integral equation is also called the weakly-singular second kind Volterra type integral equations in the form

$$ f(x) = g(x) + \lambda \int\limits_{a}^{x} \frac{1}{\sqrt{x-t}} f(t)dt,\ x \in [a,b], $$

where λ and b are constants. These equations occur in different fields of the sciences.

The solution methods of the Abel integral equations are collocation methods [9–13], Adomian decomposition methods [14–16], homotopy perturbation methods [17], quadrature methods [18–20], homotopy analysis methods [21], and Laplace transform methods [22–26]. Besides, Abel integral equations are also solved by using Chebyshev [27–29], Legendre [30,31], Taylor [32], Bernstein [33,34], Block-Pulse [35], and Laguerre [36] functions. Moreover, numerical methods on other fractional integral equations are hybrid collocation [37], smoothing technique [38], piecewise constant orthogonal functions approximation [39], the Haar wavelet method [40], the Galerkin method [41], Bernstein's approximation [42,43], the Simpson 3/8 rule method [44], mechanical quadrature [45], Legendre Pseudo spectral [46], and the iterative numerical method [47].

The aim of this study is to develop a collocation method, and to give a comparison for solving the fractional integral equations with orthogonal polynomials such as Jacobi, Legendre, Chebyshev, Hermite, and Laguerre polynomials. More details about these orthogonal polynomials can be found in [48–50]. The proposed method is simple, fast and a direct method.

2. Description of the Method

In this work, the solution of the following Volterra integral equation will be investigated:

$$ c(x)y(x) = g(x) + \lambda \int\limits_{a}^{x} K(x,t)y(t)dt,\ 0 \leq a \leq x \leq b. \tag{1} $$

If $c(x) \equiv 0$ and $c(x) \equiv 1$, Equation (1) is called Volterra integral equations of the first kind and of the second kind, respectively. Otherwise, Equation (1) is called Volterra integral equations of the third kind. Here, $K(x,t)$ is the kernel function, $g(x)$ is the free term, λ is a constant, and $y(x)$ is the unknown function. If the integral next to λ in Equation (1) is taken as in Definitions 1–4, then Equation (1) becomes a fractional integral equation in the form:

$$ c(x)y(x) = g(x) + h(x)\ I^{\alpha}y(x),\ 0 \leq a \leq x \leq b. \tag{2} $$

In this section, a collocation method which requires neither opening the kernel in series nor the calculation of the numerical integral, is developed for the Volterra integral Equation (1). At the same time, this method is valid for the fractional integral Equation (2). In the next chapter, problems according to Equation (2) will be solved, but the method is presented for the general form Equation (1).

Assume that Equation (1) has the polynomial solution in the form:

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n \, \phi_n(x), \tag{3}$$

where a_n are unknown constants and $\phi_n(x)$ are the orthogonal polynomials of degree n such as $P_n^{(\alpha,\beta)}(x),\ P_n(x),\ T_n(x),\ U_n(x),\ H_n(x),\ L_n(x)$. The matrix form of truncated series is written by

$$y_N(x) = \boldsymbol{\phi}(x)\mathbf{A}, \tag{4}$$

such that $\boldsymbol{\phi}(x) = \begin{bmatrix} \phi_0(x) & \phi_1(x) & \dots & \phi_N(x) \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} a_0 & a_1 & \dots & a_N \end{bmatrix}^T$.

Substituting matrix form (4) and the collocation points into Equation (1), and then simplifying the equation, we have the system

$$\{c(x_s)\boldsymbol{\phi}(x_s) - \lambda \, \mathbf{v}(x_s)\}\mathbf{A} = g(x_s), \; s = 0, 1, \dots, N,$$

where

$$\mathbf{v}(x) = \int_a^x K(x,t)\boldsymbol{\phi}(t)\, dt = \begin{bmatrix} v_0(x) & v_1(x) & \dots & v_N(x) \end{bmatrix}.$$

The matrix form of this linear system becomes

$$(\mathbf{C}\boldsymbol{\Phi} - \lambda\mathbf{V})\mathbf{A} = \mathbf{G}. \tag{5}$$

Here, the matrices are as follows:

$$\mathbf{C} = \begin{bmatrix} c(x_0) & 0 & \dots & 0 \\ 0 & c(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c(x_N) \end{bmatrix}, \; \boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\phi}(x_0) \\ \boldsymbol{\phi}(x_1) \\ \vdots \\ \boldsymbol{\phi}(x_N) \end{bmatrix}, \; \mathbf{V} = \begin{bmatrix} \mathbf{v}(x_0) \\ \mathbf{v}(x_1) \\ \vdots \\ \mathbf{v}(x_N) \end{bmatrix}, \; \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}.$$

Unknown matrix $\mathbf{A}$ is determined by solving system Equation (5). Therefore, the solution of integral Equation (1) is obtained by substituting these coefficients in Equation (3).

3. Numerical Examples

In this section, six numerical examples are given to illustrate the method for fractional integral Equation (2). The solutions are obtained by using Laguerre, Hermit, Jacobi, Legendre and Chebyshev polynomials, and compared with the results in the literature. These problems are solved by the codes written in Matlab R2015a MATLAB (The MathWorks, Inc., Natick, MA, USA) and Mathcad 15 (PTC, Needham, MA 02494, USA). Maximum error and mean absolute error, which are used in the examples, are given by the formulas:

$$E_{\max} = \max_{0 \le s \le N} |y(x_s) - y_N(x_s)| \text{ and } E_{\text{mean}} = \frac{1}{N+1}\sum_{s=0}^{N} |y(x_s) - y_N(x_s)|.$$

Besides, for the range $(0, R]$ and $s = 0, 1, \dots, N$, the collocation points used in the examples are in the following three formats:

(a) $x_s = \frac{s+1}{N+1} \cdot R$ (Equally spaced points)

(b) $x_s = \frac{R}{2}\left[1 - \cos\left(\frac{(s+1)\pi}{N+1}\right)\right]$ (extreme points of $T_{N+1}\left(\frac{2}{T}x - 1\right)$)

(c) $x_s = \frac{R}{2}\left[1 - \cos\left(\frac{(2s+1)\pi}{2(N+1)}\right)\right]$ (the zeros of $T_{N+1}\left(\frac{2}{T}x - 1\right)$)

If the range includes the zero and R, then $s+1$ and N+1 are respectively replaced by s and N in (a) and (b).

Example 1. Consider the following Abel's integral equation

$$\int_0^x \frac{1}{\sqrt{x-t}} y(t)dt = \frac{2}{105}\sqrt{x}\left(105 - 56x^2 + 48x^3\right),\ 0 \le x \le 1,$$

with the Riemann–Liouville fractional form

$$\sqrt{\pi} I^{1/2} y(x) = \frac{2}{105}\sqrt{x}\left(105 - 56x^2 + 48x^3\right),\ 0 \le x \le 1,$$

which has the exact solution $y(x) = x^3 - x^2 + 1$. By applying the presented method, the main matrix relation is $\mathbf{VA} = \mathbf{G}$. For $N = 3$, using Laguerre polynomials and the collocation points $x_0 = 0.25,\ x_1 = 0.5,\ x_2 = 0.75,\ x_3 = 1$, the matrices in the main equation become

$$\mathbf{V} = \begin{bmatrix} 1 & \frac{5}{6} & \frac{41}{60} & \frac{461}{840} \\ \sqrt{2} & \frac{2\sqrt{2}}{3} & \frac{2\sqrt{2}}{5} & \frac{4\sqrt{2}}{21} \\ \sqrt{3} & \frac{\sqrt{3}}{2} & \frac{3\sqrt{3}}{20} & -\frac{23\sqrt{3}}{280} \\ 2 & \frac{2}{3} & -\frac{2}{15} & -\frac{58}{105} \end{bmatrix}, \mathbf{G} = \begin{bmatrix} \frac{409}{420} \\ \frac{97\sqrt{2}}{105} \\ \frac{25\sqrt{3}}{28} \\ \frac{194}{105} \end{bmatrix}.$$

Solving the system by symbolic calculation in Mathcad 15, we obtain $\mathbf{A} = \begin{bmatrix} 5 & -14 & 16 & -6 \end{bmatrix}^T$, and then we have the exact solution. We obtained the same result by using Matlab R2015a.

By using the Tau method [10], the Chebyshev polynomials [28], the Legendre wavelets [30], the shifted Legendre collocation method [31], and the Bernstein polynomial multiwavelets [33], an exact solution was found for $N = 3$. Besides, Sohrabi [29] also obtained the exact solution by applying the Chebyshev wavelets method with $N = 4$. Yang [23] also found the exact solution by using Laplace transform symbolic calculus in Mathematica. Moreover, Maleknejad et al. [43] and Rahman et al. [36] obtained the absolute errors of order 10^{-7} and 10^{-16} by using the Bernstein's approximation and Laguerre polynomials for $N = 10$, respectively. Furthermore, Noeiaghdam et al. [25] found the absolute errors of order 10^{-7} by using the homotopy analysis transform method for $x = 1$ and $N = 20$. Therefore, our method is more accurate and faster than the other methods.

Example 2. Consider the weakly singular second kind of Volterra integral equation of the form

$$y(x) = x + \frac{4}{3}x^{3/2} - \int_0^x \frac{y(t)}{\sqrt{x-t}} dt, 0 \le x \le 1\,,$$

with the Riemann-Liouville fractional form

$$y(x) = x + \frac{4}{3}x^{3/2} - \sqrt{\pi} I^{1/2} y(x),\ 0 \le x \le 1,$$

which has the exact solution $y(x) = x$. By applying the method, the main matrix relation is obtained in the form $(\mathbf{\Phi} + \mathbf{V})\mathbf{A} = \mathbf{G}$. For $N = 1$, using Laguerre polynomials and the collocation points $x_0 = 0,\ x_1 = 1$, the matrices in the main equation become

$$\mathbf{\Phi} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \mathbf{V} = \begin{bmatrix} 0 & 0 \\ 2 & \frac{2}{3} \end{bmatrix}, \mathbf{G} = \begin{bmatrix} 0 \\ \frac{7}{3} \end{bmatrix}.$$

Solving the system by symbolic calculation in Mathcad 15 and Matlab R2015a, we obtain $\mathbf{A} = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$, and then we have the exact solution.

By using the homotopy perturbation method, Pandey et al. [17] obtained maximum absolute and relative errors of nearly 0.005 and 0.5, respectively for $N = 18$. They also used the Adomian decomposition method and found the approximate solution with $O(x^{2+N/2})$ error. Besides, they obtained the exact solution by using both modified the homotopy perturbation and modified Adomian decomposition methods. Li and Clarkson **[26]** also applied Babenko's approach to find the exact solution. In addition to these, Avazzadeh et al. [28] and Abdelkawy et al. [11] found the exact solution for $N \geq 1$ by using Chebyshev polynomials and spectral collocation methods, respectively. Moreover, Singh et al. [21] obtained analytical approximate solutions with nearly 10^{-10} absolute errors by using the homotopy analysis method for $N = 14$ and different convergence parameters. Furthermore, Kumar et al. [24] obtained the maximum absolute error 7.1×10^{-5} by using the homotopy perturbation transform method for $N = 25$. Therefore, our method is better than the homotopy methods, and is faster than the other methods that have the same accuracy.

Example 3. Consider the Volterra integral equation

$$\frac{2}{3}\pi x^3 = \int_0^x \frac{1}{\sqrt{x^2 - t^2}} y(t)dt,\ 0 < x < 2\,,$$

with the Erdélyi-Kober type fractional form

$$\frac{4}{3}\sqrt{\pi}x^3 = I^{1/2}_{0+;2,-1/2} y(x),\ 0 < x < 2,$$

which has the exact solution $y(x) = \pi x^3$. By applying the method, the main matrix equation becomes $\mathbf{VA} = \mathbf{G}$. For $N = 3$, using Legendre polynomials and the collocation points $x_0 = 0.5$, $x_1 = 1$, $x_2 = 1.5$, $x_3 = 2$, yields the matrices

$$\mathbf{V} = \begin{bmatrix} \frac{\pi}{2} & \frac{1}{2} - \frac{\pi}{2} & \frac{19\pi}{32} - \frac{3}{2} & \frac{77}{24} - \frac{31\pi}{32} \\ \frac{\pi}{2} & 1 - \frac{\pi}{2} & \frac{7\pi}{8} - 3 & \frac{23}{3} - \frac{19\pi}{8} \\ \frac{\pi}{2} & \frac{3}{2} - \frac{\pi}{2} & \frac{43\pi}{32} - \frac{9}{2} & \frac{117}{8} - \frac{151\pi}{32} \\ \frac{\pi}{2} & 2 - \frac{\pi}{2} & 2\pi - 6 & \frac{76}{3} - 8\pi \end{bmatrix}, \mathbf{G} = \begin{bmatrix} \pi/12 \\ 2\pi/3 \\ 9\pi/4 \\ 16\pi/3 \end{bmatrix}.$$

Solving this system by symbolic calculation in Matlab R2015a, we obtain $\mathbf{A} = \begin{bmatrix} 2\pi & 18\pi/5 & 2\pi & 2\pi/5 \end{bmatrix}^T$, and then we have the exact solution.

Example 4. Consider the Volterra integral equation

$$\int_0^x \frac{x^2t^3 + t^4 + 1}{(x-t)^{1/4}} y(t)dt = g(x),\ 0 \leq x \leq 1,$$

with the Erdélyi-Kober type fractional form

$$\Gamma(\frac{3}{4}) \left[x^{23/4}\, I^{3/4}_{0+;1,3} y(x) + x^{19/4}\, I^{3/4}_{0+;1,4} y(x) + x^{3/4} I^{3/4}_{0+;1,0} y(x) \right] = g(x)\,,$$

where $g(x) = \frac{32768}{100947}x^{31/4} + \frac{262144}{908523}x^{27/4} + \frac{128}{231}x^{11/4}$. The exact solution is $y(x) = x^2$. By applying the method, the main matrix equation is gained in the form $\mathbf{VA} = \mathbf{G}$. For $N = 2$, using Laguerre polynomials and the collocation points $x_0 = 1/3,\ x_1 = 2/3,\ x_2 = 1$, we solve the system by symbolic calculation in Mathcad 15. We obtain $a_0 = 2,\ a_1 = -4,\ a_2 = 2$, and then we have the exact solution. Besides, we have mean absolute errors 1.1×10^{-17} by using Matlab R2015a for $N = 2$.

Liu and Tao [18] obtained the maximum errors 1×10^{-3}, 8×10^{-3}, 4×10^{-3}, 5×10^{-5}, 5×10^{-3} by using the mid-point, trapezoidal quadrature, their average, combination results and a posteriori estimate methods for $h = 0.025$, respectively. Besides, Liu and Tao [19] presented the absolute errors 4×10^{-5}, 1×10^{-4}, 1×10^{-6}, 2×10^{-3} by the mid-point, trapezoidal quadrature, their Richardson extrapolation, and the product-integration methods, respectively, when $h = 0.025$ and $x = 1$. They also obtained the absolute errors 1×10^{-5}, 3×10^{-5}, 2×10^{-8}, 7×10^{-4} by the same methods when $h = 0.0125$ and $x = 1$. They used 40 and 80 points whereas we used 3 points to obtain the exact solution. The presented method is more accurate and faster than the mentioned quadrature methods. Moreover, Abdelkawy et al. [11] found the exact solution for $N \geq 2$ by using the spectral collocation method. Our method is more practical than these mentioned methods.

Example 5. Consider the Volterra integral equation

$$\frac{6}{25}x^{25/6} = \int_0^x \frac{1}{(x^5 - t^5)^{1/6}} y(t)dt,\ 0 < x < 2,$$

with the Erdélyi-Kober type fractional form

$$\frac{6}{5\Gamma\left(\frac{5}{6}\right)}x^4 = I^{5/6}_{0+;5,-4/5}y(x),\ 0 < x < 2,$$

which has the exact solution $y(x) = x^4$.

The maximum and mean absolute errors calculated in Matlab R2015a for $N = 4$ are given in Table 1 by using the presented method. It is observed that the errors for almost all polynomials are better using points (c). In addition, it is seen that the maximum and the mean absolute errors obtained by using the Hermit polynomials for collocation points (a) and (c), and by using the Chebyshev polynomials of the second kind for collocation point (b), are smaller than the other polynomials. Although the exact solution is a polynomial, we have the approximate solution since the integral part cannot be calculated symbolically.

Table 1. Maximum and mean absolute errors for Example 5.

Orthogonal Polynomials	**Maximum Error**			**Mean absolute Error**		
	a	**b**	**c**	**a**	**b**	**c**
Laguerre	3.47×10^{-14}	2.17×10^{-14}	1.45×10^{-14}	1.22×10^{-14}	1.17×10^{-14}	7.65×10^{-15}
Hermit	9.56×10^{-16}	4.77×10^{-15}	7.37×10^{-16}	4.12×10^{-16}	1.40×10^{-15}	2.40×10^{-16}
Jacobi (0,1)	6.18×10^{-15}	2.79×10^{-15}	1.94×10^{-15}	1.95×10^{-15}	1.59×10^{-15}	1.17×10^{-15}
Jacobi (1,1)	3.95×10^{-15}	2.12×10^{-15}	2.12×10^{-15}	1.50×10^{-15}	8.08×10^{-16}	7.23×10^{-16}
Legendre	4.78×10^{-15}	2.11×10^{-15}	1.99×10^{-15}	1.63×10^{-15}	1.03×10^{-15}	5.26×10^{-16}
Chebyshev (first kind)	6.84×10^{-15}	3.13×10^{-15}	3.09×10^{-15}	2.09×10^{-15}	1.17×10^{-15}	7.86×10^{-16}
Chebyshev (second kind)	1.48×10^{-15}	1.47×10^{-15}	3.45×10^{-15}	6.83×10^{-16}	6.67×10^{-16}	8.61×10^{-16}

Example 6. Consider the singular Volterra integral equation

$$\frac{4}{3}(\sin x)^{3/4} = \int_0^x \frac{1}{(\sin x - \sin t)^{1/4}} y(t)dt,\ 0 < x < \frac{\pi}{2},$$

which has the exact solution $y(x) = \cos x$. This equation cannot be expressed in the fractional form because of the existing definitions of the fractional integrals. However, it may belong to one of the general forms of Abel's integral equation.

By using the presented method for seven different polynomials and three different collocation points (a–c), the maximum errors are given in Tables 2–4, respectively. The results are calculated numerically in Mathcad 15. Unfortunately, the integral part cannot be calculated in Matlab R2015a.

Table 2. Maximum error for Example 6 using the collocation point (a).

Orthogonal Polynomials	$N = 2$	$N = 3$	$N = 4$	$N = 5$	$N = 6$	$N = 7$	$N = 8$	$N = 9$	$N = 10$
Laguerre	1.4×10^{-2}	1.2×10^{-3}	1.5×10^{-4}	1.3×10^{-5}	1.6×10^{-6}	5.7×10^{-5}	5.7×10^{-8}	9.9×10^{-6}	6.5×10^{-7}
Hermit	1.4×10^{-2}	1.2×10^{-3}	1.8×10^{-4}	1.2×10^{-5}	1.2×10^{-6}	6.7×10^{-8}	2.4×10^{-6}	4.5×10^{-9}	5.2×10^{-9}
Jacobi (0,1)	1.4×10^{-2}	1.2×10^{-3}	1.8×10^{-4}	1×10^{-5}	2.6×10^{-6}	6.5×10^{-8}	1.3×10^{-5}	7.7×10^{-8}	5×10^{-4}
Jacobi (1,1)	1.4×10^{-2}	1.2×10^{-3}	1.8×10^{-4}	1.3×10^{-5}	1.2×10^{-6}	3.8×10^{-6}	2.2×10^{-9}	1.2×10^{-5}	3.5×10^{-6}
Legendre	1.4×10^{-2}	1.2×10^{-3}	1.8×10^{-4}	1.3×10^{-5}	1.2×10^{-6}	3.9×10^{-6}	3.9×10^{-7}	1×10^{-5}	4.3×10^{-9}
1. Chebyshev	1.4×10^{-2}	1.2×10^{-3}	1.8×10^{-4}	9.3×10^{-6}	1.2×10^{-6}	3.9×10^{-6}	1.3×10^{-6}	1.3×10^{-5}	4.2×10^{-4}
2. Chebyshev	1.4×10^{-2}	1.2×10^{-3}	1.8×10^{-4}	1.3×10^{-5}	1.2×10^{-6}	4×10^{-6}	1.4×10^{-7}	1.3×10^{-5}	4.5×10^{-9}

Table 3. Maximum error for Example 6 using the collocation point (b).

Orthogonal Polynomials	$N = 2$	$N = 3$	$N = 4$	$N = 5$	$N = 6$	$N = 7$	$N = 8$	$N = 9$	$N = 10$
Laguerre	1.2×10^{-2}	1.4×10^{-3}	1×10^{-4}	1.8×10^{-5}	2.8×10^{-5}	2.2×10^{-8}	3.9×10^{-6}	8.8×10^{-7}	4.8×10^{-9}
Hermit	1.2×10^{-2}	1.4×10^{-3}	1×10^{-4}	8.2×10^{-6}	4.3×10^{-7}	2.4×10^{-8}	6.1×10^{-9}	7.4×10^{-9}	5.8×10^{-9}
Jacobi (0,1)	1.2×10^{-2}	1.4×10^{-3}	1×10^{-4}	8.2×10^{-6}	4.3×10^{-7}	2.5×10^{-8}	4.3×10^{-9}	4.7×10^{-6}	3.4×10^{-7}
Jacobi (1,1)	1.2×10^{-2}	1.4×10^{-3}	1×10^{-4}	8×10^{-6}	4.3×10^{-7}	4.3×10^{-7}	1.2×10^{-8}	5×10^{-7}	1.9×10^{-8}
Legendre	1.2×10^{-2}	1.4×10^{-3}	1×10^{-4}	8×10^{-6}	4.3×10^{-7}	4.5×10^{-7}	4.9×10^{-9}	6.1×10^{-7}	6.6×10^{-9}
1. Chebyshev	1.2×10^{-2}	1.4×10^{-3}	1×10^{-4}	8×10^{-6}	4.3×10^{-7}	4×10^{-7}	4.6×10^{-9}	5.2×10^{-7}	7.7×10^{-6}
2. Chebyshev	1.2×10^{-2}	1.4×10^{-3}	1×10^{-4}	8×10^{-6}	4.3×10^{-7}	4.5×10^{-7}	6×10^{-8}	5.2×10^{-7}	6.8×10^{-9}

Table 4. Maximum error for Example 6 using the collocation point (c).

Orthogonal Polynomials	$N = 2$	$N = 3$	$N = 4$	$N = 5$	$N = 6$	$N = 7$	$N = 8$	$N = 9$	$N = 10$
Laguerre	2.6×10^{-2}	3.2×10^{-3}	3.3×10^{-4}	2.9×10^{-5}	1.6×10^{-6}	1.8×10^{-7}	1.5×10^{-8}	1.6×10^{-7}	2.1×10^{-6}
Hermit	2.6×10^{-2}	3.2×10^{-3}	3.3×10^{-4}	2.4×10^{-5}	1.6×10^{-6}	3.5×10^{-7}	5.4×10^{-9}	2.6×10^{-9}	3.6×10^{-9}
Jacobi (0,1)	2.6×10^{-2}	3.2×10^{-3}	3.3×10^{-4}	2.4×10^{-5}	1.6×10^{-6}	1.1×10^{-7}	2.6×10^{-6}	3×10^{-7}	1.6×10^{-5}
Jacobi (1,1)	2.6×10^{-2}	3.2×10^{-3}	3.3×10^{-4}	2.4×10^{-5}	9.8×10^{-7}	8.4×10^{-8}	6.8×10^{-7}	2.8×10^{-7}	8.3×10^{-7}
Legendre	2.6×10^{-2}	3.2×10^{-3}	3.3×10^{-4}	2.4×10^{-5}	9.1×10^{-7}	8.3×10^{-8}	6.6×10^{-7}	2×10^{-9}	5.1×10^{-6}
1. Chebyshev	2.6×10^{-2}	3.2×10^{-3}	3.3×10^{-4}	2.4×10^{-5}	9.6×10^{-7}	9.7×10^{-8}	9.1×10^{-7}	5.5×10^{-7}	8.1×10^{-7}
2. Chebyshev	2.6×10^{-2}	3.2×10^{-3}	3.3×10^{-4}	2.4×10^{-5}	9.6×10^{-7}	8.4×10^{-8}	7×10^{-7}	2.1×10^{-8}	7.8×10^{-7}

It is observed from Tables 2–4 that the results obtained using the collocation points (b) are generally better than the collocation points (a) and (c). Furthermore, it is observed that errors have decreased gradually to $N = 7$, and then no such arrangement is observed. Therefore, it is not possible to say which polynomial gives the best results.

4. Conclusions

In the present study, a collocation method based on the orthogonal polynomials has been presented to solve Volterra integral equations which contain fractional, Abel and singular integrals. At the same time, these methods are valid for the first, second and third kind of integral equations. When we were solving these, we used three types of collocation points for seven kinds of polynomials. We solved six numerical examples, of which five were fractional integral equations and one was a singular fractional integral equation, by using the codes written in Mathcad 15 and Matlab R2015a. Then, we compared some of our solutions with the results in the literature. The maximum and the mean absolute errors are presented in Tables 1–4.

If the solution was a polynomial for these kinds of equations, we achieved the exact solution by symbolic calculation in both Mathcad 15 and Matlab R2015a. We obtained a numerical approximation in non-polynomial solutions. In the numerical examples, we obtained good approximations with low terms of N. Approximate solutions have a good degree of accuracy. The numerical results show the validity and applicability of the method. Their accuracy is high and comparable with existing methods.

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