

Completeness of Statistical Structures

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Abstract: In this survey note, we discuss the notion of completeness for statistical structures. There are at least three connections whose completeness might be taken into account, namely, the Levi-Civita connection of the given metric, the statistical connection, and its conjugate. Especially little is known on the completeness of statistical connections.

Keywords: statistical structure; affine hypersurface; affine sphere; conjugate symmetric statistical structure; sectional ∇ -curvature; complete connection

MSC: 53C05; 53A15

1. Introduction

In affine differential geometry, which is still the main source of statistical structures, the affine completeness of a nondegenerate affine hypersurface has always been meant as the completeness of the metric being the second fundamental form. In particular, Calabi's famous conjecture deals with affine completeness. Complete affine spheres are those whose Blaschke metric is complete, see, e.g., Reference [1]. This affine completeness has been opposed to Euclidean completeness, that is, completeness relative to the first fundamental form on a hypersurface. The completeness of the induced connections on affine hypersurfaces has never been studied. However, even if we restrict to the completeness of the second fundamental form and we switch from the geometry of affine hypersurfaces to the geometry of statistical structures, the situation becomes immediately much more complicated. It follows from the fact that, on a hypersurface, the induced structure has very strong properties that, in general, are not satisfied by an arbitrary statistical structure. In other words, not all statistical structures, even Ricci-symmetric, are realizable (even locally) on hypersurfaces. As examples of results on affine complete affine spheres, here we cite two classical theorems and three other theorems, being their analogs and generalizations in the category of statistical manifolds.

As for statistical connections, the first attempt to the study of their completeness was made by Noguchi [2]. He gave a procedure of constructing a complete statistical connection on a complete Riemannian manifold by using just one function. Statistical connections on compact manifolds are not necessarily complete. We provide a simple example on a torus. We also give a theorem generalizing the situation from this concrete simple example.

2. Preliminaries

We recall only those notions of statistical geometry that are needed in this note (for more information, see [3]). Let g be a positive definite Riemannian tensor field on a manifold M . Denote by $\hat{\nabla}$ the Levi-Civita connection for g . A statistical structure is a pair (g, ∇) , where ∇ is a torsion-free connection such that the following Codazzi condition is satisfied:

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z) \quad (1)$$

for all $X, Y, Z \in T_x M$, $x \in M$. A connection ∇ satisfying (1) is called a statistical connection for g . A statistical structure (g, ∇) is trivial if the statistical connection ∇ coincides with the Levi-Civita connection $\hat{\nabla}$.

For any connection ∇ one defines its conjugate connection $\bar{\nabla}$ relative to g by the following formula:

$$g(\nabla_X Y, Z) + g(Y, \bar{\nabla}_X Z) = Xg(Y, Z). \quad (2)$$

It is known that if (g, ∇) is a statistical structure, then so is $(g, \bar{\nabla})$. From now on, we assume that ∇ is a statistical connection for g .

If R is the curvature tensor for ∇ , and \bar{R} is the curvature tensor for $\bar{\nabla}$, then we have

$$g(R(X, Y)Z, W) = -g(\bar{R}(X, Y)W, Z). \quad (3)$$

Denote by Ric and $\bar{\text{Ric}}$ the corresponding Ricci tensors. Note that, in general, these Ricci tensors are not symmetric. The curvature and the Ricci tensors of $\hat{\nabla}$ are denoted by \hat{R} and $\widehat{\text{Ric}}$. The function

$$\rho = \text{tr}_g \text{Ric}(\cdot, \cdot) \quad (4)$$

is called the scalar curvature of (g, ∇) . Similarly, one can define the scalar curvature $\bar{\rho}$ for $(g, \bar{\nabla})$ but, by (3), $\rho = \bar{\rho}$. The function ρ is called the scalar statistical curvature. We also have the usual scalar curvature $\hat{\rho}$ for g .

We define the cubic form A by

$$A(X, Y, Z) = -\frac{1}{2} \nabla g(X, Y, Z), \quad (5)$$

where $\nabla g(X, Y, Z)$ stands for $(\nabla_X g)(Y, Z)$. It is clear that a statistical structure can be equivalently defined as a pair (g, A) , where A is a symmetric cubic form.

The condition characterized by the following lemma plays a crucial role in our considerations.

Lemma 1. *Let (g, ∇) be a statistical structure. The following conditions are equivalent:*

- (1) $R = \bar{R}$,
- (2) $\hat{\nabla} A$ is symmetric,
- (3) $g(R(X, Y)Z, W)$ is skew-symmetric relative to Z, W .

The family of statistical structures satisfying one of the above conditions is as important in the geometry of statistical structures as the family of affine spheres in affine differential geometry. A statistical structure satisfying Condition (2) in the above lemma was called conjugate symmetric in [4]. We adopt this name here. Note that condition $R = \bar{R}$ easily implies the symmetry of Ric .

A statistical structure is called trace-free if $\text{tr}_g A(X, \cdot, \cdot) = 0$ for every $X \in TM$. This condition is equivalent to the condition that $\nabla \nu_g = 0$, where ν_g is the volume form determined by g .

In [3], we introduced the notion of the sectional ∇ -curvature. Namely, the tensor field

$$\mathcal{R} = \frac{1}{2}(R + \bar{R}) \quad (6)$$

satisfies the following condition:

$$g(\mathcal{R}(X, Y)Z, W) = -g(\mathcal{R}(X, Y)W, Z).$$

If we denote by the same letter \mathcal{R} the $(0, 4)$ -tensor field given by $\mathcal{R}(X, Y, W, Z) = g(\mathcal{R}(W, Z)Y, X)$, then this \mathcal{R} has the same symmetries as the Riemannian $(0, 4)$ curvature tensor. Therefore, we can define the sectional ∇ -curvature by

$$k(\pi) = g(\mathcal{R}(e_1, e_2)e_2, e_1) \quad (7)$$

for a vector plane $\pi \in T_x M$, $x \in M$, where e_1, e_2 is any orthonormal basis of π . It is a well-defined notion, but it is not quite analogous to the Riemannian sectional curvature. For instance, in general, Schur's lemma does not hold for the sectional ∇ -curvature. However, if a statistical structure is conjugate-symmetric (in this case $\mathcal{R} = R$) some type of the second Bianchi identity holds and, consequently, the Schur lemma holds [3].

The theory of affine hypersurfaces in \mathbf{R}^{n+1} is a natural source of statistical structures. For the theory, we refer to [1] or [5]. We recall here only some basic facts.

Let $\mathbf{f} : M \rightarrow \mathbf{R}^{n+1}$ be a locally strongly convex hypersurface. For simplicity, assume that M is connected and orientable. Let ζ be a transversal vector field on M . The induced volume form ν_ζ on M is defined as follows:

$$\nu_\zeta(X_1, \dots, X_n) = \det(\mathbf{f}_*X_1, \dots, \mathbf{f}_*X_n, \zeta).$$

We also have the induced connection ∇ and second fundamental form g given by the Gauss formula:

$$D_X \mathbf{f}_*Y = \mathbf{f}_*\nabla_X Y + g(X, Y)\zeta,$$

where D is the standard flat connection on \mathbf{R}^{n+1} . Since the hypersurface is locally strongly convex, g is definite. By multiplying ζ by -1 , if necessary, we can assume that g is positive definite. A transversal vector field is called equiaffine if $\nabla \nu_\zeta = 0$. This condition is equivalent to the fact that ∇g is symmetric, i.e., (g, ∇) is a statistical structure. It means, in particular, that for a statistical structure obtained on a hypersurface by a choice of an equiaffine transversal vector field, the Ricci tensor of ∇ is automatically symmetric. A hypersurface equipped with an equiaffine transversal vector field is called an equiaffine hypersurface.

Recall now the notion of the shape operator and the Gauss equations. Having a chosen equiaffine transversal vector field and differentiating it, we get the Weingarten formula:

$$D_X \zeta = -\mathbf{f}_*SX.$$

The tensor field S is called the shape operator for ζ . If R is the curvature tensor for the induced connection ∇ , then

$$R(X, Y)Z = g(Y, Z)SX - g(X, Z)SY. \quad (8)$$

This is the Gauss equation for R . The Gauss equation for \bar{R} is the following:

$$\bar{R}(X, Y)Z = g(Y, SZ)X - g(X, SZ)Y. \quad (9)$$

It follows that the conjugate connection is projectively flat if $n > 2$. The conjugate connection is also projectively flat for two-dimensional surfaces equipped with an equiaffine transversal vector field, that is, that the cubic form $\bar{\nabla} \text{Ric}$ is symmetric.

We have the volume form ν_g determined by g on M . In general, this volume form is not covariant constant relative to ∇ . The central point of the classical affine differential geometry is the theorem saying that there is a unique equiaffine transversal vector field ζ , such that $\nu_\zeta = \nu_g$. This unique transversal vector field is called the affine normal vector field or the Blaschke affine normal. The second fundamental form for the affine normal is called the Blaschke metric. A hypersurface endowed with the affine Blaschke normal is called a Blaschke hypersurface. Note that conditions $\nabla \nu_\zeta = 0$ and $\nu_\zeta = \nu_g$ imply that the statistical structure on a Blaschke hypersurface is trace-free.

If the affine lines determined by the affine normal vector field meet at one point or are parallel, then the hypersurface is called an affine sphere. In the first case, the sphere is called proper, in the second one improper. The class of affine spheres is very large. There exist a lot of conditions characterizing affine spheres. For instance, a hypersurface is an affine sphere if and only if $R = \bar{R}$. Therefore, conjugate symmetric statistical manifolds can be regarded as generalizations of affine spheres. For connected affine spheres, the shape operator S is a constant multiple of the identity, i.e., $S = k \text{ id}$. In particular, for affine spheres we have:

$$R(X, Y)Z = k\{g(Y, Z)X - g(X, Z)Y\}. \quad (10)$$

It follows that the statistical sectional curvature on a connected affine sphere is constant. If, as we have already done, we choose a positive definite Blaschke metric on a locally strongly convex affine sphere, then we call the sphere elliptic if $k > 0$, parabolic if $k = 0$, and hyperbolic if $k < 0$.

As we have already mentioned, if ∇ is a connection on a hypersurface induced by an equiaffine transversal vector field, then the conjugate connection $\bar{\nabla}$ is projectively flat. Therefore, the projective flatness of the conjugate connection is a necessary condition for (g, ∇) to be realizable as the induced structure on a hypersurface equipped with an equiaffine transversal vector field. In fact, roughly speaking, it is also a sufficient condition for local realizability. Note that, if (g, ∇) is a conjugate symmetric statistical structure, then ∇ and $\bar{\nabla}$ are simultaneously projectively flat. It follows that, if (g, ∇) is conjugate symmetric, then it is locally realizable on an equiaffine hypersurface if and only if ∇ or $\bar{\nabla}$ is projectively flat, and the realization is automatically on an affine sphere.

In [3,6], a few examples of conjugate symmetric statistical structures that are not realizable (even locally) on affine spheres were produced.

3. Statistical Structures with Complete Metrics

The following theorems are attributed to Blaschke, Deicke and Calabi (see e.g., [1]).

Theorem 1. *Let $f : M \rightarrow \mathbf{R}^{n+1}$ be an elliptic affine sphere whose Blaschke metric is complete. Then, M is compact and the induced structure on M is trivial. Consequently, the affine sphere is an ellipsoid.*

Theorem 2. *Let $f : M \rightarrow \mathbf{R}^{n+1}$ be a hyperbolic or parabolic affine sphere whose Blaschke metric is complete. Then, the Ricci tensor of the metric is negative semidefinite.*

The theorems can be generalized to the case of statistical manifolds in the following manner:

Theorem 3. *Let (g, ∇) be a trace-free conjugate symmetric statistical structure on a manifold M . Assume that g is complete on M . If the sectional ∇ -curvature is bounded from below and above on M , then the Ricci tensor of g is bounded from below and above on M . If the sectional ∇ -curvature is non-negative everywhere, then the statistical structure is trivial, that is, $\nabla = \hat{\nabla}$. If the statistical sectional curvature is bounded from 0 by a positive constant then, additionally, M is compact and its first fundamental group is finite.*

Let us explain why Theorem 3 is a generalization of Theorems 1 and 2. The induced structure on an affine sphere is a conjugate symmetric trace-free statistical structure. Moreover, the statistical connection on an affine sphere is projectively flat and its ∇ -sectional curvature is constant. In Theorem 3, we do not need the projective flatness of the statistical connection, which means that the manifold with a statistical structure can be nonrealizable on any Blaschke hypersurface, even locally. Moreover, the assumption about the constant curvature is replaced by the assumption that the curvature satisfies some inequalities.

More precise and more general formulations of this theorem give the two following results:

Theorem 4. Let (g, ∇) be a trace-free conjugate symmetric statistical structure on an n -dimensional manifold M . Assume that (M, g) is complete and the sectional ∇ -curvature $k(\pi)$ satisfies the inequality

$$H_3 + \frac{n-2}{2}\varepsilon \leq k(\pi) \leq H_3 + \frac{n}{2}\varepsilon \quad (11)$$

for every tangent plane π , where H_3 is a non-positive number and ε is a non-negative function on M . Then, the Ricci tensor $\widehat{\text{Ric}}$ of g satisfies the following inequalities:

$$(n-1)H_3 + \frac{(n-1)(n-2)}{2}\varepsilon \leq \widehat{\text{Ric}} \leq -(n-1)^2H_3 + \frac{(n-1)n}{2}\varepsilon. \quad (12)$$

The scalar curvature $\hat{\rho}$ of g satisfies the following inequalities:

$$n(n-1)H_3 + \frac{(n-1)(n-2)n}{2}\varepsilon \leq \hat{\rho} \leq \frac{n^2(n-1)}{2}\varepsilon. \quad (13)$$

Theorem 5. Let (M, g) be a complete Riemannian manifold with conjugate symmetric trace-free statistical structure (g, ∇) . If the sectional ∇ -curvature is non-negative on M , then the statistical structure is trivial, i.e., $\nabla = \hat{\nabla}$. Moreover, if the sectional ∇ -curvature is bounded from 0 by a positive constant, then M is compact and its first fundamental group is finite.

Proofs of Theorems 3–5 can be found in [6].

4. Completeness of Statistical Connections

Very little is known about the completeness of statistical connections. The difference between the completeness of metrics and that of affine connections is huge. In particular, a statistical connection on a compact manifold does not have to be complete. Indeed, we can offer the following simple example:

Example 1. Take \mathbf{R}^2 with its standard flat Riemannian structure. Let U, V be the canonical frame field on \mathbf{R}^2 . Define a statistical connection ∇ as follows:

$$\nabla_U U = U, \quad \nabla_U V = -V, \quad \nabla_V V = -U. \quad (14)$$

This statistical structure can be projected on the standard torus T^2 . A curve $\gamma(t) = (x(t), y(t))$ is a ∇ -geodesic if and only if

$$\ddot{x} + (\dot{x})^2 - (\dot{y})^2 = 0, \quad \ddot{y} - 2\dot{x}\dot{y} = 0. \quad (15)$$

Let y_0 be a fixed real number. Consider the curve

$$\gamma(t) = (\ln(1-t), y_0) \quad (16)$$

for $t \in [0, 1)$. It is a ∇ -geodesic. We have $\|\dot{\gamma}(t)\| = \frac{1}{1-t} \rightarrow +\infty$ if $t \rightarrow 1$. Hence, this geodesic cannot be extended beyond 1. The connection ∇ is not complete on T^2 .

In the above example, the cubic form A of the statistical structure is $\hat{\nabla}$ -parallel. This is the reason why the statistical connection is not complete. More precisely, we have:

Theorem 6. ([7]) Let (g, ∇) be a non-trivial statistical structure such that

$$\hat{\nabla} A(U, U, U, U) \leq 0$$

for every $U \in \mathcal{UM}$, where \mathcal{UM} is the unit sphere bundle over M . The statistical connection ∇ is not complete.

As a corollary, we get:

Corollary 1. Let (g, ∇) be a statistical structure for which ∇ is complete and $(\hat{\nabla} A)(U, U, U, U) \leq 0$ for each $U \in \mathcal{UM}$. Then, the statistical structure must be trivial.

Let us now cite a positive result first proved by Noguchi [2].

Theorem 7. Let (M, g) be a complete Riemannian manifold, and A be a cubic form given by:

$$A = \text{sym}(d\sigma \otimes g) \quad (17)$$

for some function σ on M . Assume that the function σ is bounded from below on M . Then, the statistical connection of statistical structure (g, A) is complete.

In particular, any function on a compact Riemannian manifold M gives rise to a statistical structure on M whose statistical connection is complete. In fact, we have a more general fact:

Corollary 2. Let (M, g) be a compact Riemannian manifold. Each function σ on M gives rise to a statistical structure whose statistical connection and its conjugate are complete.

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