Article

Time-Space Fractional Coupled Generalized Zakharov-Kuznetsov Equations Set for Rossby Solitary Waves in Two-Layer Fluids

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Abstract: In this paper, the theoretical model of Rossby waves in two-layer fluids is studied. A single quasi-geostrophic vortex equation is used to derive various models of Rossby waves in a one-layer fluid in previous research. In order to explore the propagation and interaction of Rossby waves in two-layer fluids, from the classical quasi-geodesic vortex equations, by employing the multi-scale analysis and turbulence method, we derived a new (2+1)-dimensional coupled equations set, namely the generalized Zakharov-Kuznetsov(gZK) equations set. The gZK equations set is an extension of a single ZK equation; they can describe two kinds of weakly nonlinear waves interaction by multiple coupling terms. Then, for the first time, based on the semi-inverse method and the variational method, a new fractional-order model which is the time-space fractional coupled gZK equations set is derived successfully, which is greatly different from the single fractional equation. Finally, group solutions of the time-space fractional coupled gZK equations set are obtained with the help of the improved (G'/G)-expansion method.

Keywords: time-space fractional coupled generalized Zakharov-Kuznetsov equations set; Rossby waves; quasi-geostrophic vortex equations; two-layer fluids

1. Introduction

Although the existence of solitary waves has been known in hydrokinetics for about a century, it was not until recently that the theory was applied to wave phenomena in the atmospheric, ocean [1–6] and large lake dynamics system, such as solitary waves [7], internal gravity waves [8], internal Kelvin waves [9] and so on. Since Long(1964) derived the Korteweg de-Vries(KdV) equation on the positive pressure system, the isolated Rossby wave theory has been gradually developed [10–12], but in many complex atmospheric and oceanic systems, waves interact with each other. The problem of wave-wave interaction is a very important one in the earth fluid, and further study can deepen people's understanding of large-scale motion phenomena in the atmosphere, ocean and large lakes. There are many motion equations describing Rossby waves in a layer of fluid, such as one-dimensional KdV equation [13], MKdV equation [14], Boussinesq equation [15] and high-dimensional ZK equation [16], mZK equation [17], KP equation [18] and so on. The actual atmospheric ocean system is very complex, compared with the single equation of low dimension, the coupled equations set [19–21] of high dimension are more practical. Therefore, this paper derives the (2+1)-dimensional coupled generalized ZK(gZK) equations set from the classical quasi-geostrophic vorticity equations [22,23]. The gZK equation is a class of important high-dimensional nonlinear evolution equation in mathematical

physics, the gZK equations set as the extension of a single equation can be used to describe the interaction of nonlinear Rossby waves in two-layer fluids [24–26].

Fractional calculus have been used to model atmospheric and oceanic systems, and these models have been found to be suitable to be described by fractional differential equations [27–29]. Standard mathematical models of integer derivatives, including nonlinear models, do not work well in many cases. For fractional-order equations set, most researchers only choose the time fractional-order equations set [30,31], but few study the time-space fractional-order equations set, especially the time-space coupled fractional-order equations set. So, for the first time, we transform integral order coupled gZK equations set into fractional order coupled gZK equations set of space-time, and find that it has great research value.

Given the exact solution of nonlinear equations set [32,33] plays an important role in explaining some complex phenomena in physics, dynamics and other scientific fields, mathematicians and physicists have done a lot of researches on solving nonlinear equations set [34–41]. So far, researchers have proposed many effective solutions, such as the (G'/G) -expansion method [42], the exp-function method [43], the Kudryashov method [21], the sun-equation method [44], the functional variable method [45], the modified extended Tanh method [46], Khater method [47], Godunov-type method [48], Lie group analysis method [49,50] and so on. We choose the modified (G'/G) -expansion method to solve the time-space fractional coupled gZK equations set, and get several different kinds of solutions.

2. Derivation of Coupled gZK Equations Set

A single set of quasi-geostrophic vortex equation is usually used to derive various equations which can be used to research Rossby solitary waves [51] in a single layer of the atmosphere and ocean. For the waves in two-layer fluids, they are divided into upper and lower layers as shown in Figure 1, but the isolated waves between the two layers do not propagate separately but interact with each other, this is where the coupled equations set model comes in.

Figure 1. Model geometry for the two-layer fluids, U and V are the basic flows of the upper and lower layers, β represents the Coriolis parameter.

In the first place, for the exploration of the Rossby waves propagation and action between two-layer fluids, the coupled gZK equations set of the objective function is derived from the following two layers of quasi-geostrophic vortex equations set by using the multi-scale analysis and turbulence method

$$\begin{aligned} q_{At} + J(\psi_A, q_A) + \beta \psi_{Ax} &= 0, \\ q_{Bt} + J(\psi_B, q_B) + \beta \psi_{Bx} &= 0, \end{aligned} \quad (1)$$

where ψ_A , ψ_B are the stream functions of the upper fluid and the lower fluid respectively, $J[a, b]$ is the Jacobi operator and $J(a, b) = a_x b_y - b_x a_y$ and β represents the Coriolis parameter, as well as

$$\begin{aligned} q_A &= \psi_{Axx} + \psi_{Ayy} + F(\psi_B - \psi_A), \\ q_B &= \psi_{Bxx} + \psi_{Byy} + F(\psi_A - \psi_B). \end{aligned} \quad (2)$$

Then the Equation (1) colorredcan be expanded to the following form

$$\begin{aligned} &\psi_{Axx}t + \psi_{Ayy}t + F(\psi_B - \psi_A)_t + \psi_{Ax}[\psi_{Axx}y + \psi_{Ayy}y + F(\psi_B - \psi_A)_y] \\ &- [\psi_{Axxx} + \psi_{Ayyx} + F(\psi_B - \psi_A)_x]\psi_{Ay} + \beta\psi_{Ax} = 0, \\ &\psi_{Bxx}t + \psi_{Byy}t + F(\psi_A - \psi_B)_t + \psi_{Bx}[\psi_{Bxx}y + \psi_{Byy}y + F(\psi_A - \psi_B)_y] \\ &- [\psi_{Bxxx} + \psi_{Byyx} + F(\psi_A - \psi_B)_x]\psi_{By} + \beta\psi_{Bx} = 0, \end{aligned} \quad (3)$$

with F represents the weak coupling coefficient between two-layer fluids [22].

In the next place, so as to derive the gZK type equations set, we take the long wave approximation in the x direction, then the stream functions ψ_A and ψ_B can be divided into the basic stream functions and the disturbance stream functions, written as

$$\begin{aligned} \psi_A &= \phi_{A_0}(y) + \phi_A(x, y, t) = (U_{A_0} + c_0 y) + \phi_A(x, y, t), \\ \psi_B &= \phi_{B_0}(y) + \phi_B(x, y, t) = (U_{B_0} + c_0 y) + \phi_B(x, y, t). \end{aligned} \quad (4)$$

Suppose the coupling between the two-layer fluids is weak, and the rotation effect of the Earth is very small, therefore we can adopt the following space-time transformation

$$F = \varepsilon F_0, \quad \beta = \varepsilon^2 \beta, \quad (5)$$

$$X = \varepsilon(x - c_0 t), \quad Y = \varepsilon y, \quad T = \varepsilon^3 t, \quad (6)$$

where ε is a small parameter.

Under the premise of Equations (5) and (6), Equation (3) can be rewritten as the following form

$$\begin{aligned} &-c_0 \varepsilon^3 \phi_{AXXX} + \varepsilon^5 \phi_{AXXT} - c_0 \varepsilon^3 \phi_{AYYX} - 2c_0 \varepsilon^2 \phi_{AYyX} - c_0 \varepsilon \phi_{AyyX} + \varepsilon^5 \phi_{AYYT} + 2\varepsilon^4 \phi_{AYyT} \\ &+ \varepsilon^3 \phi_{AyyT} + \varepsilon^4 F_0(\phi_{BT} - \phi_{AT}) - c_0 \varepsilon^2 F_0(\phi_{BX} - \phi_{AX}) + \varepsilon \phi_{AX}[\varepsilon^3 \phi_{AXXY} + \varepsilon^2 \phi_{AXXy} + \\ &\phi_{A_0y}y + \varepsilon^3 \phi_{AYYY} + 3\varepsilon^2 \phi_{AYYy} + 3\varepsilon \phi_{AYyy} + \phi_{Ayyy} + \varepsilon F_0(\phi_{B_0y} - \phi_{A_0y}) + \varepsilon^2 F_0(\phi_{BY} - \phi_{Ay}) \\ &+ \varepsilon F_0(\phi_{By} - \phi_{Ay})] - [\varepsilon^3 \phi_{AXXX} + \varepsilon^3 \phi_{AYYX} + 2\varepsilon^2 \phi_{AYyX} + \varepsilon^2 F_0(\phi_{BX} - \phi_{AX})](\phi_{A_0y} \\ &+ \varepsilon \phi_{AY} + \phi_{Ay}) + \varepsilon^4 \beta_1 \phi_{AX} = 0, \\ &-c_0 \varepsilon^3 \phi_{BXXX} + \varepsilon^5 \phi_{BXXT} - c_0 \varepsilon^3 \phi_{AYYX} - 2c_0 \varepsilon^2 \phi_{BYyX} - c_0 \varepsilon \phi_{ByyX} + \varepsilon^5 \phi_{BYYT} + 2\varepsilon^4 \phi_{BYyT} \\ &+ \varepsilon^3 \phi_{ByyT} + \varepsilon^4 F_0(\phi_{AT} - \phi_{BT}) - c_0 \varepsilon^2 F_0(\phi_{AX} - \phi_{BX}) + \varepsilon \phi_{BX}[\varepsilon^3 \phi_{BXXY} + \varepsilon^2 \phi_{BXXy} + \\ &\phi_{B_0y}y + \varepsilon^3 \phi_{BYYY} + 3\varepsilon^2 \phi_{BYYy} + 3\varepsilon \phi_{BYyy} + \phi_{Byyy} + \varepsilon F_0(\phi_{A_0y} - \phi_{B_0y}) + \varepsilon^2 F_0(\phi_{AY} - \phi_{By}) \\ &+ \varepsilon F_0(\phi_{Ay} - \phi_{By})] - [\varepsilon^3 \phi_{BXXX} + \varepsilon^3 \phi_{BYYX} + 2\varepsilon^2 \phi_{BYyX} + \varepsilon^2 F_0(\phi_{AX} - \phi_{BX})](\phi_{B_0y} \\ &+ \varepsilon \phi_{BY} + \phi_{By}) + \varepsilon^4 \beta_1 \phi_{BX} = 0, \end{aligned} \quad (7)$$

where the perturbation stream functions have the following series expansion forms

$$\begin{aligned} \phi_A &= \varepsilon \phi_{A_1}(X, Y, T) + \varepsilon^2 \phi_{A_2}(X, Y, T) + \varepsilon^3 \phi_{A_3}(X, Y, T) + o(\varepsilon^4), \\ \phi_B &= \varepsilon \phi_{B_1}(X, Y, T) + \varepsilon^2 \phi_{B_2}(X, Y, T) + \varepsilon^3 \phi_{B_3}(X, Y, T) + o(\varepsilon^4). \end{aligned} \quad (8)$$

By substituting Equation (8) into Equation (7), we obtained the following equations about small parameter ε

$$\varepsilon^2 : \begin{cases} -\phi_{A_{0y}yy} \phi_{A_{1X}} - c_0 \phi_{A_{1yy}X} + \phi_{A_{0y}} \phi_{A_{1yy}X} = 0, \\ -\phi_{B_{0y}yy} \phi_{B_{1X}} - c_0 \phi_{B_{1yy}X} + \phi_{B_{0y}} \phi_{B_{1yy}X} = 0, \end{cases} \quad (9)$$

$$\varepsilon^3 : \begin{cases} -\phi_{A_{0y}yy} \phi_{A_{2X}} - c_0 \phi_{A_{2yy}X} + \phi_{A_{0y}} \phi_{A_{2yy}X} - 2\phi_{A_{1Y}X} (\phi_{A_{0y}} - c_0) \\ + \phi_{A_{1X}} \phi_{A_{1yy}y} - \phi_{A_{1yy}X} \phi_{A_{1y}} + F_0 \phi_{B_{1X}} (\phi_{A_{0y}} - c_0) + F_0 \phi_{A_{1X}} (\phi_{B_{0y}} - c_0) = 0, \\ -\phi_{B_{0y}yy} \phi_{B_{2X}} - c_0 \phi_{B_{2yy}X} + \phi_{B_{0y}} \phi_{B_{2yy}X} - 2\phi_{B_{1Y}X} (\phi_{B_{0y}} - c_0) \\ + \phi_{B_{1X}} \phi_{B_{1yy}y} - \phi_{B_{1yy}X} \phi_{B_{1y}} + F_0 \phi_{A_{1X}} (\phi_{B_{0y}} - c_0) + F_0 \phi_{B_{1X}} (\phi_{A_{0y}} - c_0) = 0, \end{cases} \quad (10)$$

$$\varepsilon^4 : \begin{cases} -\phi_{A_{0y}yy} \phi_{A_{3X}} - c_0 \phi_{A_{3yy}X} + \phi_{A_{0y}} \phi_{A_{3yy}X} - 2\phi_{A_{2Y}X} (\phi_{A_{0y}} - c_0) + \phi_{A_{2X}} \phi_{A_{1yy}y} \\ + \phi_{A_{1X}} \phi_{A_{2yy}y} - \phi_{A_{1yy}X} \phi_{A_{2y}} + F_0 \phi_{B_{2X}} (\phi_{A_{0y}} - c_0) + F_0 \phi_{A_{2X}} (\phi_{B_{0y}} - c_0) \\ + \phi_{A_{1XX}} (\phi_{A_{0y}} - c_0) + \phi_{A_{1yy}T} + \phi_{A_{1YY}X} (\phi_{A_{0y}} - c_0) + \phi_{A_{1X}} \phi_{A_{0yy}y} + 3\phi_{A_{1X}} \phi_{A_{0yy}y} \\ + 3\phi_{A_{1X}} \phi_{A_{1Y}yy} - 2c_0 \phi_{B_{1Y}X} \phi_{B_{1y}} - \phi_{B_{1yy}X} \phi_{B_{1Y}} + F_0 \phi_{B_{1X}} \phi_{A_{1y}} - F_0 \phi_{B_{1y}} \phi_{A_{1X}} = 0, \\ -\phi_{B_{0y}yy} \phi_{B_{3X}} - c_0 \phi_{B_{3yy}X} + \phi_{B_{0y}} \phi_{B_{3yy}X} - 2\phi_{B_{2Y}X} (\phi_{B_{0y}} - c_0) + \phi_{B_{2X}} \phi_{B_{1yy}y} \\ + \phi_{B_{1X}} \phi_{B_{2yy}y} - \phi_{B_{1yy}X} \phi_{B_{2y}} + F_0 \phi_{A_{2X}} (\phi_{B_{0y}} - c_0) + F_0 \phi_{B_{2X}} (\phi_{A_{0y}} - c_0) \\ + \phi_{B_{1XX}} (\phi_{B_{0y}} - c_0) + \phi_{B_{1yy}T} + \phi_{B_{1YY}X} (\phi_{B_{0y}} - c_0) + \phi_{B_{1X}} \phi_{B_{0yy}y} + 3\phi_{B_{1X}} \phi_{B_{0yy}y} \\ + 3\phi_{B_{1X}} \phi_{B_{1Y}yy} - 2c_0 \phi_{B_{1Y}X} \phi_{B_{1y}} - \phi_{B_{1yy}X} \phi_{B_{1Y}} + F_0 \phi_{B_{1X}} \phi_{A_{1y}} - F_0 \phi_{B_{1y}} \phi_{A_{1X}} = 0. \end{cases} \quad (11)$$

In Equation (9), it is easy to find that ϕ_{A_1} and ϕ_{A_2} have the following separable variables form solutions

$$\begin{aligned} \phi_{A_1} &= A_1(X, Y, T) B_1(y) \equiv A_1 B_1, \\ \phi_{B_1} &= A_2(X, Y, T) B_2(y) \equiv A_2 B_2. \end{aligned} \quad (12)$$

Substituting (12) into Equation (9), we get the following equations about variable y

$$\begin{aligned} U_{A_{0y}} B_{1y} - U_{A_{0yy}} B_1 + C_1 &= 0, \\ U_{B_{0y}} B_{2y} - U_{B_{0yy}} B_2 + C_2 &= 0, \end{aligned} \quad (13)$$

where C_1, C_2 are arbitrary constants. Applying Equations (12) and (13) to Equation (10), integrating with respect to X once, cancel out the integral function, and we get the following equations

$$\begin{aligned} 2U_{A_{0y}} (B_1 \partial_{yy} - B_1) \phi_{A_2} + B_1 [A_1^2 (B_1 B_{1yy} - B_{1y} B_{1yy}) \\ - 4U_{A_{0y}} A_{1Y} B_{1y} - 2F_0 (U_{B_{0y}} B_1 A_1 - U_{A_{0y}} B_2 A_2)] &= 0, \\ 2U_{B_{0y}} (B_2 \partial_{yy} - B_2) \phi_{B_2} + B_2 [A_2^2 (B_2 B_{2yy} - B_{2y} B_{2yy}) \\ - 4U_{B_{0y}} A_{2Y} B_{2y} - 2F_0 (U_{A_{0y}} B_2 A_2 - U_{B_{0y}} B_1 A_1)] &= 0. \end{aligned} \quad (14)$$

It's easy to see from this set of equations that

$$\begin{aligned} \phi_{A_2} &= (a_1 A_1^2 + a_2 A_{1Y} + a_3 A_1 + a_4 A_2) B_1, \\ \phi_{B_2} &= (b_1 A_2^2 + b_2 A_{2Y} + b_3 A_2 + b_4 A_1) B_2, \end{aligned} \quad (15)$$

where $a_i, b_i, i = 1, 2, 3, 4$ are functions of y

$$\begin{aligned} a_1 &= \frac{B_1 B_{1yy} - B_{1y} B_{1yy}}{4U_{A_{0y}} B_{1yy}}, \quad a_2 = -\frac{B_{1y}}{B_{1yy}}, \quad a_3 = -\frac{F_0 U_{B_{0y}}}{2U_{A_{0y}} B_{1yy}}, \quad a_4 = \frac{F_0 B_2}{2B_{1yy}}, \\ b_1 &= \frac{B_2 B_{2yy} - B_{2y} B_{2yy}}{4U_{B_{0y}} B_{2yy}}, \quad b_2 = -\frac{B_{2y}}{B_{2yy}}, \quad b_3 = -\frac{F_0 U_{A_{0y}}}{2U_{B_{0y}} B_{2yy}}, \quad b_4 = \frac{F_0 B_1}{2B_{2yy}}. \end{aligned} \quad (16)$$

In the end, after substituting Equation (12), Equation (15) and $\psi_{A_3} = \psi_{B_3} = 0$ into Equation (9), the integrating from 0 to y_0 of the resulting, the following coupled gZK equations set can be obtained though simple calculation.

$$\begin{aligned} A_{1T} + c_1(A_1A_2)_X + c_2(A_1^2)_X + c_3(A_2^2)_X + c_4A_{1XY} + c_5A_{2XY} + c_6(A_1^2)_{XY} + c_7A_{1XXX} + c_8A_{1XYY} &= 0, \\ A_{2T} + d_1(A_1A_2)_X + d_2(A_1^2)_X + d_3(A_2^2)_X + d_4A_{1XY} + d_5A_{2XY} + d_6(A_2^2)_{XY} + d_7A_{2XXX} + d_8A_{2XYY} &= 0, \end{aligned} \quad (17)$$

where

$$\begin{aligned} c_1 &= \int_0^{y_0} \left(\frac{a_4B_1B_{1yyy} + F_0B_1B_{1y}}{B_{1yy}} - a_4B_1 \right) dy, & c_2 &= \int_0^{y_0} \left(\frac{F_0U_{B_{0y}}}{B_{1yy}} - 2a_3B_1 - 2B_{1y} \right) dy, \\ c_3 &= 2 \int_0^{y_0} \frac{b_1F_0U_{A_{0y}}B_2}{B_{1yy}} dy, & c_4 &= \int_0^{y_0} \frac{a_3B_{1y} + a_2F_0U_{B_{0y}}B_1}{B_{1yy}} dy, & c_5 &= \int_0^{y_0} \frac{a_4B_{1y} + b_2F_0U_{A_{0y}}B_2}{B_{1yy}} dy, \\ c_6 &= \int_0^{y_0} \left(\frac{a_2B_1B_{1yy} - 2B_{1y}^2 + 2a_1B_{1y}}{B_{1y}} - a_2B_1 \right) dy, & c_7 &= \int_0^{y_0} \frac{U_{A_{0y}}B_1}{B_{1yy}} dy, & c_8 &= \int_0^{y_0} \frac{a_2B_{1y} + U_{A_{0y}}B_1}{B_{1yy}} dy, \\ d_1 &= \int_0^{y_0} \left(\frac{b_4B_2B_{2yy} + F_0B_2B_{2y}}{B_{2yy}} - b_4B_2 \right) dy, & d_3 &= \int_0^{y_0} \left(\frac{b_1F_0U_{A_{0y}}B_2}{B_{2yy}} - 2a_3B_2 - 2a_3B_{2y} \right) dy, \\ d_2 &= 2 \int_0^{y_0} \frac{a_1F_0U_{B_{0y}}B_1}{B_{2yy}} dy, & d_4 &= \int_0^{y_0} \frac{b_4B_{2y} + a_2F_0U_{B_{0y}}B_1}{B_{2yy}} dy, & d_5 &= \int_0^{y_0} \frac{b_3B_{2y} + b_2F_0U_{B_{0y}}B_2}{B_{2yy}} dy, \\ d_6 &= \int_0^{y_0} \left(\frac{b_2B_2B_{2yy} - 2B_{2y}^2 + 2b_1B_{2y}}{B_{2yy}} - b_2B_2 \right) dy, & d_7 &= \int_0^{y_0} \frac{U_{B_{0y}}B_2}{B_{2yy}} dy, & d_8 &= \int_0^{y_0} \frac{b_2B_{2y} + U_{B_{0y}}B_2}{B_{2yy}} dy. \end{aligned} \quad (18)$$

Remark 1. The coupled gZK equations set is the extension of a single ZK equation and a class of important nonlinear evolution equations of high dimension. They describe two kinds of weakly nonlinear waves interaction with each other and the interaction between two waves is reflected in multiple coupling terms $(A_1A_2)_X$, $(A_1^2)_{XY}$ and $(A_2^2)_{XY}$.

3. The Time-Space Fractional Coupled gZK Equations set

In previous work, we only derived a single fractional-order equation, but here we will apply the semi-inverse method and the variational method [52–54] to derive the coupled fractional-order equations set for the first time, and obtain a new fractional-order coupled equations set, namely time-space fractional coupled gZK equations set. For the ease of understanding, some definitions and properties of fractional order are introduced before demonstrating the specific derivation process.

Definition 1 ([52]). Modified Rieman-Liouville derivative

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\delta)^{-\alpha} (f(\delta) - f(0)) d\delta, & 0 < \alpha < 1, \\ (f^{(n)}(x))^{(\alpha-n)}, & n \leq \alpha < n+1, n \geq 1, \end{cases}$$

where $f(t)$ is a continuous function.

Definition 2 ([42]). Assume that $f(t)$ denotes a continuous $R \rightarrow R$ function, we use the following equality for the integral

$$D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{-1} f(\zeta) d\zeta = \frac{1}{\Gamma(1+\alpha)} \frac{d}{dt} \int_0^t f(\zeta) (d\zeta)^\alpha, \quad 0 < \alpha \leq 1.$$

Property 1 ([54]). Integral property of fractional order equation

$$\int_a^t (d\tau)^\alpha f(\tau) = \alpha \int_a^t d\tau (t-\tau)^\alpha f(\tau).$$

Property 2 ([54]). *Integration by parts property of fractional order equation*

$$\int_a^b (d\tau)^\alpha f(\tau) D_t^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} [g(t)f(t)|_a^b - \int_a^b (d\tau)^\alpha g(\tau) D_t^\alpha f(t)],$$

$$f(t), g(t) \in [a, b].$$

Introducing two potential functions $U(X, Y, T)$ and $V(X, Y, T)$, and their relationship to A_1 and A_2 is that $A_1 = U_X, A_2 = V_X$, substituting these two expressions into Equation (17) separately, the potential equations of the coupled gZK equations set have the form as

$$\begin{aligned} U_{XT} + c_1(v_X V)_X + c_2(U_X^2)_X + c_3(V_X^2)_X + c_4 U_{XXY} \\ + c_5 V_{XXY} + c_6(U_X^2)_{XY} + c_7 U_{XXXX} + c_8 U_{XXYY} = 0, \\ V_{XT} + d_1(V_X U)_X + d_2(U_X^2)_X + d_3(V_X^2)_X + d_4 U_{XXY} \\ + d_5 V_{XXY} + d_6(V_X^2)_{XY} + d_7 V_{XXXX} + d_8 V_{XXYY} = 0. \end{aligned} \quad (19)$$

Further, the semi-inverse method was applied to derive the Lagrangian equations of coupled gZK equations, functional of the Equation (19) could be written as

$$\begin{aligned} J(U, V) = \int_R dX \int_R dY \int_T dT \{ U[m_1 U_{XT} + m_2 c_1 (U_X V)_X + m_3 c_2 U_X U_{XX} + m_4 c_3 V_X V_{XX} \\ + m_5 c_4 U_{XXY} + m_6 c_5 V_{XXY} + m_7 c_6 (U_X^2)_{XY} + m_8 c_7 U_{XXXX} + m_9 c_8 U_{XXYY}] \\ + V[n_1 V_{XT} + n_2 d_1 (U_X V)_X + n_3 d_2 U_X U_{XX} + n_4 d_3 V_X V_{XX} + n_5 d_4 U_{XXY} \\ + n_6 d_5 V_{XXY} + n_7 d_6 (V_X^2)_{XY} + n_8 d_7 V_{XXXX} + n_9 d_8 V_{XXYY}] \}, \end{aligned} \quad (20)$$

with $m_i, n_i, i = 1, \dots, 9$ are Lagrange coefficients and will be calculated later to determine the exact values. Carrying out the integration by parts in Equation (20) and taking $U_X|_R = U_Y|_R = U_T|_T = V_X|_R = V_Y|_R = V_T|_T = 0$, the functional be rewritten as

$$\begin{aligned} J(U, V) = \int_R dX \int_R dY \int_T dT \{ [-m_1 U_X U_T - m_2 c_1 U_X^2 V_X - \frac{1}{2} m_3 c_2 U_X^3 - \frac{1}{2} m_4 c_3 U_X V_X^2 \\ - m_5 c_4 U_X U_{XY} - m_6 c_5 U_X V_{XY} - m_7 c_6 (U_X^2)_{XY} - m_8 c_7 U_{XX}^2 - m_9 c_8 U_{XY}^2] \\ - [n_1 V_X V_T + n_2 d_1 V_X^2 U_X + \frac{1}{2} n_3 d_2 V_X^3 - \frac{1}{2} n_4 d_3 V_X U_X^2 + n_5 d_4 V_X V_{XY} \\ + n_6 d_5 V_X U_{XY} + n_7 d_6 V_X^2 V_{XY} + n_8 d_7 V_{XX}^2 + n_9 d_8 V_{XY}^2] \}. \end{aligned} \quad (21)$$

Using the variational method for this functional equation, integrating by parts to optimize this variational, the resulting forms are expressed as

$$\begin{aligned} -2m_1 U_{XT} - 2m_2 c_1 (U_X V)_X - 3m_3 c_2 (U_X^2)_X - 3m_4 c_3 (V_X^2)_X - 2m_5 c_4 U_{XXY} \\ - m_6 c_5 V_{XXY} - 4m_7 c_6 (U_X^2)_{XY} + m_8 c_7 U_{XXXX} - m_9 c_8 U_{XXYY} = 0, \\ -2n_1 V_{XT} - 2n_2 d_1 (V_X U)_X - 3n_3 d_2 (U_X^2)_X - 3n_4 d_3 (V_X^2)_X - 2n_5 d_4 U_{XXY} \\ - n_6 d_5 V_{XXY} - 4n_7 d_6 (V_X^2)_{XY} + n_8 d_7 V_{XXXX} - n_9 d_8 V_{XXYY} = 0. \end{aligned} \quad (22)$$

Since the Equations (22) and (19) are equal, the values of all the Lagrangian constants in the equations can be obtained, $m_1 = m_2 = m_5 = n_1 = n_2 = n_5 = -\frac{1}{2}, m_3 = m_4 = n_3 = n_4 = -\frac{1}{3}, m_7 = n_7 = -\frac{1}{4}, m_8 = n_8 = 1, m_6 = m_9 = n_6 = n_9 = -1$. Thus, the Lagrangian forms [55,56] of the integer order coupled gZK equations set is,

$$\begin{aligned}
I_1 &= \frac{1}{2}U_X U_T + \frac{1}{2}c_1 U_X^2 V_X + \frac{1}{3}c_2 U_X^3 + \frac{1}{3}c_3 U_X^2 V_X + c_4 U_X U_{XY} \\
&\quad + \frac{1}{2}c_5 U_X V_{XY} + \frac{1}{4}c_6 (U_X^2) U_{XY} - c_7 U_{XX}^2 + c_8 U_{XY}^2 = 0, \\
I_2 &= \frac{1}{2}V_X V_T + \frac{1}{2}d_1 V_X^2 U_X + \frac{1}{3}d_2 V_X^3 + \frac{1}{3}d_3 V_X^2 U_X + d_4 V_X V_{XY} \\
&\quad + \frac{1}{2}d_5 V_X U_{XY} + \frac{1}{4}d_6 (V_X^2) V_{XY} - d_7 V_{XX}^2 + d_8 V_{XY}^2 = 0.
\end{aligned} \tag{23}$$

The fractional variational problem of Lagrange was obtained [57]. A natural generalization of Agrawal's approach [53,58–60], was applied to the fractional calculus of constrained systems. In order to obtain time-space fractional gZK equation set, we use the Lagrangian to minimize certain functionals which will naturally contain fractional derivative terms. Analogously, based on the Definition 1 and Agrawal's method, the Lagrangian forms of the time-space fractional coupled gZK equations are given as

$$\begin{aligned}
F_1 &= \frac{1}{2}D_T^\alpha U \times D_X^\beta U + \frac{1}{2}c_1 (D_X^\beta U)^2 D_X^\beta V + \frac{1}{3}c_2 (D_X^\beta U)^3 + \frac{1}{3}c_3 (D_X^\beta V)^2 D_X^\beta U + c_4 D_X^\beta U \times D_{XY}^{\beta\omega} U \\
&\quad + \frac{1}{2}c_5 D_X^\beta U \times D_{XY}^{\beta\omega} V + \frac{1}{4}c_6 (D_X^\beta U)^2 D_{XY}^{\beta\omega} V - c_7 (D_X^{2\beta} U)^2 + c_8 (D_{XY}^{\beta\omega} U)^2 = 0, \\
F_2 &= \frac{1}{2}D_T^\alpha V \times D_X^\beta V + \frac{1}{2}d_1 (D_X^\beta V)^2 D_X^\beta U + \frac{1}{3}d_2 (D_X^\beta V)^3 + \frac{1}{3}d_3 (D_X^\beta U)^2 D_X^\beta V + d_4 D_X^\beta V \times D_{XY}^{\beta\omega} V \\
&\quad + \frac{1}{2}d_5 D_X^\beta V \times D_{XY}^{\beta\omega} U + \frac{1}{4}d_6 (D_X^\beta V)^2 D_{XY}^{\beta\omega} U - d_7 (D_X^{2\beta} V)^2 + d_8 (D_{XY}^{\beta\omega} V)^2 = 0,
\end{aligned} \tag{24}$$

where $D_{XY}^{\beta\omega} f = D_Y^\omega [D_X^\beta f]$, $D_X^{2\beta} f = D_X^\beta [D_X^\beta f]$.

It's similar to what we did for the integral order equation, the functional of the time-space coupled gZK equations set has the form

$$J_F(U, V) = \int_R (dX)^\beta \int_R (dY)^\omega \int_T (dT)^\alpha (F_1 + F_2). \tag{25}$$

and the variation of functional Equation (24) leads to

$$\begin{aligned}
\delta J_F(U, V) &= \int_R (dX)^\beta \int_R (dY)^\omega \int_T (dT)^\alpha [D_T^\alpha (\frac{\partial F_1}{\partial D_T^\alpha U}) + D_X^\beta (\frac{\partial F_1}{\partial D_X^\beta U}) + D_Y^\omega (\frac{\partial F_1}{\partial D_{XY}^{\beta\omega} U}) \\
&\quad - D_X^{2\beta} (\frac{\partial F_1}{\partial D_X^{2\beta} U})] \delta J_U + \int_R (dX)^\beta \int_R (dY)^\omega \int_T (dT)^\alpha [D_T^\alpha (\frac{\partial F_2}{\partial D_T^\alpha V}) \\
&\quad + D_X^\beta (\frac{\partial F_2}{\partial D_X^\beta V}) + D_Y^\omega (\frac{\partial F_2}{\partial D_{XY}^{\beta\omega} V}) - D_X^{2\beta} (\frac{\partial F_2}{\partial D_X^{2\beta} V})] \delta J_V.
\end{aligned} \tag{26}$$

According to the properties introduced at the beginning, integrating the Equation (26) by parts and making $\delta J_F(U, V) = 0$, optimizing the variation of the function, the following form Euler-Lagrange equations [53] for the time-space fractional coupled gZK equations set can be given

$$\begin{aligned}
D_T^\alpha (\frac{\partial F_1}{\partial D_T^\alpha U}) + D_X^\beta (\frac{\partial F_1}{\partial D_X^\beta U}) + D_Y^\omega (\frac{\partial F_1}{\partial D_{XY}^{\beta\omega} U}) - D_X^{2\beta} (\frac{\partial F_1}{\partial D_X^{2\beta} U}) &= 0, \\
D_T^\alpha (\frac{\partial F_2}{\partial D_T^\alpha V}) + D_X^\beta (\frac{\partial F_2}{\partial D_X^\beta V}) + D_Y^\omega (\frac{\partial F_2}{\partial D_{XY}^{\beta\omega} V}) - D_X^{2\beta} (\frac{\partial F_2}{\partial D_X^{2\beta} V}) &= 0.
\end{aligned} \tag{27}$$

The last step is to plug expressions for F_1, F_2 given by Equation (24) and fractional potential functions $D_X^\beta U(X, Y, T) = u(X, Y, T), D_X^\beta V(X, Y, T) = v(X, Y, T)$ in this equation, the final equations set is

$$\begin{aligned} D_T^\alpha u + c_1 D_X^\beta (uv) + c_2 D_X^\beta (u^2) + c_3 D_X^\beta (v^2) + c_4 D_X^\beta D_Y^\omega u \\ + c_5 D_X^\beta D_Y^\omega v + c_6 D_X^\beta D_Y^\omega (u^2) + c_7 D_X^{3\beta} u + c_8 D_X^\beta D_Y^{2\omega} v = 0, \\ D_T^\alpha v + d_1 D_X^\beta (uv) + d_2 D_X^\beta (u^2) + d_3 D_X^\beta (v^2) + d_4 D_X^\beta D_Y^\omega u \\ + d_5 D_X^\beta D_Y^\omega v + d_6 D_X^\beta D_Y^\omega (v^2) + d_7 D_X^{3\beta} v + d_8 D_X^\beta D_Y^{2\omega} v = 0. \end{aligned} \quad (28)$$

It is the time-space fractional coupled gZK equations set. This new set of fractional-order equations will promote the study of fractional-order nonlinear equations and has great significance for the future research.

4. Solutions of Time-Space Fractional Coupled gZK Equations Set

In the previous section, the integral order coupled equations set is transformed into the fractional order coupled equations set. To further explore the Rossby solitary waves interaction between two-layer fluids, we solved the time-space fractional coupled gZK equations set by improved (G'/G) -expansion method [42,43] in this section.

Firstly, by using the following fractional traveling wave transformations

$$u(X, Y, T) = \phi_1(\xi), \quad v(X, Y, T) = \phi_2(\xi) \quad (29)$$

$$\xi = \frac{k_1 X^\beta}{\Gamma(1+\beta)} + \frac{k_2 Y^\omega}{\Gamma(1+\omega)} - \frac{\sigma T^\alpha}{\Gamma(1+\alpha)} \quad (30)$$

where the k_1, k_2, σ are constants, and using chain rule, we have the equations

$$\begin{aligned} D_T^\alpha u &= \rho_T' \frac{\partial \phi_1}{\partial \xi} D_T^\alpha, & D_T^\alpha v &= \rho_T' \frac{\partial \phi_2}{\partial \xi} D_T^\alpha, \\ D_X^\beta u &= \rho_X' \frac{\partial \phi_1}{\partial \xi} D_X^\beta, & D_X^\beta v &= \rho_X' \frac{\partial \phi_2}{\partial \xi} D_X^\beta, \\ D_Y^\omega u &= \rho_Y' \frac{\partial \phi_1}{\partial \xi} D_Y^\omega, & D_Y^\omega v &= \rho_Y' \frac{\partial \phi_2}{\partial \xi} D_Y^\omega, \end{aligned} \quad (31)$$

with ρ_T, ρ_X, ρ_Y are the fractal indexes, without loss of generality we can make $\rho_T = \rho_X = \rho_Y = l$, thereinto l is a constant.

Put the Equation (29) with Equations (30) and (31) into the Equation (28), the time-space fractional coupled equations set can be reduce to the ordinary differential coupled equations set

$$\begin{aligned} -\sigma \phi_1' + k_1(c_1 \phi_2 \phi_1' + c_1 \phi_1 \phi_2' + c_2 \phi_1 \phi_1' + c_3 \phi_2 \phi_2') + k_1 k_2 l [c_4 \phi_1'' + c_5 \phi_2'' \\ + c_6 \phi_1 \phi_1'' + c_6 (\phi_1')^2] + (c_7 k_1^3 l^2 + c_8 k_1 k_2^2 l^2) \phi_1''' = 0, \\ -\sigma \phi_2' + k_1(d_1 \phi_2 \phi_1' + d_1 \phi_1 \phi_2' + d_2 \phi_1 \phi_1' + d_3 \phi_2 \phi_2') + k_1 k_2 l [d_4 \phi_1'' + d_5 \phi_2'' \\ + d_6 \phi_2 \phi_2'' + d_6 (\phi_2')^2] + (d_7 k_1^3 l^2 + d_8 k_1 k_2^2 l^2) \phi_2''' = 0. \end{aligned} \quad (32)$$

Secondly, suppose Equation (32) have the solutions in relation to (G'/G) as follows

$$\phi_1(\xi) = e_0 + e_1 \left(\frac{G'}{G} \right), \quad \phi_2(\xi) = f_0 + f_1 \left(\frac{G'}{G} \right). \quad (33)$$

where e_0, e_1, f_0, f_1 are computed later, $G = G(\xi)$ satisfies the second ordinary differential equation

$$G'' + mG' + nG = 0. \quad (34)$$

thereinto the apostrophe represents derivative with respect to ξ and m, n are parameters.

By substituting Equation (33) with Equation (34) into (32), collecting all terms with the same order of (G'/G) , equating each coefficient of the resulting polynomial to zero, we can obtain a set of algebraic equations for $k_1, k_2, \sigma, l, e_0, e_1, f_0, f_1, m$ and n .

The last, due to the very complicated coefficients in the equations, the calculation process is extremely complicated, in order to calculate accurate results, take $c_1 = d_1 = c_6 = d_6 = 2, c_2 = c_3 = d_2 = d_3 = 6, c_4 = c_5 = c_7 = c_8 = d_4 = d_5 = d_7 = d_8 = 1$. Solving the algebraic equations system and subsequently substituting these constants $k_1, k_2, \sigma, l, e_0, e_1, f_0, f_1, m$ and n , we get the solutions of the time-space fractional gZK coupled equations set that we want in following as

Case 1:

$$\sigma = \frac{(H_1 l k_1 + 8k_1)(I_1 H_1 + H_3) - 4(2H_2 H_1 + H_3)k_1 - I_1 H_1 l + H_4 H_1 k_2 l}{4k_2^2 l}, \quad e_0 = \frac{I_1 H_1 + H_2}{4},$$

$$e_1 = e_1, \quad f_0 = -\frac{I_1 H_1 + H_3}{4k_2^2 l}, \quad f_1 = e_1, \quad l = l, \quad m = -\frac{H_1}{4k_2}, \quad n = 0,$$
(35)

$$\begin{cases} u_1 = \frac{I_1 H_1 + H_3}{4} \pm e_1 \frac{H_2}{8k_2} \left[\frac{C_1 \sinh(\pm \frac{H_2}{8k_2} \xi) + C_2 \cosh(\pm \frac{H_2}{8k_2} \xi)}{C_1 \cosh(\pm \frac{H_2}{8k_2} \xi) + C_2 \sinh(\pm \frac{H_2}{8k_2} \xi)} \right] + \frac{H_1}{8k_2}, \\ v_1 = \frac{I_1 H_1 + H_2}{4} \pm f_1 \frac{H_2}{8k_2} \left[\frac{C_1 \sinh(\pm \frac{H_2}{8k_2} \xi) + C_2 \cosh(\pm \frac{H_2}{8k_2} \xi)}{C_1 \cosh(\pm \frac{H_2}{8k_2} \xi) + C_2 \sinh(\pm \frac{H_2}{8k_2} \xi)} \right] + \frac{H_1}{8k_2}, \end{cases}$$
(36)

where C_1, C_2 are arbitrary constants, $H_1 = 2k_2 e_1 - 3(k_1^2 + k_2^2)l, I_1 = -(k_1^2 + k_2^2)l^2, H_2 = 2e_1 k_2 l + 5I_1, H_3 = k_2^2 l - 4e_1 k_2, H_4 = l k_1 k_2 - 4e_1 k_1$.

Case 2:

$$\sigma = \frac{-16H_5 k_1 + I_2 k_1 k_2 l^2 + I_3 k_2 l - 4l^2 k_1 k_2^2}{k_2 l}, \quad e_0 = \frac{H_5}{k_2 l}, \quad e_1 = -\frac{H_6}{2k_2},$$

$$f_0 = -\frac{H_5}{k_2 l}, \quad f_1 = -\frac{H_6}{2k_2}, \quad l = l, \quad m = 0, \quad n = \frac{1}{2},$$
(37)

$$\begin{cases} u_2 = \frac{H_5}{k_2 l} - \frac{H_6 \sqrt{2}}{4k_2} \left[\frac{-C_1 \sin(\frac{\sqrt{2}}{2} \xi) + C_2 \cos(\frac{\sqrt{2}}{2} \xi)}{C_1 \cos(\frac{\sqrt{2}}{2} \xi) + C_2 \sin(\frac{\sqrt{2}}{2} \xi)} \right] - \frac{\sqrt{2}}{2}, \\ v_2 = -\frac{H_5}{k_2 l} - \frac{H_6 \sqrt{2}}{4k_2} \left[\frac{-C_1 \sin(\frac{\sqrt{2}}{2} \xi) + C_2 \cos(\frac{\sqrt{2}}{2} \xi)}{C_1 \cos(\frac{\sqrt{2}}{2} \xi) + C_2 \sin(\frac{\sqrt{2}}{2} \xi)} \right] - \frac{\sqrt{2}}{2}, \end{cases}$$
(38)

where C_1, C_2 are arbitrary constants, $H_5 = \frac{k_1 k_2^2 l - 2[3(k_1^2 + k_2^2)l + 4k_2]k_1}{k_2}, I_2 = -(2k_1^2 + 3k_2^2)k_1 l^2 + 4k_1 k_2 l, I_3 = 4(k_1^3 + k_1 k_2^2)l^2, H_6 = -3(k_1^2 + k_2^2)l + 4k_2$.

Case 3:

$$\sigma = \frac{I_4 n}{H_7^2}, \quad e_0 = -12 \frac{H_8}{H_7}, \quad e_1 = \frac{4k_2^2}{H_7}, \quad f_0 = 36 \frac{H_8}{H_7},$$

$$f_1 = 0, \quad l = \frac{12k_2}{H_7}, \quad m = -2n + 1, \quad n = n,$$
(39)

(i) When $4n^2 - 8n + 2 > 0$, the hyperbolic solutions as:

$$\begin{cases} u_{31} = -12 \frac{H_8}{H_7} + \frac{4k_2^2 \sqrt{4n^2 - 8n + 2}}{2H_7} \left[\frac{C_1 \sinh(\sqrt{4n^2 - 8n + 2} \xi) + C_2 \cosh(\sqrt{4n^2 - 8n + 2} \xi)}{C_1 \cosh(\sqrt{4n^2 - 8n + 2} \xi) + C_2 \sinh(\sqrt{4n^2 - 8n + 2} \xi)} \right] - \frac{1 - 2n}{2}, \\ v_{31} = 36 \frac{H_8}{H_7} - \frac{1 - 2n}{2}, \end{cases}$$
(40)

(ii) When $4n^2 - 8n + 2 > 0$, the trigonometric solutions as:

$$\begin{cases} u_{32} = -12 \frac{H_8}{H_7} - \frac{4k_2^2 \sqrt{4n^2 + 2}}{2H_7} \left[\frac{-C_1 \sin(\sqrt{4n^2 - 8n + 2}\xi) + C_2 \cos(\sqrt{4n^2 - 8n + 2}\xi)}{C_1 \cos(\sqrt{4n^2 - 8n + 2}\xi) + C_2 \sin(\sqrt{4n^2 - 8n + 2}\xi)} \right] - \frac{1 - 2n}{2}, \\ v_{32} = 36 \frac{H_8}{H_7} - \frac{1 - 2n}{2}, \end{cases} \quad (41)$$

(iii) When $4n^2 - 8n + 2 > 0$, the solutions as:

$$\begin{cases} u_{33} = \left(\frac{C_2}{C_1 + C_2\xi} - \frac{1 - 2n}{2} \right), \\ v_{33} = \left(\frac{C_1}{C_2 + C_1\xi} - \frac{1 - 2n}{2} \right), \end{cases} \quad (42)$$

where C_1, C_2 are arbitrary constants, $I_4 = 288k_1k_2^2(k_1^2 + k_2^2)$, $H_7 = 9k_1^2 + 7k_2^2$, $H_8 = 2k_1^2n + 2k_2^2n - k_1^2 - k_2^2$.

The exact solutions obtained by the research show that the fluctuation relationship of each wave not only contains its own wave number and amplitude, but also contains the amplitude of another wave, which explains the main characteristics of nonlinear waves interaction [61]. On the other hand, the interaction between Rossby waves has a great influence on the propagation stability of waves. When both waves are unstable, and they are still unstable after the interaction. When at least one wave is stable, the two waves may be stable or unstable through the interaction, which is related to the values of the coupling term coefficients.

5. Conclusions

In this paper, based on the quasi-geostrophic vortex equation set, we obtain the (2+1)-dimensional coupled gZK equations set for the first time, which can describe Rossby solitary waves interactions in two-layer fluids. Next, according to the new model and using the semi-inverse method and the fractional variational principle, a new (2+1)-dimensional time-space fractional coupled gZK equations set is obtained. Then, we solved the (2+1)-dimensional time-space fractional coupled gZK equations set. The coupled gZK equations set is the evolution of a single gZK equation in two-layer fluids, which is of great significance for the study of Rossby waves propagation and interaction. How Rossby solitary waves described by coupled equations set interacts specifically and how the energy changes during the interaction, which are our research aim in the future.

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