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Complex Intuitionistic Fuzzy Graphs with Application in Cellular Network Provider Companies

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Abstract: In recent years, a mathematical approach of blending different aspects is on the way, which as a result gives a more generalized approach. Following the above mathematical approach, we combine two very powerful techniques, namely complex intuitionistic fuzzy sets and graph theory, and introduce the notion of complex intuitionistic fuzzy graphs. Then, we introduce certain notions including union, join and composition of complex intuitionistic fuzzy graphs, through which one can easily manipulate the complex intuitionistic fuzzy graphs in decision making problems. We elucidate these operations with some examples. We also describe the homomorphisms of complex intuitionistic fuzzy graphs. Finally, we provide an application in cellular network provider companies for the testing of our approach.

Keywords: complex intuitionistic fuzzy sets; intuitionistic fuzzy graphs; complex intuitionistic fuzzy graphs

1. Introduction

We divide the Introduction Section into four main paragraphs. In the first paragraph, we provide some details about the fuzzy sets. In the second paragraph, detail is given about the complex version of fuzzy sets, namely complex fuzzy sets, which is an extension of fuzzy sets. In the third paragraph, detail is given about graph theory in terms of different types of fuzzy sets. In the fourth paragraph, we give our presented approach by combining the two different approaches given in the second and third paragraphs.

Fuzzy set theory was conferred by Zadeh [1] to solve difficulties in dealing with uncertainties. Since then, the theory of fuzzy sets and fuzzy logic have been examined by many researchers to solve many real life problems involving ambiguous and uncertain environment. Atanassov [2] proposed the extended form of fuzzy set by adding a new component, called "intuitionistic fuzzy sets" (*IF*-sets). The idea of *IF*-sets is more meaningful as well as intensive due to the presence of degree of truth and falsity membership. Applications of these sets have been broadly studied in other aspects such as image processing [3], multi-criteria decision making [4], pattern recognition [5], etc.

Buckley [6] and Nguyen et al. [7] combined complex numbers with fuzzy sets. On the other hand, Ramot et al. [8,9] extended the range of membership to "unit circle in the complex plane", unlike others who limited the range to [0, 1]. Zhang et al. [10] studied some operation properties and δ -equalities of complex fuzzy sets. Some applications of complex fuzzy sets have been considered in reasoning schemes [11], image restoration [12] and decision making [13]. Further, this concept has

been studied in intuitionistic fuzzy sets [14]. Alkouri and Salleh studied some operations on complex Atanassov's intuitionistic fuzzy sets in [15] and also studied complex Atanassov's intuitionistic fuzzy relation in [16]. Ali et al. [17] introduced complex intuitionistic fuzzy classes.

Fuzzy graphs were narrated by Rosenfeld [18] and Mordeson [19]. After that, some opinion on "fuzzy graphs" were given by Bhattacharya [20]. He showed that none of the concepts of crisp graph theory have similarities in fuzzy graphs. Thirunavukarasu et al. [21] extended this concept for complex fuzzy graphs. Shannon and Atanassov [22] and Akram and Davvaz [23] defined intuitionistic fuzzy graphs. Later, several authors worked on intuitionistic fuzzy graphs and added many useful results to this area, for instance, Akram and Akmal [24], Alshehri and Akram [25], Karunambigai et al. [26], Myithili et al. [27], Nagoorgani et al. [28] and Parvathi et al. [29,30]. See also [31–35].

Inspired by the fact that complex intuitionistic fuzzy sets generalize intuitionistic fuzzy sets, in this paper, we provide the new idea of complex intuitionistic fuzzy graphs with some fundamental operations. We also describe homomorphisms of complex intuitionistic fuzzy graphs. Finally, we provide an application.

2. Preliminaries and Basic Definitions

Definition 1. [8] A complex fuzzy set (CFS) A, defined on a universe of discourse X is an object of the form

$$\mathcal{A} = \{ (x, u_{\mathcal{A}}(x)e^{i\omega_{\mathcal{A}}(x)}) : x \in \mathcal{X} \},\$$

where $i = \sqrt{-1}$, $u_{\mathcal{A}}(x) \in [0, 1]$ and $0 \le \omega_{\mathcal{A}}(x) \le 2\pi$.

Definition 2. [14] A complex intuitionistic fuzzy set (cif-set) A, defined on a universe of discourse X is an object of the form

$$\mathcal{A} = \{ (x, \mu_{\mathcal{A}}(x)e^{i\alpha_{\mathcal{A}}(x)}, v_{\mathcal{A}}(x)e^{i\beta_{\mathcal{A}}(x)}) : x \in \mathcal{X} \},\$$

where $i = \sqrt{-1}$, $\mu_{\mathcal{A}}(x)$, $v_{\mathcal{A}}(x) \in [0, 1]$, $\alpha_{\mathcal{A}}(x)$, $\beta_{\mathcal{A}}(x) \in [0, 2\pi]$ and $0 \le \mu_{\mathcal{A}}(x) + v_{\mathcal{A}}(x) \le 1$.

Definition 3. [14] Let A and B be two cif-sets in X, where

$$\mathcal{A} = \{ (x, \mu_{\mathcal{A}}(x)e^{i\alpha_{\mathcal{A}}(x)}, v_{\mathcal{A}}(x)e^{i\beta_{\mathcal{A}}(x)}) : x \in \mathcal{X} \}$$

and $\mathcal{B} = \{ (x, \mu_{\mathcal{B}}(x)e^{i\alpha_{\mathcal{B}}(x)}, v_{\mathcal{B}}(x)e^{i\beta_{\mathcal{B}}(x)}) : x \in \mathcal{X} \}.$

Then, $A \cup B$ *is given as*

$$\mathcal{A} \cup \mathcal{B} = \{ (x, \mu_{\mathcal{A} \cup \mathcal{B}}(x)e^{i\alpha_{\mathcal{A} \cup \mathcal{B}}(x)}, v_{\mathcal{A} \cup \mathcal{B}}(x)e^{i\beta_{\mathcal{A} \cup \mathcal{B}}(x)} : x \in \mathcal{X} \}$$

where

$$\mu_{\mathcal{A}\cup\mathcal{B}}(x)e^{i\alpha_{\mathcal{A}\cup\mathcal{B}}(x)} = [\mu_{\mathcal{A}}(x) \lor \mu_{\mathcal{B}}(x)]e^{i\{\alpha_{\mathcal{A}}(x)\lor \alpha_{\mathcal{B}}(x)\}},$$
$$v_{\mathcal{A}\cup\mathcal{B}}(x)e^{i\beta_{\mathcal{A}\cup\mathcal{B}}(x)} = [v_{\mathcal{A}}(x) \land v_{\mathcal{B}}(x)]e^{i\{\beta_{\mathcal{A}}(x)\land \beta_{\mathcal{B}}(x)\}},$$

Definition 4. [16] Let A and B be two cif-sets in X, where

$$\mathcal{A} = \{ (x, \mu_{\mathcal{A}}(x)e^{i\alpha_{\mathcal{A}}(x)}, v_{\mathcal{A}}(x)e^{i\beta_{\mathcal{A}}(x)}) : x \in \mathcal{X} \}$$

and $\mathcal{B} = \{ (x, \mu_{\mathcal{B}}(x)e^{i\alpha_{\mathcal{B}}(x)}, v_{\mathcal{B}}(x)e^{i\beta_{\mathcal{B}}(x)}) : x \in \mathcal{X} \}.$

Then, for all $x \in \mathcal{X}$:

- (1) $\mathcal{A} \subset \mathcal{B}$ if and only if $\mu_{\mathcal{A}}(x) < \mu_{\mathcal{B}}(x), v_{\mathcal{A}}(x) > v_{\mathcal{B}}(x)$ for amplitude terms and $\alpha_{\mathcal{A}}(x) < \alpha_{\mathcal{B}}(x), \beta_{\mathcal{A}}(x) > \beta_{\mathcal{B}}(x)$ for phase terms.
- (2) $\mathcal{A} = \mathcal{B}$ if and only if $\mu_{\mathcal{A}}(x) = \mu_{\mathcal{B}}(x)$, $v_{\mathcal{A}}(x) = v_{\mathcal{B}}(x)$ for amplitude terms and $\alpha_{\mathcal{A}}(x) = \alpha_{\mathcal{B}}(x)$, $\beta_{\mathcal{A}}(x) = \beta_{\mathcal{B}}(x)$ for phase terms.

Definition 5. A graph is an ordered pair $G^* = (V, E)$, where V is the set of vertices of G^* and E is the set of edges of G^* .

3. Complex Intuitionistic Fuzzy Graphs

In this section, we provide definition and operations of complex intuitionistic fuzzy graphs.

Definition 6. A complex intuitionistic fuzzy graph (cif-graph) with an underlaying set V is defined to be a pair $\mathbb{G} = (\mathcal{A}, \mathcal{B})$, where \mathcal{A} is a cif-set on V and \mathcal{B} is a cif-set on $E \subseteq V \times V$ such that

 $\mu_{\mathcal{B}}(xy)e^{i\alpha_{\mathcal{B}}(xy)} \le \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(y)\}e^{i\min\{\alpha_{\mathcal{A}}(x), \alpha_{\mathcal{A}}(y)\}}$ $v_{\mathcal{B}}(xy)e^{i\beta_{\mathcal{B}}(xy)} \le \max\{v_{\mathcal{A}}(x), v_{\mathcal{A}}(y)\}e^{i\max\{\beta_{\mathcal{A}}(x), \beta_{\mathcal{A}}(y)\}}$

for all $x, y \in V$.

Definition 7. Let $\mathbb{G} = (\mathcal{A}, \mathcal{B})$ be a cif-graph. The order of a cif-graph is defined by

$$O(\mathbb{G}) = \left(\sum_{x \in V} \mu_{\mathcal{A}}(x) e^{\sum_{x \in V} \alpha_{\mathcal{A}}(x)}, \sum_{x \in V} \upsilon_{\mathcal{A}}(x) e^{\sum_{x \in V} \beta_{\mathcal{A}}(x)}\right)$$

The degree of a vertex x in \mathbb{G} *is defined by*

$$deg(x) = \left(\sum_{x \in E} \mu_{\mathcal{B}}(xy) e^{\sum_{x \in E} \alpha_{\mathcal{B}}(xy)}, \sum_{x \in E} \upsilon_{\mathcal{B}}(xy) e^{\sum_{x \in E} \beta_{\mathcal{B}}(xy)}\right)$$

Example 1. Consider a graph $\mathbb{G}^* = (V, E)$ such that $V = \{a, b, c, d\}$, $E = \{ab, ac, bc, cd\}$. Let \mathcal{A} be a *cif-subset of V and let B be a cif-subset of E \subset V × V, as given:*

$$\mathcal{A} = \left(\frac{(0.2e^{i\ 1.3\pi}, 0.4e^{i\ 0.5\pi})}{a}, \frac{(1.0e^{i\ 1.5\pi}, 0.0e^{i\ 0.5\pi})}{b}, \frac{(0.7e^{i\ 0.3\pi}, 0.2e^{i\ 1.5\pi})}{c}, \frac{(0.8e^{i\ 1.1\pi}, 0.1e^{i\ 0.7\pi})}{d}\right).$$
$$\mathcal{B} = \left(\frac{(0.2e^{i\ 0.9\pi}, 0.3e^{i\ 0.4\pi})}{ab}, \frac{(0.1e^{i\ 0.2\pi}, 0.3e^{i\ 0.9\pi})}{ac}, \frac{(0.1e^{i\ 0.1\pi}, 0.2e^{i\ 0.5\pi})}{bc}, \frac{(0.5e^{i\ 0.2\pi}, 0.1e^{i\ 0.5\pi})}{cd}\right).$$

- (*i*) By routine calculations, it can be observed that the graph shown in Figure 1 is a cif-graph.
- (*ii*) Order of cif-graph = $O(\mathbb{G}) = (2.7e^{i \ 4.2\pi}, 0.7e^{i \ 3.2\pi})$
- (iii) Degree of each vertex in \mathbb{G} is

$$deg(a) = (0.3e^{i \ 1.1\pi}, 0.6e^{i \ 1.3\pi}),$$

$$deg(b) = (0.3e^{i \ 1.0\pi}, 0.5e^{i \ 0.9\pi}),$$

$$deg(c) = (0.7e^{i \ 0.5\pi}, 0.6e^{i \ 1.9\pi}),$$

$$deg(d) = (0.5e^{i \ 0.2\pi}, 0.1e^{i \ 0.5\pi}).$$

Definition 8. The Cartesian product $\mathbb{G}_1 \times \mathbb{G}_2$ of two cif-graphs is defined as a pair $\mathbb{G}_1 \times \mathbb{G}_2 = (\mathcal{A}_1 \times \mathcal{A}_2, \mathcal{B}_1 \times \mathcal{B}_2)$, such that:

1.
$$\mu_{\mathcal{A}_{1}\times\mathcal{A}_{2}}(x_{1},x_{2})e^{i\alpha_{\mathcal{A}_{1}\times\mathcal{A}_{2}}(x_{1},x_{2})} = \min\{\mu_{\mathcal{A}_{1}}(x_{1}),\mu_{\mathcal{A}_{2}}(x_{2})\}e^{i\min\{\alpha_{\mathcal{A}_{1}}(x_{1}),\alpha_{\mathcal{A}_{2}}(x_{2})\}} \\ v_{\mathcal{A}_{1}\times\mathcal{A}_{2}}(x_{1},x_{2})e^{i\beta_{\mathcal{A}_{1}\times\mathcal{A}_{2}}(x_{1},x_{2})} = \max\{v_{\mathcal{A}_{1}}(x_{1}),v_{\mathcal{A}_{2}}(x_{2})\}e^{i\max\{\beta_{\mathcal{A}_{1}}(x_{1}),\beta_{\mathcal{A}_{2}}(x_{2})\}} for all x_{1},x_{2} \in V, \\ 2. \quad \mu_{\mathcal{B}_{1}\times\mathcal{B}_{2}}((x,x_{2})(x,y_{2}))e^{i\alpha_{\mathcal{B}_{1}\times\mathcal{B}_{2}}((x,x_{2})(x,y_{2}))} = \min\{\mu_{\mathcal{A}_{1}}(x),\mu_{\mathcal{B}_{2}}(x_{2}y_{2})\}e^{i\min\{\alpha_{\mathcal{A}_{1}}(x),\alpha_{\mathcal{B}_{2}}(x_{2}y_{2})\}} \\ \end{cases}$$

 $v_{\mathcal{B}_1 \times \mathcal{B}_2}((x, x_2)(x, y_2))e^{i\beta_{\mathcal{B}_1 \times \mathcal{B}_2}((x, x_2)(x, y_2))} = \max\{v_{\mathcal{A}_1}(x), v_{\mathcal{B}_2}(x_2 y_2))\}e^{i\max\{\beta_{\mathcal{A}_1}(x), \beta_{\mathcal{B}_2}(x_2 y_2)\}}$ for all $x \in V_1$, and $x_2 y_2 \in E_2$,

3.
$$\mu_{\mathcal{B}_1 \times \mathcal{B}_2}((x_1, z)(y_1, z))e^{i\alpha_{\mathcal{B}_1 \times \mathcal{B}_2}((x_1, z)(y_1, z))} = \min\{\mu_{\mathcal{B}_1}(x_1y_1), \mu_{\mathcal{A}_2}(z)\}e^{i\min\{\alpha_{\mathcal{B}_1}(x_1y_1), \alpha_{\mathcal{A}_2}(z)\}} \\ v_{\mathcal{B}_1 \times \mathcal{B}_2}((x_1, z)(y_1, z))e^{i\beta_{\mathcal{B}_1 \times \mathcal{B}_2}((x_1, z)(y_1, z))} = \max\{v_{\mathcal{B}_1}(x_1y_1), v_{\mathcal{A}_2}(z)\}e^{i\max\{\beta_{\mathcal{B}_1}(x_1y_1), \beta_{\mathcal{A}_2}(z)\}} \\ \text{for all } z \in V_2, \text{ and } x_1y_1 \in E_1.$$



Figure 1. Complex intuitionistic fuzzy graph **G**.

Definition 9. Let \mathbb{G}_1 and \mathbb{G}_2 be two cif-graphs. The degree of a vertex in $\mathbb{G}_1 \times \mathbb{G}_2$ can be defined as follows: for any $(x_1, x_2) \in V_1 \times V_2$,

$$d_{\mathbb{G}_{1}\times\mathbb{G}_{2}}(x_{1},x_{2}) = \begin{pmatrix} \sum_{\substack{(x_{1},x_{2})(y_{1},y_{2})\in E}} \mu_{\mathcal{B}_{1}\times\mathcal{B}_{2}}((x_{1},x_{2})(y_{1},y_{2}))e^{(x_{1},x_{2})(y_{1},y_{2})\in E} \alpha_{\mathcal{B}_{1}\times\mathcal{B}_{2}}((x_{1},x_{2})(y_{1},y_{2}))e^{(x_{1},x_{2})(y_{1},y_{2})\in E} \beta_{\mathcal{B}_{1}\times\mathcal{B}_{2}}((x_{1},x_{2})(y_{1},y_{2}))e^{(x_{1},x_{2})(y_{1},y_{2})\in E} \beta_{\mathcal{B}_{2}\times\mathcal{B}_{2}}((x_{1},x_{2})(y_{1},y_{2}))e^{(x_{1},x_{2})(y_{1},y_{2})(y_{1},y_{2})}((x_{1},x_{2})(y_{1},y_{2}))e^{(x_{1},x_{2})(y_{1},y_{2})}((x_{1},x_{2})(y_{1},y_{2}))e^{(x_{1},x_{2})(y_{1},y_{2})}((x_{1},x_{2})(y_{1},y_{2}))e^{(x_{1},x_{2})(y_{1},y_{2})}((x_{1},x_{2})(y_{1},y_{2}))e^{(x_{1},x_{2})(y_{1},y_{2})}((x_{1},x_{2})(y_{1},y_{2}))e^{(x_{1},x_{2})(y_{1},y_{2})}((x_{1},x_{2})(y_{1},y_{2}))e^{(x_{1},x_{2})(y_{1},y_{2})}((x_{1},x_{2})(y_{1},y_{2}))e^{(x_$$

Example 2. Consider the two cif-graphs \mathbb{G}_1 and \mathbb{G}_2 , as shown in Figures 2 and 3.





Figure 3. Complex intuitionistic fuzzy graph \mathbb{G}_2 .

Then, their corresponding Cartesian product $\mathbb{G}_1 \times \mathbb{G}_2$ is shown in Figure 4.



Figure 4. Complex intuitionistic fuzzy graph of $\mathbb{G}_1 \times \mathbb{G}_2$.

Proposition 1. The Cartesian product of two cif-graphs is a cif-graph.

Proof. The conditions for $A_1 \times A_2$ are obvious, therefore, we verify only conditions for $B_1 \times B_2$. Let $x \in V_1$, and $x_2y_2 \in E_2$. Then,

$$\begin{split} &\mu_{\mathcal{B}_{1}\times\mathcal{B}_{2}}((x,x_{2})(x,y_{2}))e^{i\alpha_{\mathcal{B}_{1}\times\mathcal{B}_{2}}((x,x_{2})(x,y_{2}))} \\ &= \min\{\mu_{\mathcal{A}_{1}}(x),\mu_{\mathcal{B}_{2}}(x_{2}y_{2}))\}e^{i\min\{\alpha_{\mathcal{A}_{1}}(x),\alpha_{\mathcal{B}_{2}}(x_{2}y_{2})\}} \\ &\leq \min\{\mu_{\mathcal{A}_{1}}(x),\min\{\mu_{\mathcal{A}_{2}}(x_{2}),\mu_{\mathcal{A}_{2}}(y_{2})\}\}e^{i\min\{\alpha_{\mathcal{A}_{1}}(x),\min\{\alpha_{\mathcal{A}_{2}}(x_{2}),\alpha_{\mathcal{A}_{2}}(y_{2})\}\}} \\ &= \min\{\min\{\mu_{\mathcal{A}_{1}}(x),\mu_{\mathcal{A}_{2}}(x_{2})\},\min\{\mu_{\mathcal{A}_{1}}(x),\mu_{\mathcal{A}_{2}}(y_{2})\}\}e^{i\min\{\alpha_{\mathcal{A}_{1}}\times\mathcal{A}_{2}}(x,y_{2})},\min\{\alpha_{\mathcal{A}_{1}}(x),\alpha_{\mathcal{A}_{2}}(x,y_{2})\}} \\ &= \min\{\mu_{\mathcal{A}_{1}\times\mathcal{A}_{2}}(x,x_{2}),\mu_{\mathcal{A}_{1}\times\mathcal{A}_{2}}(x,y_{2})\}e^{i\min\{\alpha_{\mathcal{A}_{1}\times\mathcal{A}_{2}}(x,x_{2}),\alpha_{\mathcal{A}_{1}\times\mathcal{A}_{2}}(x,y_{2})\}}, \end{split}$$

$$\begin{split} v_{\mathcal{B}_{1}\times\mathcal{B}_{2}}((x,x_{2})(x,y_{2}))e^{i\gamma_{\mathcal{B}_{1}\times\mathcal{B}_{2}}((x,x_{2})(x,y_{2}))} \\ &= \max\{v_{\mathcal{A}_{1}}(x),v_{\mathcal{B}_{2}}(x_{2}y_{2}))\}e^{i\max\{\gamma_{\mathcal{A}_{1}}(x),\gamma_{\mathcal{B}_{2}}(x_{2}y_{2})\}} \\ &\leq \max\{v_{\mathcal{A}_{1}}(x),\max\{v_{\mathcal{A}_{2}}(x_{2}),v_{\mathcal{A}_{2}}(y_{2})\}\}e^{i\max\{\gamma_{\mathcal{A}_{1}}(x),\max\{\gamma_{\mathcal{A}_{2}}(x_{2}),\gamma_{\mathcal{A}_{2}}(y_{2})\}\}} \\ &= \max\{\max\{v_{\mathcal{A}_{1}}(x),v_{\mathcal{A}_{2}}(x_{2})\},\max\{v_{\mathcal{A}_{1}}(x),v_{\mathcal{A}_{2}}(y_{2})\}\}e^{i\max\{\max\{\gamma_{\mathcal{A}_{1}}(x),\gamma_{\mathcal{A}_{2}}(x_{2})\},\max\{\gamma_{\mathcal{A}_{1}}(x),\gamma_{\mathcal{A}_{2}}(x_{2})\}\}} \\ &= \max\{v_{\mathcal{A}_{1}\times\mathcal{A}_{2}}(x,x_{2}),v_{\mathcal{A}_{1}\times\mathcal{A}_{2}}(x,y_{2})\}e^{i\max\{\gamma_{\mathcal{A}_{1}}\times\mathcal{A}_{2}}(x,x_{2}),\gamma_{\mathcal{A}_{1}\times\mathcal{A}_{2}}(x,y_{2})\}}, \end{split}$$

Similarly, we can prove it for $z \in V_2$ and $x_1y_1 \in E_1$. \Box

Definition 10. The composition $\mathbb{G}_1 \circ \mathbb{G}_2$ of two cif-graphs is defined as a pair $\mathbb{G}_1 \circ \mathbb{G}_2 = (\mathcal{A}_1 \circ \mathcal{A}_2, \mathcal{B}_1 \circ \mathcal{B}_2)$, such that:

- 1. $\mu_{\mathcal{A}_{1}\circ\mathcal{A}_{2}}(x_{1},x_{2})e^{i\alpha_{\mathcal{A}_{1}\circ\mathcal{A}_{2}}(x_{1},x_{2})} = \min\{\mu_{\mathcal{A}_{1}}(x_{1}),\mu_{\mathcal{A}_{2}}(x_{2})\}e^{i\min\{\alpha_{\mathcal{A}_{1}}(x_{1}),\alpha_{\mathcal{A}_{2}}(x_{2})\}}$
- $v_{\mathcal{A}_{1}\circ\mathcal{A}_{2}}(x_{1},x_{2})e^{i\beta_{\mathcal{A}_{1}\circ\mathcal{A}_{2}}(x_{1},x_{2})} = \max\{v_{\mathcal{A}_{1}}(x_{1}),v_{\mathcal{A}_{2}}(x_{2})\}e^{i\max\{\beta_{\mathcal{A}_{1}}(x_{1}),\beta_{\mathcal{A}_{2}}(x_{2})\}} \text{ for all } x_{1},x_{2} \in V,$ 2. $\mu_{\mathcal{B}_{1}\circ\mathcal{B}_{2}}((x,x_{2})(x,y_{2}))e^{i\alpha_{\mathcal{B}_{1}\circ\mathcal{B}_{2}}((x,x_{2})(x,y_{2}))} = \min\{\mu_{\mathcal{A}_{1}}(x),\mu_{\mathcal{B}_{2}}(x_{2}y_{2})\}e^{i\min\{\alpha_{\mathcal{A}_{1}}(x),\alpha_{\mathcal{B}_{2}}(x_{2}y_{2})\}}$ $v_{\mathcal{B}_{1}\circ\mathcal{B}_{2}}((x,x_{2})(x,y_{2}))e^{i\beta_{\mathcal{B}_{1}\circ\mathcal{B}_{2}}((x,x_{2})(x,y_{2}))} = \max\{v_{\mathcal{A}_{1}}(x),v_{\mathcal{B}_{2}}(x_{2}y_{2})\}e^{i\max\{\beta_{\mathcal{A}_{1}}(x),\beta_{\mathcal{B}_{2}}(x_{2}y_{2})\}}$
- for all $x \in V_1$, and $x_2y_2 \in E_2$, 3. $\mu_{\mathcal{B}_1 \circ \mathcal{B}_2}((x_1, z)(y_1, z))e^{i\alpha_{\mathcal{B}_1 \circ \mathcal{B}_2}((x_1, z)(y_1, z))} = \min\{\mu_{\mathcal{B}_1}(x_1y_1), \mu_{\mathcal{A}_2}(z)\}e^{i\min\{\alpha_{\mathcal{B}_1}(x_1y_1), \alpha_{\mathcal{A}_2}(z)\}}$ $v_{\mathcal{B}_1 \circ \mathcal{B}_2}((x_1, z)(y_1, z))e^{i\beta_{\mathcal{B}_1 \circ \mathcal{B}_2}((x_1, z)(y_1, z))} = \max\{v_{\mathcal{B}_1}(x_1y_1), v_{\mathcal{A}_2}(z)\}e^{i\max\{\beta_{\mathcal{B}_1}(x_1y_1), \beta_{\mathcal{A}_2}(z)\}}$ for all $z \in V_2$, and $x_1y_1 \in E_1$.
- 4. $\mu_{\mathcal{B}_{1} \circ \mathcal{B}_{2}}((x_{1}, x_{2})(y_{1}, y_{2}))e^{i\alpha_{\mathcal{B}_{1} \circ \mathcal{B}_{2}}((x_{1}, x_{2})(y_{1}, y_{2}))} = \min\{\mu_{\mathcal{A}_{2}}(x_{2}), \mu_{\mathcal{A}_{2}}(y_{2}), \mu_{\mathcal{B}_{1}}(x_{1}y_{1})\}e^{i\min\{\alpha_{\mathcal{A}_{2}}(x_{2}), \alpha_{\mathcal{A}_{2}}(y_{2}), \alpha_{\mathcal{B}_{1}}(x_{1}y_{1})\}}e^{i\min\{\alpha_{\mathcal{A}_{2}}(x_{2}), \alpha_{\mathcal{A}_{2}}(y_{2}), \alpha_{\mathcal{B}_{1}}(x_{1}y_{1})\}}e^{i\min\{\alpha_{\mathcal{A}_{2}}(x_{2}), \alpha_{\mathcal{B}_{1}}(x_{1}y_{1})\}}e^{i\min\{\alpha_{\mathcal{A}_{2}}(x_{1}y_{1}), \alpha_{\mathcal{B}_{1}}(x_{1}y_{1})\}}e^{i\min\{\alpha_{\mathcal{A}_{2}}(x_{1}y_{1}), \alpha_{\mathcal{B}_{1}}(x_{1}y_{1})\}}e^{i\min\{\alpha_{\mathcal{A}_{2}}(x_{1}y_{1}), \alpha_{\mathcal{B}_{1}}(x_{1}y_{1})\}}e^{i\min\{\alpha_{\mathcal{A}_{2}}(x_{1}y_{1}), \alpha_{\mathcal{B}_{1}}(x_{1}y_{1})\}}e^{i\min\{\alpha_{\mathcal{A}_{2}}(x_{1}y_{1}), \alpha_{\mathcal{B}_{1}}(x_{1}y_{1}), \alpha_{\mathcal{B}_{1}}(x_{1}y_{1}), \alpha_{\mathcal{B}_{1}}(x_{$

 $v_{\mathcal{B}_{1} \circ \mathcal{B}_{2}}((x_{1}, x_{2})(y_{1}, y_{2}))e^{i\beta_{\mathcal{B}_{1} \circ \mathcal{B}_{2}}((x_{1}, x_{2})(y_{1}, y_{2}))} = \max\{v_{\mathcal{A}_{2}}(x_{2}), v_{\mathcal{A}_{2}}(y_{2}), v_{\mathcal{B}_{1}}(x_{1}y_{1})\}e^{i\max\{\beta_{\mathcal{A}_{2}}(x_{2}), \beta_{\mathcal{A}_{2}}(y_{2}), \beta_{\mathcal{B}_{1}}(x_{1}y_{1})\}} for all x_{2}, y_{2} \in V_{2}, x_{2} \neq y_{2} and x_{1}y_{1} \in E_{1}.$

Definition 11. Let \mathbb{G}_1 and \mathbb{G}_2 be two cif-graphs. The degree of a vertex in $\mathbb{G}_1 \circ \mathbb{G}_2$ can be defined as follows: for any $(x_1, x_2) \in V_1 \times V_2$,

$$d_{\mathbb{G}_{1}\circ\mathbb{G}_{2}}(x_{1},x_{2}) = \begin{pmatrix} \sum_{\substack{(x_{1},x_{2})(y_{1},y_{2})\in E}} \mu_{\mathcal{B}_{1}\circ\mathcal{B}_{2}}((x_{1},x_{2})(y_{1},y_{2}))e^{(x_{1},x_{2})(y_{1},y_{2})\in E} \beta_{\mathcal{B}_{1}\circ\mathcal{B}_{2}}((x_{1},x_{2})(y_{1},y_{2}))e^{(x_{1},x_{2})(y_{1},y_{2})\in E} \beta_{\mathcal{B}_{1}\circ\mathcal{B}_{2}}((x_{1},x_{2})(y_{1},y_{2}))e^{(x_{1},x_{2})(y_{1},y_{2})}e^{(x_{1},x_{2$$

Example 3. Consider the two cif-graphs, as shown in Figure 5.

$$a(0.3e^{i\,0.5\pi}, 0.1e^{i\,0.7\pi}) \qquad b(0.7e^{i\,1.5\pi}, 0.3e^{i\,0.9\pi})$$

$$(0.1e^{i\,0.3\pi}, 0.3e^{i\,0.5\pi})$$



Figure 5. Complex intuitionistic fuzzy graphs of \mathbb{G}_1 and \mathbb{G}_2 .

Then, their composition $\mathbb{G}_1 \circ \mathbb{G}_2$ is shown in Figure 6.



Figure 6. Complex intuitionistic fuzzy graph of $\mathbb{G}_1 \circ \mathbb{G}_2$.

Proposition 2. *The composition of two cif-graphs is a cif-graph.*

Definition 12. The union $\mathbb{G}_1 \cup \mathbb{G}_2 = (\mathcal{A}_1 \cup \mathcal{A}_2, \mathcal{B}_1 \cup \mathcal{B}_2)$ of two cif-graphs is defined as follows:

1. $\mu_{\mathcal{A}_1\cup\mathcal{A}_2}(x)e^{i\alpha_{\mathcal{A}_1\cup\mathcal{A}_2}(x)} = \mu_{\mathcal{A}_1}(x)e^{i\alpha_{\mathcal{A}_1}(x)},$

 $v_{\mathcal{A}_1\cup\mathcal{A}_2}(x)e^{i\beta_{\mathcal{A}_1\cup\mathcal{A}_2}(x)} = v_{\mathcal{A}_1}(x)e^{i\beta_{\mathcal{A}_1}(x)}$, for $x \in V_1$ and $x \notin V_2$.

2.
$$\mu_{\mathcal{A}_1 \cup \mathcal{A}_2}(x) e^{i\alpha_{\mathcal{A}_1 \cup \mathcal{A}_2}(x)} = \mu_{\mathcal{A}_2}(x) e^{i\alpha_{\mathcal{A}_2}(x)},$$

 $v_{\mathcal{A}_1\cup\mathcal{A}_2}(x)e^{i\beta_{\mathcal{A}_1\cup\mathcal{A}_2}(x)} = v_{\mathcal{A}_2}(x)e^{i\beta_{\mathcal{A}_2}(x)}$, for $x \in V_2$ and $x \notin V_1$.

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3. $\mu_{\mathcal{A}_1\cup\mathcal{A}_2}(x)e^{i\alpha_{\mathcal{A}_1\cup\mathcal{A}_2}(x)} = \max\{\mu_{\mathcal{A}_1}(x),\mu_{\mathcal{A}_2}(x)\}e^{i\max\{\alpha_{\mathcal{A}_1}(x),\alpha_{\mathcal{A}_2}(x)\}},$

 $v_{\mathcal{A}_1\cup\mathcal{A}_2}(x)e^{i\beta_{\mathcal{A}_1\cup\mathcal{A}_2}(x)} = \min\{v_{\mathcal{A}_1}(x), v_{\mathcal{A}_2}(x)\}e^{i\min\{\beta_{\mathcal{A}_1}(x), \beta_{\mathcal{A}_2}(x)\}}, \text{for } x \in V_1 \cap V_2.$ $\mu_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)e^{i\alpha_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)} = \mu_{\mathcal{B}_1}(xy)e^{i\alpha_{\mathcal{B}_1}(xy)},$

- 4. $\mu_{\mathcal{B}_1 \cup \mathcal{B}_2}(xy)e^{i\alpha_{\mathcal{B}_1 \cup \mathcal{B}_2}(xy)} = \mu_{\mathcal{B}_1}(xy)e^{i\alpha_{\mathcal{B}_1}(xy)},$ $v_{\mathcal{B}_1 \cup \mathcal{B}_2}(xy)e^{i\beta_{\mathcal{B}_1 \cup \mathcal{B}_2}(xy)} = v_{\mathcal{B}_1}(xy)e^{i\beta_{\mathcal{B}_1}(xy)}, \text{ for } xy \in E_1 \text{ and } xy \notin E_2.$ 5. $\mu_{\mathcal{B}_1 \cup \mathcal{B}_2}(xy)e^{i\alpha_{\mathcal{B}_1 \cup \mathcal{B}_2}(xy)} = \mu_{\mathcal{B}_2}(xy)e^{i\alpha_{\mathcal{B}_2}(xy)},$
- 5. $\mu_{\mathcal{B}_1 \cup \mathcal{B}_2}(xy)e^{i\alpha_{\mathcal{B}_1 \cup \mathcal{B}_2}(xy)} = \mu_{\mathcal{B}_2}(xy)e^{i\alpha_{\mathcal{B}_2}(xy)},$ $v_{\mathcal{B}_1 \cup \mathcal{B}_2}(xy)e^{i\beta_{\mathcal{B}_1 \cup \mathcal{B}_2}(xy)} = v_{\mathcal{B}_2}(xy)e^{i\beta_{\mathcal{B}_2}(xy)}, \text{ for } xy \in V_2 \text{ and } xy \notin V_1.$
- 6. $\mu_{\mathcal{B}_1 \cup \mathcal{B}_2}(xy)e^{i\alpha_{\mathcal{B}_1 \cup \mathcal{B}_2}(xy)} = \max\{\mu_{\mathcal{B}_1}(xy), \mu_{\mathcal{B}_2}(xy)\}e^{i\max\{\alpha_{\mathcal{B}_1}(xy), \alpha_{\mathcal{B}_2}(xy)\}},$

 $v_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)e^{i\beta_{\mathcal{B}_1\cup\mathcal{B}_2}(xy)} = \min\{v_{\mathcal{B}_1}(xy), v_{\mathcal{B}_2}(xy)\}e^{i\min\{\beta_{\mathcal{B}_1}(xy),\beta_{\mathcal{B}_2}(xy)\}}, \text{for } xy \in V_1 \cap V_2.$

Example 4. Consider the two cif-graphs, as shown in Figure 7.



Figure 7. Complex intuitionistic fuzzy graphs of \mathbb{G}_1 and \mathbb{G}_2 .

Then, their corresponding union $\mathbb{G}_1 \cup \mathbb{G}_2$ *is shown in Figure 8.*



Figure 8. Complex intuitionistic fuzzy graph of $\mathbb{G}_1 \cup \mathbb{G}_2$.

Proposition 3. *The union of two cif-graphs is a cif-graph.*

Definition 13. The join $\mathbb{G}_1 + \mathbb{G}_2 = (\mathcal{A}_1 + \mathcal{A}_2, \mathcal{B}_1 + \mathcal{B}_2)$ of two cif-graphs, where $V_1 \cap V_2 = \emptyset$, is defined as follows:

 $1. \quad \begin{cases} \ \mu_{\mathcal{A}_{1}+\mathcal{A}_{2}}(x)e^{i\alpha_{\mathcal{A}_{1}+\mathcal{A}_{2}}(x)} = \mu_{\mathcal{A}_{1}\cup\mathcal{A}_{2}}(x)e^{i\alpha_{\mathcal{A}_{1}\cup\mathcal{A}_{2}}(x)} \\ v_{\mathcal{A}_{1}+\mathcal{A}_{2}}(x)e^{i\beta_{\mathcal{A}_{1}+\mathcal{A}_{2}}(x)} = v_{\mathcal{A}_{1}\cup\mathcal{A}_{2}}(x)e^{i\beta_{\mathcal{A}_{1}\cup\mathcal{A}_{2}}(x)} \quad \text{if } x \in V_{1}\cup V_{2}, \end{cases}$

2.
$$\begin{cases} \mu_{\mathcal{B}_{1}+\mathcal{B}_{2}}(xy)e^{i\alpha_{\mathcal{B}_{1}+\mathcal{B}_{2}}(x)} = \mu_{\mathcal{B}_{1}\cup\mathcal{B}_{2}}(xy)e^{i\alpha_{\mathcal{B}_{1}\cup\mathcal{B}_{2}}(x)} \\ \nu_{\mathcal{B}_{1}+\mathcal{B}_{2}}(xy)e^{i\beta_{\mathcal{B}_{1}+\mathcal{B}_{2}}(x)} = \nu_{\mathcal{B}_{1}\cup\mathcal{B}_{2}}(xy)e^{i\beta_{\mathcal{B}_{1}\cup\mathcal{B}_{2}}(x)} & \text{if } xy \in E_{1} \cap E_{2}, \end{cases}$$

3.
$$\begin{cases} \mu_{\mathcal{B}_1+\mathcal{B}_2}(xy)e^{i\alpha_{\mathcal{B}_1+\mathcal{B}_2}(xy)} = \min\{\mu_{\mathcal{A}_1}(x), \mu_{\mathcal{A}_2}(y)\}e^{i\min\{\alpha_{\mathcal{A}_1}(x), \alpha_{\mathcal{A}_2}(y)\}}\\ v_{\mathcal{B}_1+\mathcal{B}_2}(xy)e^{i\beta_{\mathcal{B}_1+\mathcal{B}_2}(xy)} = \max\{v_{\mathcal{A}_1}(x), v_{\mathcal{A}_2}(y)\}e^{i\max\{\beta_{\mathcal{A}_1}(x), \beta_{\mathcal{A}_2}(y)\}} & \text{if } xy \in \acute{E}, \text{ where } \acute{E} \text{ is the set of all edges joining the vertices of } V_1 \text{ and } V_2. \end{cases}$$



Figure 9. Complex intuitionistic fuzzy graphs of \mathbb{G}_1 and \mathbb{G}_2 .

Then, their corresponding join $\mathbb{G}_1 + \mathbb{G}_2$ *is shown in Figure 10.*



Figure 10. Complex intuitionistic fuzzy graph of $\mathbb{G}_1 + \mathbb{G}_2$.

Proposition 4. *The join of two cif-graphs is a cif-graph.*

Proposition 5. Let $\mathbb{G}_1 = (\mathcal{A}_1, \mathcal{B}_1)$ and $\mathbb{G}_2 = (\mathcal{A}_2, \mathcal{B}_2)$ be cif-graphs of the graphs G_1^* and G_2^* and let $V_1 \cap V_2 = \emptyset$. Then, union $\mathbb{G}_1 \cup \mathbb{G}_2 = (\mathcal{A}_1 \cup \mathcal{A}_2, \mathcal{B}_1 \cup \mathcal{B}_2)$ is a cif-graph of G^* if and only if \mathbb{G}_1 and \mathbb{G}_2 are cif-graphs of the graphs G_1^* and G_2^* , respectively.

Proof. Suppose that $\mathbb{G}_1 \cup \mathbb{G}_2$ is a cif-graph. Let $xy \in E_1$. Then, $xy \notin E_2$ and $x, y \in V_1 - V_2$. Thus,

$$\begin{split} \mu_{\mathcal{B}_{1}}(xy)e^{i\alpha_{\mathcal{B}_{1}}(x)} &= \mu_{\mathcal{B}_{1}\cap\mathcal{B}_{2}}(xy)e^{i\alpha_{\mathcal{B}_{1}\cap\mathcal{B}_{2}}(x)} \\ &\leq \min(\mu_{\mathcal{A}_{1}\cap\mathcal{A}_{2}}(x),\mu_{\mathcal{A}_{1}\cap\mathcal{A}_{2}}(y))e^{i\min(\alpha_{\mathcal{A}_{1}\cap\mathcal{A}_{2}}(x),\alpha_{\mathcal{A}_{1}\cap\mathcal{A}_{2}}(y))} \\ &= \min(\mu_{\mathcal{A}_{1}}(x),\mu_{\mathcal{A}_{1}}(y))e^{i\min(\alpha_{\mathcal{A}_{1}}(x),\alpha_{\mathcal{A}_{1}}(y))}. \end{split}$$

$$\begin{aligned} v_{\mathcal{B}_{1}}(xy)e^{i\beta_{\mathcal{B}_{1}}(x)} &= v_{\mathcal{B}_{1}\cap\mathcal{B}_{2}}(xy)e^{i\beta_{\mathcal{B}_{1}\cap\mathcal{B}_{2}}(x)} \\ &\leq \max(v_{\mathcal{A}_{1}\cap\mathcal{A}_{2}}(x),v_{\mathcal{A}_{1}\cap\mathcal{A}_{2}}(y))e^{i\max(\beta_{\mathcal{A}_{1}\cap\mathcal{A}_{2}}(x),\beta_{\mathcal{A}_{1}\cap\mathcal{A}_{2}}(y))} \\ &= \max(v_{\mathcal{A}_{1}}(x),v_{\mathcal{A}_{1}}(y))e^{i\max(\beta_{\mathcal{A}_{1}}(x),\beta_{\mathcal{A}_{1}}(y))}. \end{aligned}$$

This shows that $\mathbb{G}_1 = (\mathcal{A}_1, \mathcal{B}_1)$ is a cif-graph. Similarly, we can show that $\mathbb{G}_2 = (\mathcal{A}_2, \mathcal{B}_2)$ is a cif-graph. The converse part is obvious. \Box

Proposition 6. Let $\mathbb{G}_1 = (\mathcal{A}_1, \mathcal{B}_1)$ and $\mathbb{G}_2 = (\mathcal{A}_2, \mathcal{B}_2)$ be cif-graphs of the graphs G_1^* and G_2^* and let $V_1 \cap V_2 = \emptyset$. Then, join $\mathbb{G}_1 + \mathbb{G}_2 = (\mathcal{A}_1 + \mathcal{A}_2, \mathcal{B}_1 + \mathcal{B}_2)$ is a cif-graph of G^* if and only if \mathbb{G}_1 and \mathbb{G}_2 are *cif-graphs of the graphs* G_1^* *and* G_2^* *, respectively.*

Proof. The proof is similar to the proof of Proposition 5. \Box

4. Isomorphisms of cif-Graphs

In this section, we discuss isomorphisms of cif-graphs.

Definition 14. Let $\mathbb{G}_1 = (\mathcal{A}_1, \mathcal{B}_1)$ and $\mathbb{G}_2 = (\mathcal{A}_2, \mathcal{B}_2)$ be two cif-graphs. A homomorphism $g : \mathbb{G}_1 \to \mathbb{G}_2$ is a mapping $g: V_1 \rightarrow V_2$ such that:

1.
$$\begin{cases} \mu_{\mathcal{A}_{1}}(x_{1})e^{i\alpha_{\mathcal{A}_{1}}(x_{1})} \leq \mu_{\mathcal{A}_{2}}(g(x_{1}))e^{i\alpha_{\mathcal{A}_{2}}(g(x_{1}))} \\ v_{\mathcal{A}_{1}}(x_{1})e^{i\beta_{\mathcal{A}_{1}}(x_{1})} \leq v_{\mathcal{A}_{2}}(g(x_{1}))e^{i\beta_{\mathcal{A}_{2}}(g(x_{1}))} & \text{for all } x_{1} \in V_{1}, \end{cases}$$

2. $\begin{cases} \mu_{\mathcal{B}_{1}}(x_{1}y_{1})e^{i\alpha_{\mathcal{B}_{1}}(x_{1}y_{1})} \leq \mu_{\mathcal{B}_{2}}(g(x_{1})g(y_{1}))e^{i\alpha_{\mathcal{B}_{2}}(g(x_{1})g(y_{1}))} \\ v_{\mathcal{B}_{1}}(x_{1}y_{1})e^{i\beta_{\mathcal{B}_{1}}(x_{1}y_{1})} \leq v_{\mathcal{B}_{2}}(g(x_{1})g(y_{1}))e^{i\beta_{\mathcal{B}_{2}}(g(x_{1})g(y_{1}))} \\ A \text{ bijective homomorphism with the property} \\ \end{cases} \int \mu_{\mathcal{A}_{1}}(x_{1})e^{i\alpha_{\mathcal{A}_{1}}(x_{1})} = \mu_{\mathcal{A}_{1}}(g(x_{1}))e^{i\alpha_{\mathcal{A}_{2}}(g(x_{1}))} \end{cases}$

3.
$$\begin{cases} \mu_{\mathcal{A}_{1}}(x_{1})e^{i\alpha_{\mathcal{A}_{1}}(x_{1})} = \mu_{\mathcal{A}_{2}}(g(x_{1}))e^{i\alpha_{\mathcal{A}_{2}}(g(x_{1}))} \\ v_{\mathcal{A}_{1}}(x_{1})e^{i\beta_{\mathcal{A}_{1}}(x_{1})} = v_{\mathcal{A}_{2}}(g(x_{1}))e^{i\beta_{\mathcal{A}_{2}}(g(x_{1}))} \\ A \text{ bijective homomorphism with the property} \\ & \int \mu_{\mathcal{B}_{1}}(x_{1}y_{1})e^{i\alpha_{\mathcal{B}_{1}}(x_{1}y_{1})} = \mu_{\mathcal{B}_{2}}(g(x_{1})g(y_{1}))e^{i\alpha_{\mathcal{B}_{2}}(g(x_{1})g(y_{1}))} \\ \text{ for all } x_{1}y_{1} \in \mathbb{F}_{1} \text{ is called a set} \end{cases}$$

4.
$$\begin{cases} \mu_{\mathcal{B}_1}(x_1y_1)e^{-1} & = -\mu_{\mathcal{B}_2}(g(x_1)g(y_1))e^{-2} \\ v_{\mathcal{B}_1}(x_1y_1)e^{i\beta_{\mathcal{B}_1}(x_1y_1)} &= v_{\mathcal{B}_2}(g(x_1)g(y_1))e^{i\beta_{\mathcal{B}_2}(g(x_1)g(y_1))} \\ \text{co-isomorphism. A bijective mapping } g: \mathbb{G}_1 \to \mathbb{G}_2 \text{ satisfying 3 and 4 is called an isomorphism.} \end{cases}$$

Example 6. Consider two cif-graphs, as shown in Figure 11.



Figure 11. Complex intuitionistic fuzzy graphs of \mathbb{G}_1 and \mathbb{G}_2 .

Then, it is easy to see that the mapping $g: V_1 \to V_2$ defined by $g(a_1) = b_2$ and $g(b_1) = a_2$ is a weak isomorphism.

Proposition 7. An isomorphism between cif-graphs is an equivalence relation.

Proof. The reflexivity and symmetry are obvious. To prove the transitivity, we let $f : V_1 \to V_2$ and $g: V_2 \to V_3$ be the isomorphisms of \mathbb{G}_1 onto \mathbb{G}_2 and \mathbb{G}_2 onto \mathbb{G}_3 , respectively. Then, $g \circ f: V_1 \to V_3$ is a bijective map from V_1 to V_3 , where $(g \circ f)(x_1) = g(f(x_1))$ for all $x_1 \in V_1$. Since a map $f: V_1 \to V_2$ defined by $f(x_1) = x_2$ for $x_1 \in V_1$ is an isomorphism. Now

$$\mu_{\mathcal{A}_1}(x_1)e^{i\alpha_{\mathcal{A}_1}(x_1)} = \mu_{\mathcal{A}_2}(f(x_1))e^{i\alpha_{\mathcal{A}_2}(f(x_1))}$$

= $\mu_{\mathcal{A}_2}(x_2)e^{i\alpha_{\mathcal{A}_2}(x_2)}$ for all $x_1 \in V_1 \cdots (A_1)$,

$$\begin{aligned} v_{\mathcal{A}_1}(x_1)e^{i\beta_{\mathcal{A}_1}(x_1)} &= v_{\mathcal{A}_2}(f(x_1))e^{i\beta_{\mathcal{A}_2}(f(x_1))} \\ &= v_{\mathcal{A}_2}(x_2)e^{i\beta_{\mathcal{A}_2}(x_2)} \text{ for all } x_1 \in V_1 \cdots (A_2), \end{aligned}$$

$$\mu_{\mathcal{B}_1}(x_1y_1)e^{i\alpha_{\mathcal{B}_1}(x_1y_1)} = \mu_{\mathcal{B}_2}(f(x_1)f(y_1))e^{i\alpha_{\mathcal{B}_2}(f(x_1)f(y_1))}$$
$$= \mu_{\mathcal{B}_2}(x_2y_2)e^{i\alpha_{\mathcal{B}_2}(x_2y_2)} \text{ for all } x_1y_1 \in E_1 \cdots (B_1).$$

$$v_{\mathcal{B}_1}(x_1y_1)e^{i\beta_{\mathcal{B}_1}(x_1y_1)} = v_{\mathcal{B}_2}(f(x_1)f(y_1))e^{i\beta_{\mathcal{B}_2}(f(x_1)f(y_1))}$$

= $v_{\mathcal{B}_2}(x_2y_2)e^{i\beta_{\mathcal{B}_2}(x_2y_2)}$ for all $x_1y_1 \in E_1 \cdots (B_2)$.

Since a map $g: V_2 \to V_3$ defined by $g(x_2) = x_3$ for $x_2 \in V_2$ is an isomorphism,

$$\mu_{\mathcal{A}_{2}}(x_{2})e^{i\alpha_{\mathcal{A}_{2}}(x_{2})} = \mu_{\mathcal{A}_{3}}(g(x_{2}))e^{i\alpha_{\mathcal{A}_{3}}(g(x_{2}))}$$
$$= \mu_{\mathcal{A}_{3}}(x_{3})e^{i\alpha_{\mathcal{A}_{3}}(x_{3})} \text{ for all } x_{2} \in V_{2}\cdots(C_{1}),$$

$$\begin{aligned} v_{\mathcal{A}_{2}}(x_{2})e^{i\beta_{\mathcal{A}_{2}}(x_{2})} &= v_{\mathcal{A}_{3}}(g(x_{2}))e^{i\beta_{\mathcal{A}_{3}}(g(x_{2}))} \\ &= v_{\mathcal{A}_{3}}(x_{3})e^{i\beta_{\mathcal{A}_{3}}(x_{3})} \text{ for all } x_{2} \in V_{2} \cdots (C_{2}), \end{aligned}$$

$$\mu_{\mathcal{B}_2}(x_2y_2)e^{i\alpha_{\mathcal{B}_2}(x_2y_2)} = \mu_{\mathcal{B}_3}(g(x_2)g(y_2))e^{i\alpha_{\mathcal{B}_3}(g(x_2)g(y_2))}$$
$$= \mu_{\mathcal{B}_3}(x_3y_3)e^{i\alpha_{\mathcal{B}_3}(x_3y_3)} \text{ for all } x_2y_2 \in E_2 \cdots (D_1).$$

$$v_{\mathcal{B}_2}(x_2y_2)e^{i\beta_{\mathcal{B}_2}(x_2y_2)} = v_{\mathcal{B}_3}(g(x_2)g(y_2))e^{i\beta_{\mathcal{B}_3}(g(x_2)g(y_2))}$$

= $v_{\mathcal{B}_3}(x_3y_3)e^{i\beta_{\mathcal{B}_3}(x_3y_3)}$ for all $x_2y_2 \in E_2 \cdots (D_2)$.

From (A_1) , (C_1) and $f(x_1) = x_2, x_1 \in V_1$, we have

$$\begin{split} \mu_{\mathcal{A}_1}(x_1) e^{i\alpha_{\mathcal{A}_1}(x_1)} &= \mu_{\mathcal{A}_2}(f(x_1)) e^{i\alpha_{\mathcal{A}_2}(f(x_1))} = \mu_{\mathcal{A}_2}(x_2) e^{i\alpha_{\mathcal{A}_2}(x_2)} \\ &= \mu_{\mathcal{A}_3}(g(x_2)) e^{i\alpha_{\mathcal{A}_3}(g(x_2))} \\ &= \mu_{\mathcal{A}_3}(g(f(x_1))) e^{i\alpha_{\mathcal{A}_3}(g(f(x_1)))}. \end{split}$$

From (A_2) , (C_2) and $f(x_1) = x_2, x_1 \in V_1$, we have

$$\begin{aligned} v_{\mathcal{A}_{1}}(x_{1})e^{i\beta_{\mathcal{A}_{1}}(x_{1})} &= v_{\mathcal{A}_{2}}(f(x_{1}))e^{i\beta_{\mathcal{A}_{2}}(f(x_{1}))} = v_{\mathcal{A}_{2}}(x_{2})e^{i\beta_{\mathcal{A}_{2}}(x_{2})} \\ &= v_{\mathcal{A}_{3}}(g(x_{2}))e^{i\beta_{\mathcal{A}_{3}}(g(x_{2}))} \\ &= v_{\mathcal{A}_{3}}(g(f(x_{1})))e^{i\beta_{\mathcal{A}_{3}}(g(f(x_{1})))}. \end{aligned}$$

From (B_1) and (D_1) , we have

$$\begin{split} \mu_{\mathcal{B}_1}(x_1y_1)e^{i\alpha_{\mathcal{B}_1}(x_1y_1)} &= \mu_{\mathcal{B}_2}(f(x_1)f(y_1))e^{i\alpha_{\mathcal{B}_2}(f(x_1)f(y_1))} = \mu_{\mathcal{B}_2}(x_2y_2)e^{i\alpha_{\mathcal{B}_2}(x_2y_2)} \\ &= \mu_{\mathcal{B}_3}(g(x_2)g(y_2))e^{i\alpha_{\mathcal{B}_3}(g(x_2)g(y_2))} \\ &= \mu_{\mathcal{B}_3}(g(f(x_1))g(f(y_1)))e^{i\alpha_{\mathcal{B}_3}(g(f(x_1))g(f(y_1)))}. \end{split}$$

From (B_2) and (D_2) , we have

$$\begin{aligned} v_{\mathcal{B}_1}(x_1y_1)e^{i\beta_{\mathcal{B}_1}(x_1y_1)} &= v_{\mathcal{B}_2}(f(x_1)f(y_1))e^{i\beta_{\mathcal{B}_2}(f(x_1)f(y_1))} = v_{\mathcal{B}_2}(x_2y_2)e^{i\beta_{\mathcal{B}_2}(x_2y_2)} \\ &= v_{\mathcal{B}_3}(g(x_2)g(y_2))e^{i\beta_{\mathcal{B}_3}(g(x_2)g(y_2))} \\ &= v_{\mathcal{B}_3}(g(f(x_1))g(f(y_1)))e^{i\beta_{\mathcal{B}_3}(g(f(x_1))g(f(y_1)))}, \end{aligned}$$

for all $x_1y_1 \in E_1$. Therefore, $g \circ f$ is an isomorphism between \mathbb{G}_1 and \mathbb{G}_3 . This completes the proof. \Box

Proposition 8. A weak isomorphism (co-isomorphism) between cif-graphs is a partial ordering relation.

Proof. The reflexivity and transitivity are obvious. To prove the anti-symmetry, we let $f : V_1 \to V_2$ be a strong isomorphism of \mathbb{G}_1 onto \mathbb{G}_2 . Then, f is a bijective map defined by $f(x_1) = x_2$ for all $x_1 \in V_1$ satisfying

$$\begin{split} \mu_{\mathcal{A}_{1}}(x_{1})e^{i\alpha_{\mathcal{A}_{1}}(x_{1})} &= \mu_{\mathcal{A}_{2}}(f(x_{1}))e^{i\alpha_{\mathcal{A}_{2}}(f(x_{1}))} \text{ for all } x_{1} \in V_{1}, \\ v_{\mathcal{A}_{1}}(x_{1})e^{i\beta_{\mathcal{A}_{1}}(x_{1})} &= v_{\mathcal{A}_{2}}(f(x_{1}))e^{i\beta_{\mathcal{A}_{2}}(f(x_{1}))} \text{ for all } x_{1} \in V_{1}, \\ \mu_{\mathcal{B}_{1}}(x_{1}y_{1})e^{i\alpha_{\mathcal{B}_{1}}(x_{1}y_{1})} &\leq \mu_{\mathcal{B}_{2}}(f(x_{1})f(y_{1}))e^{i\alpha_{\mathcal{B}_{2}}(f(x_{1})f(y_{1}))} \text{ for all } x_{1}y_{1} \in E_{1}\cdots(I_{1}). \\ v_{\mathcal{B}_{1}}(x_{1}y_{1})e^{i\beta_{\mathcal{B}_{1}}(x_{1}y_{1})} &\leq v_{\mathcal{B}_{2}}(f(x_{1})f(y_{1}))e^{i\beta_{\mathcal{B}_{2}}(f(x_{1})f(y_{1}))} \text{ for all } x_{1}y_{1} \in E_{1}\cdots(I_{2}). \end{split}$$

Let $g : V_2 \to V_1$ be a strong isomorphism of \mathbb{G}_2 onto \mathbb{G}_1 . Then, g is a bijective map defined by $g(x_2) = x_1$ for all $x_2 \in V_2$ satisfying

$$\begin{split} \mu_{\mathcal{A}_{2}}(x_{2})e^{i\alpha_{\mathcal{A}_{2}}(x_{2})} &= \mu_{\mathcal{A}_{1}}(g(x_{2}))e^{i\alpha_{\mathcal{A}_{1}}(g(x_{2}))} \text{ for all } x_{2} \in V_{2}, \\ v_{\mathcal{A}_{2}}(x_{2})e^{i\beta_{\mathcal{A}_{2}}(x_{2})} &= v_{\mathcal{A}_{1}}(g(x_{2}))e^{i\beta_{\mathcal{A}_{1}}(g(x_{2}))} \text{ for all } x_{2} \in V_{2}, \\ \mu_{\mathcal{B}_{2}}(x_{2}y_{2})e^{i\alpha_{\mathcal{B}_{2}}(x_{2}y_{2})} &\leq \mu_{\mathcal{B}_{1}}(g(x_{2})g(y_{2}))e^{i\alpha_{\mathcal{B}_{1}}(g(x_{2})g(y_{2}))} \text{ for all } x_{2}y_{2} \in E_{2} \cdots (J_{1}). \\ v_{\mathcal{B}_{2}}(x_{2}y_{2})e^{i\beta_{\mathcal{B}_{2}}(x_{2}y_{2})} &\leq v_{\mathcal{B}_{1}}(g(x_{2})g(y_{2}))e^{i\beta_{\mathcal{B}_{1}}(g(x_{2})g(y_{2}))} \text{ for all } x_{2}y_{2} \in E_{2} \cdots (J_{2}). \end{split}$$

The inequalities (I_1) , (J_1) and (I_2) , (J_2) hold on the finite sets V_1 and V_2 only when \mathbb{G}_1 and \mathbb{G}_2 have the same number of edges and the corresponding edges have same weight. Hence, \mathbb{G}_1 and \mathbb{G}_2 are identical. Therefore, $g \circ f$ is a strong isomorphism between \mathbb{G}_1 and \mathbb{G}_3 . This completes the proof. \Box

5. Complement of cif-Graphs

In this section, we discuss complements of cif-graphs.

Definition 15. The complement of a weak cif-graph $\mathbb{G} = (\mathcal{A}, \mathcal{B})$ of $G^* = (V, E)$ is a weak cif-graph $\overline{\mathbb{G}} = (\overline{\mathcal{A}}, \overline{\mathcal{B}})$ on $\overline{G^*}$, is defined by

(i)
$$\overline{V} = V$$
,

(ii)
$$\begin{cases} \mu_{\overline{\mathcal{A}}}(x)e^{i\alpha_{\overline{\mathcal{A}}}(x)} = \mu_{\mathcal{A}}(x)e^{i\alpha_{\mathcal{A}}(x)} \\ v_{\overline{\mathcal{A}}}(x)e^{i\beta_{\overline{\mathcal{A}}}(x)} = v_{\mathcal{A}}(x)e^{i\beta_{\mathcal{A}}(x)} & \text{for all } x \in V, \end{cases}$$

$$(iii) \begin{cases} \mu_{\overline{B}}(xy)e^{i\alpha_{\overline{B}}(xy)} = \begin{cases} 0 & \text{if } \mu_{B}(xy)e^{i\alpha_{B}(xy)} \neq 0\\ \min\{\mu_{A}(x),\mu_{A}(y)\}e^{i\min\{\alpha_{A}(x),\alpha_{A}(y)\}} & \text{if } \mu_{B}(xy)e^{i\alpha_{B}(xy)} = 0,\\ \text{if } \nu_{\overline{B}}(xy)e^{i\beta_{\overline{B}}(xy)} = \begin{cases} 0 & \text{if } \nu_{B}(xy)e^{i\beta_{B}(xy)} \neq 0\\ \max\{\nu_{A}(x),\nu_{A}(y)\}e^{i\max\{\beta_{A}(x),\beta_{A}(y)\}} & \text{if } \nu_{B}(xy)e^{i\beta_{B}(xy)} = 0. \end{cases} \end{cases}$$

Example 7. *Consider a cif-graph* G*, as shown in Figure* 12*.*



Figure 12. Complex intuitionistic fuzzy graph of **G**.

Then, the complement $\overline{\mathbb{G}}$ of \mathbb{G} is shown in Figure 13.



Figure 13. Complex intuitionistic fuzzy graph of $\overline{\mathbb{G}}$.

Definition 16. A cif-graph G is called self complementary if $\overline{\mathbb{G}} \approx \mathbb{G}$.

The following propositions are obvious.

Proposition 9. Let $\mathbb{G} = (\mathcal{A}, \mathcal{B})$ be a self complementary cif-graph. Then,

$$\sum_{x \neq y} \mu_{\mathcal{B}}(xy) e^{i\alpha_{\mathcal{B}}(xy)} = \sum_{x \neq y} \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(y)\} e^{i\min\{\alpha_{\mathcal{A}}(x), \alpha_{\mathcal{A}}(y)\}}$$
$$\sum_{x \neq y} v_{\mathcal{B}}(xy) e^{i\beta_{\mathcal{B}}(xy)} = \sum_{x \neq y} \max\{v_{\mathcal{A}}(x), v_{\mathcal{A}}(y)\} e^{i\max\{\beta_{\mathcal{A}}(x), \beta_{\mathcal{A}}(y)\}}.$$

Proposition 10. Let $\mathbb{G} = (\mathcal{A}, \mathcal{B})$ be a cif-graph. If

$$\mu_{\mathcal{B}}(xy)e^{i\alpha_{\mathcal{B}}(xy)} = \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(y)\}e^{i\min\{\alpha_{\mathcal{A}}(x), \alpha_{\mathcal{A}}(y)\}}$$
$$v_{\mathcal{B}}(xy)e^{i\beta_{\mathcal{B}}(xy)} = \max\{v_{\mathcal{A}}(x), v_{\mathcal{A}}(y)\}e^{i\max\{\beta_{\mathcal{A}}(x), \beta_{\mathcal{A}}(y)\}},$$

 $x, y \in V$, then G is self complementary.

Proposition 11. Let \mathbb{G}_1 and \mathbb{G}_2 be cif-graphs. If there is a strong isomorphism between \mathbb{G}_1 and \mathbb{G}_2 , then there is a strong isomorphism between $\overline{\mathbb{G}_1}$ and $\overline{\mathbb{G}_2}$.

Proof. Let *f* be a strong isomorphism between \mathbb{G}_1 and \mathbb{G}_2 . Then, $f : V_1 \to V_2$ is a bijective map that satisfies

$$f(x_1) = x_2 \text{ for all } x_1 \in V_1,$$

$$\mu_{\mathcal{A}_1}(x_1)e^{i\alpha_{\mathcal{A}_1}(x_1)} = \mu_{\mathcal{A}_2}(f(x_1))e^{i\alpha_{\mathcal{A}_2}(f(x_1))} \text{ for all } x \in V_1.$$

$$v_{\mathcal{A}_1}(x_1)e^{i\beta_{\mathcal{A}_1}(x_1)} = v_{\mathcal{A}_2}(f(x_1))e^{i\beta_{\mathcal{A}_2}(f(x_1))} \text{ for all } x \in V_1.$$

$$\mu_{\mathcal{A}_1}(x_1y_1)e^{i\alpha_{\mathcal{A}_1}(x_1y_1)} \le \mu_{\mathcal{A}_2}(f(x_1)f(y_1))e^{i\alpha_{\mathcal{A}_2}(f(x_1)f(y_1))} \text{ for all } x_1y_1 \in E_1.$$
$$v_{\mathcal{A}_1}(x_1y_1)e^{i\beta_{\mathcal{A}_1}(x_1y_1)} \le v_{\mathcal{A}_2}(f(x_1)f(y_1))e^{i\beta_{\mathcal{A}_2}(f(x_1)f(y_1))} \text{ for all } x_1y_1 \in E_1.$$

Since $f : V_1 \to V_2$ is a bijective map, $f^{-1} : V_2 \to V_1$ is also bijective map such that $f^{-1}(x_2) = x_1$ for all $x_2 \in V_2$. Thus

$$\begin{aligned} &\mu_{\mathcal{A}_1}(f^{-1}(x_2))e^{i\alpha_{\mathcal{A}_1}(f^{-1}(x_2))} = \mu_{\mathcal{A}_2}(x_2)e^{i\alpha_{\mathcal{A}_2}(x_2)} & \text{for all } x_2 \in V_2. \\ &v_{\mathcal{A}_1}(f^{-1}(x_2))e^{i\beta_{\mathcal{A}_1}(f^{-1}(x_2))} = v_{\mathcal{A}_2}(x_2)e^{i\beta_{\mathcal{A}_2}(x_2)} & \text{for all } x_2 \in V_2. \end{aligned}$$

By definition of complement, we have

$$\begin{split} \mu_{\overline{\mathcal{B}}_{1}}(x_{1}y_{1})e^{i\alpha_{\overline{\mathcal{B}}_{1}}(x_{1}y_{1})} &= \min\{\mu_{\mathcal{A}_{1}}(x_{1}), \mu_{\mathcal{A}_{1}}(y_{1})\}e^{i\min\{\alpha_{\mathcal{A}_{1}}(x_{1}), \alpha_{\mathcal{A}_{1}}(y_{1})\}} \\ &\leq \min\{\mu_{\mathcal{A}_{2}}(f(x_{2})), \mu_{\mathcal{A}_{2}}(f(y_{2}))\}e^{i\min\{\alpha_{\mathcal{A}_{2}}(f(x_{2})), \alpha_{\mathcal{A}_{2}}(f(y_{2}))\}} \\ &= \min\{\mu_{\mathcal{A}_{2}}(x_{2}), \mu_{\mathcal{A}_{2}}(y_{2})\}e^{i\min\{\alpha_{\mathcal{A}_{2}}(x_{2}), \alpha_{\mathcal{A}_{2}}(y_{2})\}} \\ &= \mu_{\mathcal{B}_{2}}(x_{2}y_{2})e^{i\alpha_{\mathcal{B}_{2}}(x_{2}y_{2})}. \end{split}$$

$$v_{\overline{\mathcal{B}_{1}}}(x_{1}y_{1})e^{i\beta_{\overline{\mathcal{B}_{1}}}(x_{1}y_{1})} = \max\{v_{\mathcal{A}_{1}}(x_{1}), v_{\mathcal{A}_{1}}(y_{1})\}e^{i\max\{\beta_{\mathcal{A}_{1}}(x_{1}), \beta_{\mathcal{A}_{1}}(y_{1})\}} \\ \leq \max\{v_{\mathcal{A}_{2}}(f(x_{2})), v_{\mathcal{A}_{2}}(f(y_{2}))\}e^{i\max\{\beta_{\mathcal{A}_{2}}(f(x_{2})), \beta_{\mathcal{A}_{2}}(f(y_{2}))\}} \\ = \max\{v_{\mathcal{A}_{2}}(x_{2}), v_{\mathcal{A}_{2}}(y_{2})\}e^{i\max\{\beta_{\mathcal{A}_{2}}(x_{2}), \beta_{\mathcal{A}_{2}}(y_{2})\}} \\ = v_{\mathcal{B}_{2}}(x_{2}y_{2})e^{i\beta_{\mathcal{B}_{2}}(x_{2}y_{2})}.$$

Thus, $f^{-1}: V_2 \to V_1$ is a bijective map which is a strong isomorphism between $\overline{\mathbb{G}_1}$ and $\overline{\mathbb{G}_2}$. This ends the proof. \Box

The following Proposition is obvious.

Proposition 12. Let \mathbb{G}_1 and \mathbb{G}_2 be cif-graphs. Then, $\mathbb{G}_1 \cong \mathbb{G}_2$ if and only if $\overline{\mathbb{G}_1} \cong \overline{\mathbb{G}_2}$.

Proposition 13. Let \mathbb{G}_1 and \mathbb{G}_2 be cif-graphs. If there is a co-strong isomorphism between \mathbb{G}_1 and \mathbb{G}_2 , then there is a homomorphism between $\overline{\mathbb{G}_1}$ and $\overline{\mathbb{G}_2}$.

6. Application

Intuitionistic fuzzy sets are the valuable generalization of fuzzy sets. We combine complex intuitionistic fuzzy sets with the graph theory. Complex intuitionistic fuzzy graphs have many

applications in database theory, expert systems, neural networks, decision making problems, GIS-based road networks, facility location problems and so on. In the following, we propose an assumption based application that can be utilized in a physical way.

Consider a cellular company that has a plan to fix the minimum number of towers in a city, such that the maximum numbers of the users can be attracted. For this purpose, the following are some of the parameters that can be taken in account:

- Suitable place to fix a tower
- Transportation means
- Users
- Connectivity with the main server
- Urban area or hilly area
- Any other existing cellular network
- Available recourses
- Expenditures and outcomes

Suppose a team selected five places where they are interested in placing a tower, so that they can facilitate maximum numbers of the users. They observe the following two situations:

- 1. Fixing a tower exactly at the chosen place from the selected five places
- 2. Fixing a tower between any two of the selected five places.

For Situation 1, we proceed as follows:

Let $V = \{C_1, C_2, C_3, C_4, C_5\}$ be the set of places where the team is interested in fixing a tower as a vertex set. Suppose that 60% of the experts on the team believe that C_1 should have a tower and 10% of the experts believe that there is no need to fix tower at the place C_1 after carefully observing the different parameters. Thus, in this way, we can find the amplitude term for both membership and non-membership functions. Now, the phase term that represents the period needs to be found. Let 40% of the experts believe that in a particular time C_1 can attract the maximum number of users (Profit) and 30% of the experts have the opposite opinion. We model this information as $\langle C_1 : 0.6e^{i 0.4\pi}, 0.1e^{i 0.3\pi} \rangle$.

Thus, the team finalizes its opinion about the place C_1 . Now, they visit the place C_2 . After careful observation, they model the information as $\langle C_2 : 0.7e^{i} 0.2\pi, 0.2e^{i} 0.4\pi \rangle$. It means that 70% of the experts are in the favor of C_2 , even though it will produce only 20% of profit, while 20% are opposed to C_2 , even though it will produce 40% profit. Similarly, they model all the other places as $\langle C_3 : 0.5e^{i} 0.6\pi, 0.7e^{i} 0.4\pi \rangle$, $\langle C_4 : 0.7e^{i} 0.8\pi, 0.5e^{i} 0.7\pi \rangle$ and $\langle C_5 : 0.4e^{i} 0.4\pi, 0.9e^{i} 0.2\pi \rangle$. We denote this model as

 $A = \begin{cases} \langle C_1 : 0.6e^{i \ 0.4\pi}, 0.1e^{i \ 0.3\pi} \rangle \\ \langle C_2 : 0.7e^{i \ 0.2\pi}, 0.2e^{i \ 0.4\pi} \rangle \\ \langle C_3 : 0.5e^{i \ 0.6\pi}, 0.7e^{i \ 0.4\pi} \rangle \\ \langle C_4 : 0.7e^{i \ 0.8\pi}, 0.5e^{i \ 0.7\pi} \rangle \\ \langle C_5 : 0.4e^{i \ 0.4\pi}, 0.9e^{i \ 0.2\pi} \rangle \end{cases}$

The complex membership of the vertices denotes the positive characteristics and complex non-membership of the vertices denotes the negative characteristics of a certain parameter for a certain place. Now, finding the absolute values, we have

$$\begin{split} |C_1| &= (0.6, 0.1), \\ |C_2| &= (0.7, 0.2), \\ |C_3| &= (0.5, 0.7), \\ |C_4| &= (0.7, 0.5), \\ |C_5| &= (0.4, 0.9). \end{split}$$

To find the optimal choice, we find the score function of the absolute values of C_1 , C_2 , C_3 , C_4 , C_5 . Thus, we have

$$S(C_1) = 0.6 - 0.1 = 0.5,$$

$$S(C_2) = 0.7 - 0.2 = 0.5,$$

$$S(C_3) = 0.5 - 0.7 = -0.2,$$

$$S(C_4) = 0.7 - 0.5 = 0.2,$$

$$S(C_5) = 0.4 - 0.9 = -0.5.$$

Since the scores for C_1 and C_2 are equal, we find the accuracies of C_1 and $C_2 : H(C_1) = 0.6 + 0.1 = 0.7$ and $H(C_2) = 0.7 + 0.2 = 0.9$, thus $C_1 > C_2$, which is the most suitable choice to fix a tower. This is the application of complex intuitionistic fuzzy graph, where it has no edge, as shown in Figure 14.



Figure 14. Complex intuitionistic fuzzy graph with no edge.

Now, for Situation 2, we proceed as follows:

If a tower is fixed between places C_1 and C_2 , it will represent the edge C_1C_2 of the vertex C_1 , C_2 . To find the model of C_1C_2 , we use Definition 6 and find that $\langle C_1C_2 : 0.6e^{i\ 0.4\pi}, 0.2e^{i\ 0.4\pi} \rangle$. Similarly, we find the other edges and we denote this model as

$$B = \begin{cases} \langle C_1 C_2 : 0.6e^{i \ 0.4\pi}, 0.2e^{i \ 0.4\pi} \rangle \\ \langle C_1 C_3 : 0.5e^{i \ 0.4\pi}, 0.7e^{i \ 0.4\pi} \rangle \\ \langle C_1 C_4 : 0.6e^{i \ 0.4\pi}, 0.5e^{i \ 0.7\pi} \rangle \\ \langle C_2 C_4 : 0.7e^{i \ 0.2\pi}, 0.5e^{i \ 0.7\pi} \rangle \\ \langle C_1 C_5 : 0.4e^{i \ 0.4\pi}, 0.9e^{i \ 0.3\pi} \rangle \\ \langle C_2 C_3 : 0.5e^{i \ 0.2\pi}, 0.7e^{i \ 0.4\pi} \rangle \\ \langle C_2 C_5 : 0.4e^{i \ 0.2\pi}, 0.9e^{i \ 0.4\pi} \rangle \\ \langle C_3 C_4 : 0.5e^{i \ 0.6\pi}, 0.7e^{i \ 0.4\pi} \rangle \\ \langle C_4 C_5 : 0.4e^{i \ 0.4\pi}, 0.9e^{i \ 0.4\pi} \rangle \end{cases}$$

If we consider the edge $\langle C_1 C_2 : 0.6e^{i \ 0.4\pi}, 0.2e^{i \ 0.4\pi} \rangle$. In this case, the amplitude term shows that 60% of the experts believe that there should be a tower between these two places and 20% of the experts believe the opposite. The phase terms show that 40% of the experts believe that in a certain time if a tower is fixed between these two places it will produce maximum profit, while 40% of the experts believe the opposite. Absolute values of the edges are:

$$|C_1C_2| = (0.6, 0.2), |C_1C_3| = (0.5, 0.7)$$
$$|C_1C_4| = (0.6, 0.5), |C_2C_4| = (0.7, 0.5)$$
$$|C_1C_5| = (0.4, 0.9), |C_2C_3| = (0.5, 0.7)$$
$$|C_2C_5| = (0.4, 0.9), |C_3C_4| = (0.5, 0.9)$$
$$|C_3C_5| = (0.4, 0.9), |C_4C_5| = (0.4, 0.9)$$

To find the optimal choice, we find the score function of the absolute values of the edges. Thus, we have

$$\begin{split} S(C_1C_2) &= 0.4, S(C_1C_3) = -0.2, \\ S(C_1C_4) &= 0.1, S(C_2C_4) = 0.2, \\ S(C_1C_5) &= -0.5, S(C_2C_3) = -0.2, \\ S(C_2C_5) &= -0.5, S(C_3C_4) = -0.4, \\ S(C_3C_5) &= -0.5, S(C_4C_5) = -0.5. \end{split}$$

 $S(C_1C_2) = 0.4$ is the greatest, and hence most suitable choice to fix the tower. This is the case where complex intuitionistic fuzzy graph has edges, as shown in Figure 15.



Figure 15. Complex intuitionistic fuzzy graph with edges.

7. Conclusions

We defined cif-graphs and accomplished the notion of union of cif-graphs, Cartesian product of cif-graphs, join of cif-graphs and composition of cif-graphs. Our presented approach is the generalization of fuzzy graphs. We aim to extend our work in the following directions: One can see in the Section 6, that handling different parameters is one of the most difficult tasks, and since soft sets are very useful tools where one can handle more parameters in a practical way, we will define the complex fuzzy soft graphs that will generalize the idea of fuzzy graphs, soft graphs and fuzzy soft graphs. On the other side, since intuitionistic fuzzy sets generalize the concept of fuzzy sets, we will try to produce a model related with complex intuitionistic fuzzy soft graph, which is the generalization of complex fuzzy graphs and complex fuzzy soft graphs.

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