Shannon Type Inequalities for Kapur's Entropy

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Abstract: In the paper, by methods of the theory of majorization, the authors establish the Schur m -convexity and Shannon type inequalities for Kapur's entropy.

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1. Introduction and Main Results

Let $p = (p_1, p_2, \dots, p_n)$ be a probability vector, that is, $p_i \geq 0$ for $1 \leq i \leq n$ and $\sum_{i=1}^n p_i = 1$. The quantity

$$H(p) = - \sum_{i=1}^n p_i \ln p_i,$$

where, if $p_i = 0$, we put $p_i \ln p_i = 0$ in the sum, is known as entropy of p , or the Shannon information entropy of p .

Theorem 1 ([1] (p. 101)). Let $p = (p_1, p_2, \dots, p_n)$ be a probability vector. Then $H(p)$ is Schur-concave on $[0, 1]^n$. In particular, we have

$$H(1, 0, \dots, 0) \leq H(p) \leq H(p_0) = \ln n,$$

where $p_0 = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$.

Let p be a probability vector and $t \in (0, \infty)$. The quantity

$$H_t(p) = H_t(p_1, p_2, \dots, p_n) = - \frac{1}{\sum_{i=1}^n p_i^t} \sum_{i=1}^n p_i^t \ln p_i^t$$

is known [2] as Kapur's entropy of order 1 and type $t \in (0, \infty)$. It is easy to see that

$$H_t(p_0) = - \ln A_n(p_0^t) = t \ln n,$$

where $p^t = (p_1^t, p_2^t, \dots, p_n^t)$ and $A_n(p^t) = \frac{1}{n} \sum_{i=1}^n p_i^t$. It is easy to see that the inequality $H_1(p) \leq \ln n$ holds for each probability vector p . However, the inequality

$$H_t(p) \leq \ln n \tag{1}$$

does not hold for each $t \in (0, \infty)$ and each probability vector p . Stolarsky [3] shows that the inequality (1) holds for $t \geq t_0(n)$, where $t_0(2) = \frac{1}{2}$. Hereafter, Clausen [4] proved that if $n > 3$ and $t = t_0(n)$, then the inequality (1) holds for each probability vector p . Thus, with respect to $t \in (0, \infty)$, the function $H_t(p)$ is not strictly Schur-convex. For more information about this topic, the reader is referred to the papers [2,4–14] and the closely related references therein.

In this paper, we will establish the Schur m -convexity and some Shannon type inequalities for Kapur's entropy $H_t(p)$.

Our main results are the following two theorems.

Theorem 2. Let $p \in (0, \infty)^n$, $t \in (0, \infty)$, and $\lambda_1, \lambda_2, \dots, \lambda_n \in [0, 1]$ with $\sum_{i=1}^n \lambda_i = 1$ for $n \geq 2$. Then the function $H_t(p)$ is Schur m -concave on $(0, \infty)^n$ for $m = t$ and

$$H_t(p) \leq H_t(M_t(1, \lambda; p), M_t(2, \lambda; p), \dots, M_t(n, \lambda; p)) \leq -\ln A_n(p^t) \quad (2)$$

with the equality in (2) holding for $p_0 = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$, where

$$M_t(k, \lambda; p) = \left(\sum_{i=1}^n \lambda_i p_{i+k-1}^t \right)^{1/t}, \quad k = 1, 2, \dots, n \quad (3)$$

with $p_{n+i} = p_i$ for $i = 1, 2, \dots, n-1$.

Theorem 3. Let $p \in (0, 1)^n$, $t \in (0, 1]$, and $\lambda_1, \lambda_2, \dots, \lambda_n \in [0, 1]$ with $\sum_{i=1}^n p_i = \sum_{i=1}^n \lambda_i = 1$ for $n \geq 1$.

1. If $0 < t < \frac{1}{2}$ and $p \in [q, 1]$ with $\frac{1}{q} \geq n$, then

$$\ln n \leq - \sum_{k=1}^n [M_1(k, \lambda; p)]^t \ln M_1(k, \lambda; p).$$

2. If $0 < t < \frac{1}{2}$ and $p \in (0, q]^n$ with $\frac{1}{q} \leq n$, or if $\frac{1}{2} \leq t \leq 1$ and $p \in (0, 1]^n$, then

$$\frac{H_t(p)}{t} \leq -\frac{1}{n} \sum_{k=1}^n [M_1(k, \lambda; p)]^t \ln M_1(k, \lambda; p) \leq \min \left\{ \frac{1}{t}, n^{1-t} \right\} \ln n,$$

where $q = e^{(2t-1)/t(1-t)}$ for $0 < t < \frac{1}{2}$ and $M_1(k, \lambda; p)$ is defined by (3).

2. Definitions and Lemmas

For proving our main results, we need several definitions and lemmas below.

It is well known that a function $\varphi(x_1, x_2, \dots, x_n)$ of n variables is said to be symmetric if its value is unchanged for any permutation of its n variables $x_1, x_2, \dots, x_n$.

Definition 1 ([1,15]). Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$.

1. A tuple x is said to be majorized by y (in symbols $x \prec y$) if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \dots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geq x_{[2]} \geq \dots \geq x_{[n]}$ and $y_{[1]} \geq y_{[2]} \geq \dots \geq y_{[n]}$ are rearrangements of x and y in a descending order.
2. A set $\Omega \subseteq \mathbb{R}^n$ is called convex if $(\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2, \dots, \lambda x_n + (1-\lambda)y_n) \in \Omega$ for each $x, y \in \Omega$ and $\lambda \in [0, 1]$.
3. Let $\Omega \subseteq \mathbb{R}^n$. A function $\varphi : \Omega \rightarrow \mathbb{R}$ is said to be Schur-convex on Ω if $x \prec y$ on Ω implies $\varphi(x) \leq \varphi(y)$. A function φ is said to be Schur-concave on Ω if and only if $-\varphi$ is Schur-convex.

Definition 2 ([16]). Let $\Omega \subseteq (0, \infty)^n$ and $\varphi : \Omega \rightarrow (0, \infty)$.

1. The set Ω is said to be geometrically convex if $(x_1^\lambda y_1^{1-\lambda}, x_2^\lambda y_2^{1-\lambda}, \dots, x_n^\lambda y_n^{1-\lambda}) \in \Omega$ for each $x, y \in \Omega$ and $\lambda \in [0, 1]$.
2. The function φ is said to be Schur-geometrically convex on Ω if the majorization $\ln x = (\ln x_1, \ln x_2, \dots, \ln x_n) \prec \ln y = (\ln y_1, \ln y_2, \dots, \ln y_n)$ implies $\varphi(x) \leq \varphi(y)$ for each $x, y \in \Omega$.

Definition 3 ([17]). Let $\Omega \subseteq (0, \infty)^n$ and $\varphi : \Omega \rightarrow (0, \infty)$.

1. The set Ω is said to be harmonically convex if $\frac{xy}{\lambda x + (1-\lambda)y} \in \Omega$ for each $x, y \in \Omega$ and $\lambda \in [0, 1]$.
2. The function φ is said to be Schur-harmonically convex on Ω if the majorization $\frac{1}{x} = (\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}) \prec \frac{1}{y} = (\frac{1}{y_1}, \frac{1}{y_2}, \dots, \frac{1}{y_n})$ implies $\varphi(x) \leq \varphi(y)$ for each $x, y \in \Omega$.

Definition 4 ([18–20]). Let $f_m(x) = \begin{cases} \frac{x^m-1}{m}, & m \neq 0; \\ \ln x, & m = 0. \end{cases}$ A function $\varphi : \Omega \subseteq (0, \infty)^n \rightarrow \mathbb{R}$ is said to be Schur m -power convex on Ω if the majorization relation $(f_m(x_1), f_m(x_2), \dots, f_m(x_n)) \prec (f_m(y_1), f_m(y_2), \dots, f_m(y_n))$ on Ω implies that $\varphi(x) \leq \varphi(y)$.

Remark 1. It is not difficult to see that, when $m \neq 0$,

$$\left(\frac{x_1^m - 1}{m}, \frac{x_2^m - 1}{m}, \dots, \frac{x_n^m - 1}{m} \right) \prec (x_1^m - 1, x_2^m - 1, \dots, x_n^m - 1) \prec (x_1^m, x_2^m, \dots, x_n^m).$$

When taking $m = 1, 0, -1$, we derive that $f_1(x) = x - 1$, $f_0(x) = \ln x$, and $f_{-1}(x) = 1 - \frac{1}{x}$ in Definition 4. Therefore, we can derive from Definition 4 the Schur-convexity, Schur-geometric convexity, and Schur-harmonic convexity, respectively.

Lemma 1 ([1,15,21]). Let $\Omega \subseteq \mathbb{R}^n$ be a symmetric convex set with nonempty interior and let $\varphi : \Omega \rightarrow (0, \infty)$ be a continuous symmetric function which is differentiable in the interior Ω° . Then φ is a Schur-convex function on Ω if and only if

$$(x_1 - x_2) \left[\frac{\partial \varphi(x)}{\partial x_1} - \frac{\partial \varphi(x)}{\partial x_2} \right] \geq 0, \quad x \in \Omega^\circ.$$

Lemma 2 ([18–20]). Let $\Omega \subseteq (0, \infty)^n$ be a symmetric set with nonempty interior Ω° and let $\varphi : \Omega \rightarrow \mathbb{R}$ be continuous on Ω and differentiable in Ω° . Then φ is Schur m -power convex on Ω if and only if φ is symmetric on Ω such that

$$\frac{x_1^m - x_2^m}{m} \left[x_1^{1-m} \frac{\partial \varphi(x)}{\partial x_1} - x_2^{1-m} \frac{\partial \varphi(x)}{\partial x_2} \right] \geq 0, \quad m \neq 0$$

and

$$(\ln x_1 - \ln x_2) \left[x_1 \frac{\partial \varphi(x)}{\partial x_1} - x_2 \frac{\partial \varphi(x)}{\partial x_2} \right] \geq 0, \quad m = 0$$

for $x \in \Omega^\circ$.

For more information about various convexity named after Schur, please refer to [22–34] and closely related references therein.

3. Shannon Type Inequalities for Kapur's Entropy

Now we are in a position to prove our main results.

Proof of Theorem 2. Calculating the partial derivatives of function $H_t(p)$ in p_1 and p_2 yields

$$\frac{\partial H_t(p)}{\partial p_1} = \frac{-tp_1^{t-1}}{[p_1^t + p_2^t + A(p)]^2} [p_1^t + p_2^t + A(p) - B(p) + (p_2^t + A(p)) \ln p_1^t - p_2^t \ln p_2^t]$$

and

$$\frac{\partial H_t(p)}{\partial p_2} = \frac{-tp_2^{t-1}}{[p_1^t + p_2^t + A(p)]^2} [p_1^t + p_2^t + A(p) - B(p) + (p_1^t + A(p)) \ln p_2^t - p_1^t \ln p_1^t],$$

where $A(p) = \sum_{i=3}^n p_i^t$ and $B(p) = \sum_{i=3}^n p_i^t \ln p_i^t$ under the condition that any empty sum is understood as null.

Letting $t = m$ in Definition 4 and utilizing Lemma 2 give

$$\begin{aligned} & \frac{p_1^m - p_2^m}{m} \left[p_1^{1-m} \frac{\partial H_t(p)}{\partial p_1} - p_2^{1-m} \frac{\partial H_t(p)}{\partial p_2} \right] \\ &= -\frac{t(p_1^m - p_2^m)}{m(p_1^t + p_2^t + A(p))^2} [(p_1^t + p_2^t + A(p) - B(p))(p_1^{t-m} - p_2^{t-m}) \\ & \quad + A(p)(p_1^{t-m} \ln p_1^t - p_2^{t-m} \ln p_2^t) + p_1^{t-m} p_2^{t-m} (p_1^m + p_2^m) (\ln p_1^t - \ln p_2^t)] \\ &= -\frac{p_1^t - p_2^t}{(p_1^t + p_2^t + A(p))^2} [A(p)(\ln p_1^t - \ln p_2^t) + (p_1^t + p_2^t)(\ln p_1^t - \ln p_2^t)] \\ &= -\frac{1}{(p_1^t + p_2^t + A(p))^2} (p_1^t - p_2^t)(\ln p_1^t - \ln p_2^t) \sum_{i=1}^n p_i^t \\ &\leq 0. \end{aligned}$$

Hence, the function $H_t(p)$ is Schur m -concave on $(0, \infty)^n$ for $m = t$.

For $\ell_1, \ell_2, \dots, \ell_k \in \{1, 2, \dots, n\}$ and $1 \leq k \leq n$, we have

$$\sum_{j=1}^k p_{[j]}^t - \sum_{j=1}^k \sum_{i=1}^n \lambda_i p_{i+\ell_j-1}^t = \sum_{j=1}^k p_{[j]}^t - \sum_{i=1}^n \lambda_i \sum_{j=1}^k p_{i+\ell_j-1}^t \geq \sum_{j=1}^k p_{[j]}^t - \sum_{i=1}^n \lambda_i \sum_{j=1}^k p_{[j]}^t = 0.$$

Let $y_j = \sum_{i=1}^n \lambda_i p_{i+j-1}^t$ for $j = 1, 2, \dots, n$. If there exists $1 \leq k_0 < n$ such that $\sum_{j=1}^{k_0} y_{[j]} \geq k_0 A_n(p^t)$ and $\sum_{j=1}^{k_0+1} y_{[j]} < (k_0 + 1) A_n(p^t)$, that is,

$$\sum_{i=1}^n p_i^t = \sum_{j=1}^{k_0+1} y_{[j]} + \sum_{k_0+2}^n y_{[j]} < (k_0 + 1) A_n(p^t) + (n - k_0 - 1) A_n(p^t) = \sum_{i=1}^n p_i^t,$$

Then, it follows that

$$(A_n(p^t), A_n(p^t), \dots, A_n(p^t)) \prec ([M_t(1, \lambda; p)]^t, [M_t(2, \lambda; p)]^t, \dots, [M_t(n, \lambda; p)]^t) \prec (p_1^t, p_2^t, \dots, p_n^t).$$

Further by virtue of Definition 4, we have

$$H_t(p) \leq H_t(M_t(1, \lambda; p), M_t(2, \lambda; p), \dots, M_t(n, \lambda; p)) \leq -\ln A_n(p^t).$$

The proof of Theorem 2 is thus complete. $\square$

Proof of Theorem 3. For all $p \in (0, 1)^n$ and $t \in (0, 1]$ with $\sum_{i=1}^n p_i = 1$, by the easily-understandable inequality

$$A_n(p^t) \geq \frac{1}{n} \sum_{i=1}^n p_i = \frac{1}{n} \quad (4)$$

for $0 < t \leq 1$, we obtain

$$-\frac{1}{n^{1-t}} \sum_{i=1}^n p_i^t \ln p_i^t \leq H_t(p) \leq -\sum_{i=1}^n p_i^t \ln p_i^t.$$

Let $F_t(p) = -\sum_{i=1}^n p_i^t \ln p_i^t$. Then

$$\frac{\partial F_t(p)}{\partial p_1} = -tp_1^{t-1}(1 + \ln p_1^t) \quad \text{and} \quad \frac{\partial F_t(p)}{\partial p_2} = -tp_2^{t-1}(1 + \ln p_2^t).$$

By Lemma 1, we obtain

$$(p_1 - p_2) \left[\frac{\partial F_t(p)}{\partial p_1} - \frac{\partial F_t(p)}{\partial p_2} \right] = t(p_1 - p_2) [g(p_2) - g(p_1)],$$

where $g(p) = p^{t-1} + p^{t-1} \ln p^t$ for $p \in (0, 1]$. Then

1. when $0 < t < \frac{1}{2}$, the function $g(p)$ is increasing on $(0, q]$ and decreasing on $[q, 1)$;
2. when $\frac{1}{2} \leq t \leq 1$, the function $g(p)$ is increasing on $(0, 1]$.

Therefore, by Lemma 1, we find that

1. if $0 < t < \frac{1}{2}$, the function $F_t(p)$ is Schur-convex on $[q, 1)^n$;
2. if $0 < t < \frac{1}{2}$, the function $F_t(p)$ is Schur-concave on $(0, q]^n$;
3. if $\frac{1}{2} \leq t \leq 1$, the function $F_t(p)$ is Schur-concave on $(0, 1]^n$.

For $\lambda_1, \lambda_2, \dots, \lambda_n \in [0, 1]$ with $\sum_{i=1}^n \lambda_i = 1$, we have

$$\left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n} \right) \prec \left(\sum_{i=1}^n \lambda_i p_i, \sum_{i=1}^n \lambda_i p_{i+1}, \dots, \sum_{i=1}^n \lambda_i p_{i+n-1} \right) \prec p.$$

By Definition 1, we see that

1. if $0 < t < \frac{1}{2}$ and $p \in [q, 1)^n$ with $\frac{1}{q} \geq n$, then

$$H_t(p) \geq n^{t-1} F_t(p) \geq -\frac{1}{n^{1-t}} \sum_{k=1}^n [M_1(k, \lambda; p)]^t \ln [M_1(k, \lambda; p)]^t \geq \frac{t \ln n}{n};$$

2. if $0 < t < \frac{1}{2}$ and $p \in (0, q]^n$ with $\frac{1}{q} \leq n$, then

$$H_t(p) \leq F_t(p) \leq -\sum_{k=1}^n [M_1(k, \lambda; p)]^t \ln [M_1(k, \lambda; p)]^t \leq tn^{1-t} \ln n; \quad (5)$$

3. if $\frac{1}{2} \leq t \leq 1$ and $p \in (0, 1)^n$, the inequality (5) still holds.

The proof of Theorem 3 is thus complete. $\square$

4. Remarks

Finally we list several remarks on our main results and closely related ones.

Remark 2. Let $p \in (0, \infty)^n$, $t \geq 1$, $\lambda_1, \lambda_2, \dots, \lambda_n \in [0, 1]$ with $\sum_{i=1}^n \lambda_i = 1$ for $n \geq 2$. Since $[A_n(p)]^t \leq A_n(p^t)$ for $t \geq 1$, from Theorem 2, it follows that

$$H_t(p) \leq H_t(M_t(1, \lambda; p), M_t(2, \lambda; p), \dots, M_t(n, \lambda; p)) \leq -\ln A_n(p^t) \leq -t \ln A_n(p)$$

with equality for $p_1 = p_2 = \dots = p_n > 0$, where $M_1(k, \lambda; p)$ is defined by (3).

Remark 3. Let $p \in (0, 1)^n$, $t \in (0, \infty)$, and $\lambda_1, \lambda_2, \dots, \lambda_n \in [0, 1]$ with $\sum_{i=1}^n p_i = \sum_{i=1}^n \lambda_i = 1$ and $n \geq 2$. From Remark 2 and the inequality (4), we conclude the following results.

1. If $t \geq 1$, we have

$$H_t(p) \leq H_t(M_t(1, \lambda; p), M_t(2, \lambda; p), \dots, M_t(n, \lambda; p)) \leq -\ln A_n(p^t) \leq t \ln n \quad (6)$$

with equality for $p_0 = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$.

2. If $0 < t < 1$, we have

$$H_t(p) \leq H_t(M_t(1, \lambda; p), M_t(2, \lambda; p), \dots, M_t(n, \lambda; p)) \leq -\ln A_n(p^t) \leq \ln n.$$

In particular, when $\lambda_1 = \lambda_2 = \frac{1}{2}$ and $\lambda_3 = \dots = \lambda_n = 0$,

1. if $t \geq 1$, then

$$H_t(p) \leq H_t\left(\sqrt[t]{\frac{p_1^t + p_2^t}{2}}, \sqrt[t]{\frac{p_2^t + p_3^t}{2}}, \dots, \sqrt[t]{\frac{p_n^t + p_1^t}{2}}\right) \leq -\ln A_n(p^t) \leq t \ln n$$

with equality for $p_0 = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$;

2. if $0 < t < 1$, then

$$H_t(p) \leq H_t\left(\sqrt[t]{\frac{p_1^t + p_2^t}{2}}, \sqrt[t]{\frac{p_2^t + p_3^t}{2}}, \dots, \sqrt[t]{\frac{p_n^t + p_1^t}{2}}\right) \leq -\ln A_n(p^t) \leq \ln n.$$

Remark 4. When $0 < t < 1$, the inequality (6) does not necessarily hold. For $t = 0.369$, $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 0$, and $p = (0.3, 0.69, 0.01)$, we have

$$0.4053 \dots = t \ln n < H_t(p) = 0.4215 \dots < \ln n = 1.0986 \dots$$

Remark 5. Under conditions of Theorems 2 and 3, if $0 < t < \frac{1}{2}$ and $p \in [q, 1)$ with $\frac{1}{q} \geq n$, then

$$\begin{aligned} \frac{t \ln n}{n} &\leq -\frac{1}{n} \sum_{k=1}^n [M_1(k, \lambda; p)]^t \ln [M_1(k, \lambda; p)]^t \leq H_t(p) \\ &\leq H_t(M_t(1, \lambda; p), M_t(2, \lambda; p), \dots, M_t(n, \lambda; p)) \leq -\ln A_n(p^t) \leq \min\{1, tn^{1-t}\} \ln n. \end{aligned}$$

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