

Fisher-Type Fixed Point Results in b -Metric Spaces

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Abstract: In this paper, we prove some common fixed-point theorems for two self-mappings in the context of a complete b -metric space by proposing a new contractive type condition. Further, we derive a result for three self-mappings in the same setting. We provide two examples to demonstrate the validity of the obtained results.

Keywords: b -metric space; common fixed point; weakly compatible

MSC: 47H10; 54H25

1. Introduction and Preliminaries

It would not be wrong to say that fixed-point theory was a results of the investigation of the existence and uniqueness of a solution of certain differential equations. In this aspect, the roots of metric fixed-point theory can be attributed back to the results of renowned mathematicians, J. Liouville [1], E. Picard [2] and H. Poincaré [3]. Roughly speaking, these famous mathematicians used the method of successive approximation to solve the differential equation that they dealt with. In 1922, Banach [4] reported an elegant fixed-point theorem which was an abstraction of the successive approximation method. Formally, metric fixed-point theory appeared with this renowned result of Banach [4] that is mostly known as the Banach Contraction Mapping Principle in the literature. As an another historical note, we emphasize that Banach's original proof is in the framework of normed space. The corresponding form in the setting of metric space was formulated by Caccioppoli [5]. After then, in some sources, it has been called Banach–Caccioppoli fixed-point theorem.

Due to its origin, fixed-point theory has a wide application potential in most quantitative sciences. Many real world problems can be easily characterized in the framework of fixed-point theory. For example, a fixed-point equation $h(x) = x$ can be considered $H(x) = x - h(x) = 0$. Regarding the requirements of the applications; intrinsically, the structure of Banach–Caccioppoli fixed-point theorem has been improved in several aspects. On one hand, the properties of the operator have been weakened, while on the other hand, metric space structure has been replaced by some other abstract spaces. Among all, in 1993, Czerwik [6] suggested a successful and proper generalization of the metric space notion by introducing the concept of b -metric space. Following this famous result in the setting of b -metric spaces, several extensions in distinct aspects have been released in this direction (see e.g., [7–15] and references therein). In this paper, we study certain common fixed-point theorems for three maps in the setting of complete b -metric spaces.

Firstly, we recall the notion of b -metric.

Definition 1 (Czerwik [6]). Let X be a nonempty set. A function $d : X \times X \rightarrow [0, \infty)$ is called a b -metric if the following axioms are fulfilled:

- (b1) d is reflexive, that is, $d(x, y) = 0$ if and only if $x = y$.
- (b2) d has a symmetry, that is, $d(x, y) = d(y, x)$ for all $x, y \in X$.
- (b3) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$, where $s \geq 1$.

Here, (X, d) is called a b -metric space, in short, bMS .

Remark 1. In case of $s = 1$, the b -metric coincide the standard metric. Notice also that b -metric does not need to be continuous in general. In this manuscript, we deal with continuous b -metrics only [16].

The following is a constructive example of b -metric.

Example 1. Let $X = \{x_i : 1 \leq i \leq M\}$ for some $M \in \mathbb{N}$ and $s \geq 2$. Define $d : X \times X \rightarrow \infty$ as

$$d(x_i, x_j) = \begin{cases} 0 & \text{if } i = j, \\ s & \text{if } (i, j) = (1, 2) \text{ or } (i, j) = (2, 1), \\ 1 & \text{otherwise.} \end{cases}$$

Consequently, we derive that

$$d(x_i, x_j) \leq \frac{s}{2}[d(x_i, x_k) + d(x_k, x_j)],$$

for all $i, j, k \in \{1, M\}$. Thus, (X, d) forms a b -metric for $s > 2$ where the ordinary triangle inequality does not hold.

For more examples for b -metric, we may refer e.g., [7,8,10–15] and the corresponding references therein.

Example 2. (See e.g., [12]) For $0 < q < 1$, the space $L^q[0, 1]$ of all real-valued functions $f(t)$, $t \in [0, 1]$ such that $\int_0^1 |f(t)|^q dt < \infty$, endowed with

$$d(f, h) := \left(\int_0^1 |f(t) - h(t)|^q dt \right)^{1/q}, \text{ for each } x, y \in L^p[0, 1],$$

forms a b -metric space. Notice that $s = 2^{1/q}$.

Definition 2. (see e.g., [17,18]) Suppose that f and g are self mappings on a non-empty set X . A point x is names as a coincidence point of f and g incase $fx = gx$, for x in X . Moreover, z is called a point of coincidence of f and g whenever $z = fx = gx$ for some x in X . In addition, f and g are said to be weakly compatible, if

$$fx = gx \Rightarrow f(gx) = g(fx)$$

holds for every $x \in X$.

Proposition 1. (see Lemma 3 in [19]) Let f, g, h be self mappings on a non-empty set X and $v \in X$ is the a unique coincidence point of f, g and h . These self-mappings, f, g, h , have a unique common fixed point if $\{f, h\}$ and $\{g, h\}$ are weakly compatible.

Definition of comparison function, defined by Rus [20], is the following:

Definition 3. [20,21] A function $\phi : [0, \infty) \rightarrow [0, \infty)$ is called a comparison function if it is increasing and $\phi^n(t) \rightarrow 0$ as $n \rightarrow \infty$ for every $t \in [0, \infty)$, where ϕ^n is the n -th iterate of ϕ .

Lemma 1. ([20,21]) If $\phi : [0, \infty) \rightarrow [0, \infty)$ is a comparison function, then

1. each iterate ϕ^k of ϕ , $k \geq 1$ is also a comparison function;
2. ϕ is continuous at 0;
3. $\phi(t) < t$ for all $t > 0$.

Definition 4. Let $s \geq 1$ be a real number. A function $\psi : [0, \infty) \rightarrow [0, \infty)$ is called a (b) -comparison function if

1. ψ is increasing;
2. there exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent nonnegative series $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\psi^{k+1}(t) \leq as^k\psi^k(t) + v_k$, for $k \geq k_0$ and any $t \geq 0$.

Let $\Psi = \{\psi : [0, \infty) \rightarrow [0, \infty) : \psi \text{ is } b\text{-comparison function}\}$. Note that in case of $s = 1$, a (b) -comparison function is named as (c) -comparison.

Lemma 2. ([22]) For $\phi \in \Psi$,

1. the series $\sum_{k=0}^{\infty} s^k \phi^k(t)$ converges for any $t \in [0, \infty)$;
2. the function $b_s : [0, \infty) \rightarrow [0, \infty)$ defined as $b_s = \sum_{k=0}^{\infty} s^k \phi^k(t)$ is increasing and continuous at $t = 0$.

Remark 2. On account of Lemma 2 and Lemma 1, any (b) -comparison function, we have ψ satisfies $\psi(t) < t$.

Fisher [23] proved the following existence theorem:

Theorem 1. [23] Let T be a mapping of the complete metric space X into itself satisfying the inequality

$$[d(Tx, Ty)]^2 \leq a(d(x, Tx)d(y, Ty)) + b(d(x, Ty)d(y, Tx))$$

$\forall x, y \in X$, $0 \leq a < 1$, $0 \leq b$ then T has a fixed point in X .

In 1980, Pachpatte [24] extended the result of Fisher [23] in the following way.

Theorem 2. [24] Let T be a mapping of the complete metric space X into itself satisfying the inequality

$$[d(Tx, Ty)]^2 \leq a[d(x, Tx)d(y, Ty) + d(x, Ty)d(y, Tx)] + b[d(x, Tx)d(y, Tx) + d(x, Ty)d(y, Ty)]$$

$\forall x, y \in X$, where $a, b \geq 0$ and $a + 2b < 1$ then T has a unique fixed point in X .

This trend has been followed by Sharma and Sahu [25], Popa [26], Ali and Arshad [27] and so on. By following the trend, initiated by [23] and succeeded by [24–27], we introduce a new type contraction for three maps via auxiliary function ψ . We examined the existence and uniqueness of a common fixed point for such contractions in the framework of b -metric space.

2. Main Results

Let (X, d) be a complete b -metric space and let f, g, h be mappings from X into itself satisfying the condition:

$$f(X) \cup g(X) \subseteq h(X). \quad (1)$$

Let $x_0 \in X$. By (1) there exists a point $x_1 \in X$ such that $hx_1 = fx_0$ and for x_1 there exists $x_2 \in X$ such that $hx_2 = gx_1$. Inductively we can define the sequences $\{x_n\}$ and y_n in X such that

$$y_{2n} = hx_{2n+1} = fx_{2n}, y_{2n+1} = hx_{2n+2} = gx_{2n+1} \quad \forall n \geq 0. \quad (2)$$

Lemma 3. Let f, g, h be mappings from a b -metric space (X, d) into itself satisfying (1) and such that for all $x, y \in X$

$$d(fx, gy)]^2 \leq \psi(F(x, y)), \quad (3)$$

where, $\psi \in \Psi$ and

$$F(x, y) = \max\{d(fx, gy)d(hx, fx), d(fx, gy)d(hy, gy), d(hy, fx)d(hx, gy), \frac{1}{2s}d(hy, gy)d(hx, gy)\},$$

$\psi \in \Psi$. Then, the sequence $\{y_n\}$ defined by (2) is a Cauchy sequence in X .

Proof. For an arbitrary $x_0 \in X$, we shall construct a sequence $\{x_n\}$ and $\{y_n\}$ in (2). If there exists n_0 such that $y_{2n_0} = y_{2n_0+1}$ we obtain : $hx_{2n_0+1} = fx_{2n_0} = hx_{2n_0+2} = gx_{2n_0+1}$ and hence, x_{2n_0+1} forms a common fixed point of h and g .

Without loss of generality, we suppose that $y_{2n} \neq y_{2n+1}$. Accordingly, from (2) and (3) we find that

$$[d(y_{2n}, y_{2n+1})]^2 = [d(fx_{2n}, gx_{2n+1})]^2 \leq \psi(F(x_{2n}, x_{2n+1})) \quad (4)$$

where

$$\begin{aligned} F(x_{2n}, x_{2n+1}) &= \max \{d(fx_{2n}, gx_{2n+1})d(hx_{2n}, fx_{2n}), d(fx_{2n}, gx_{2n+1})d(hx_{2n+1}, gx_{2n+1}), \\ &\quad d(hx_{2n+1}, fx_{2n})d(hx_{2n}, gx_{2n+1}), \frac{1}{2s}d(hx_{2n+1}, gx_{2n+1})d(hx_{2n}, gx_{2n+1})\} \\ &\leq \max \{d(hx_{2n+1}, hx_{2n+2})d(hx_{2n}, hx_{2n+1}), d(hx_{2n+1}, hx_{2n+2})d(hx_{2n+1}, hx_{2n+2}) \\ &\quad d(hx_{2n+1}, hx_{2n+1})d(hx_{2n}, hx_{2n+2}), \frac{1}{2s}d(hx_{2n+1}, hx_{2n+2})d(hx_{2n}, hx_{2n+2})\} \\ &\leq \max \{d(y_{2n}, y_{2n+1})d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})d(y_{2n}, y_{2n+1}) \\ &\quad d(y_{2n}, y_{2n})d(y_{2n-1}, y_{2n+1}), \frac{1}{2s}d(y_{2n}, y_{2n+1})d(y_{2n-1}, y_{2n+1})\} \\ &\leq \max \{d(y_{2n}, y_{2n+1})d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})d(y_{2n}, y_{2n+1}) \\ &\quad d(y_{2n}, y_{2n})d(y_{2n-1}, y_{2n+1}), \frac{1}{2}d(y_{2n}, y_{2n+1})[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})]\}. \end{aligned}$$

Suppose $d(y_{2n_0-1}, y_{2n_0}) < d(y_{2n_0}, y_{2n_0+1})$ for some n_0 . Since the function then the inequality (4) turns into

$$[d(y_{2n_0}, y_{2n_0+1})]^2 \leq \psi([d(y_{2n_0}, y_{2n_0+1})]^2) < [d(y_{2n_0}, y_{2n_0+1})]^2,$$

which is a contradiction. Thus, we have

$$d(y_{2n}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n}) \text{ for all } n \in \mathbb{N}.$$

Keeping in mind that ψ is non-decreasing, and by taking the inequality (4) into account and employing Remark 2 recursively, we conclude also that

$$\begin{aligned} [d(y_{2n}, y_{2n+1})]^2 &\leq \psi([d(y_{2n-1}, y_{2n})]^2) < [d(y_{2n-1}, y_{2n})]^2 \\ &\leq \psi^2([d(y_{2n-2}, y_{2n-1})]^2) < [d(y_{2n-2}, y_{2n-1})]^2 \\ &\dots \\ &\leq \psi^{2n}([d(y_0, y_1)]^2). \end{aligned}$$

By using the same arguments, similarly, we find that

$$d(y_{2n-1}, y_{2n}) \leq d(y_{2n-2}, y_{2n-1}),$$

and moreover,

$$\begin{aligned} [d(y_{2n-1}, y_{2n})]^2 &\leq \psi([d(y_{2n-2}, y_{2n-1})]^2) < [d(y_{2n-2}, y_{2n-1})]^2 \\ &\leq \psi^2([d(y_{2n-3}, y_{2n-2})]^2) < [d(y_{2n-3}, y_{2n-2})]^2 \\ &\dots \\ &\leq \psi^{2n-1}([d(y_0, y_1)]^2). \end{aligned}$$

As a result, for all $n \in \mathbb{N}$, we get

$$[d(y_n, y_{n+1})]^2 \leq \psi([d(y_{n-1}, y_n)]^2) < [d(y_{n-1}, y_n)]^2 \leq \dots < \psi^n([d(y_0, y_1)]^2). \quad (5)$$

On the account of Lemma 2, we conclude that

$$\lim_{n \rightarrow \infty} d(y_{n+1}, y_n) = 0. \quad (6)$$

Now, we shall indicate that the sequence $\{y_n\}$ is Cauchy.

By using the modified triangle inequality (b3) recursively, and keeping the fact that $(\alpha + \beta)^2 \leq 2(\alpha^2 + \beta^2)$ in mind, we observe the following estimation for the distance $d(y_n, y_{n+k})$ for $k \geq 1$ and $s \geq 1$

$$\begin{aligned} [d(y_n, y_{n+k})]^2 &\leq [s(d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+k}))]^2 \\ &\leq 2s^2[d(y_n, y_{n+1})]^2 + 2s^2[d(y_{n+1}, y_{n+k})]^2 \\ &\leq 2s^2[d(y_n, y_{n+1})]^2 + 2s^2 \{s[d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+k})]\}^2 \\ &\leq 2s^2[d(y_n, y_{n+1})]^2 + (2s^2)^2[d(y_{n+1}, y_{n+2})]^2 + (2s^2)^2[d(y_{n+2}, y_{n+k})]^2 \\ &\vdots \\ &\leq 2s^2[d(y_n, y_{n+1})]^2 + (2s^2)^2[d(y_{n+1}, y_{n+2})]^2 + \dots + (2s^2)^k[d(y_{n+k-1}, y_{n+k})]^2 \end{aligned} \quad (7)$$

Applying (5) and (7) we derive that

$$\begin{aligned} [d(y_n, y_{n+k})]^2 &\leq (2s^2)\psi^n([d(y_0, y_1)]^2) + (2s^2)^2\psi^{n+1}([d(y_0, y_1)]^2) + \dots + (2s^2)^k\psi^{n+k-1}([d(y_0, y_1)]^2) \\ &= \frac{1}{(2s^2)^{n-1}} \left((2s^2)^n\psi^n([d(y_0, y_1)]^2) + (2s^2)^{n+1}\psi^{n+1}([d(y_0, y_1)]^2) + \dots \right. \\ &\quad \left. + (2s^2)^{n+k-1}\psi^{n+k-1}([d(y_0, y_1)]^2) \right). \end{aligned} \quad (8)$$

Consequently, we have

$$d^2(y_n, y_{n+k}) \leq \frac{1}{(2s^2)^{n-1}} [P_{n+k-1} - P_{n-1}], \quad n \geq 1, k \geq 1, \quad (9)$$

where $P_n = \sum_{j=0}^n (2s^2)^j \psi^j([d(y_0, y_1)]^2)$, $n \geq 1$. On the account of Lemma 2, we guarantee that the series $\sum_{j=0}^{\infty} (2s^2)^j \psi^j([d(y_0, y_1)]^2)$ is convergent. Since $s \geq 1$, letting limit $n \rightarrow \infty$ in (9) we deduce that

$$\lim_{n \rightarrow \infty} d^2(y_n, y_{n+k}) \leq \lim_{n \rightarrow \infty} \frac{1}{(2s^2)^{n-1}} [P_{n+k-1} - P_{n-1}] = 0. \quad (10)$$

We find that the constructive sequence $\{y_n\}$ is Cauchy in (X, d) . $\square$

Theorem 3. Let (X, d) be a complete b -metric space, f, g, h be self mappings of X satisfying the conditions (1) and (3). We suppose also that $h(X)$ is a closed subspace of X . Then the maps f, g and h have a coincidence point z in X . Moreover, if the pairs $\{f, h\}$ and $\{g, h\}$ are weakly compatible then f, g and h have a unique common fixed point in X .

Proof. Let us consider now the sequence $\{y_n\}$ defined by (2). By Lemma 3, we have that $\{y_n\}$ is a Cauchy sequence in X and since X is complete, the sequence $\{y_n\}$ converges to a point z in X . But, $h(X)$ is complete, being a closed subspace of X and since $f(X) \cup g(X) \subseteq h(X)$, the subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ which are contained in $h(X)$ must have a limit z in $h(X)$, i.e.

$$\lim_{n \rightarrow \infty} f x_{2n} = \lim_{n \rightarrow \infty} g x_{2n+1} = \lim_{n \rightarrow \infty} h x_{2n+1} = \lim_{n \rightarrow \infty} h x_{2n+2} = z.$$

Let $u \in h^{-1}z$. Then $hu = z$ and we suppose that $gu \neq z$. From (3) we have

$$[d(fx_{2n}, gu)]^2 \leq \psi(F(x_{2n}, u)), \quad (11)$$

where

$$F(x_{2n}, u) = \max \{ [d(fx_{2n}, gu)d(hx_{2n}, fx_{2n})], [d(fx_{2n}, gu)d(hu, gu)] \\ [d(hu, fx_{2n})d(hx_{2n}, gu)], \frac{1}{2s} [d(hu, gu)d(hx_{2n}, gu)] \}.$$

Keeping Remark 2 in mind and by taking $\limsup$ in (11) as $n \rightarrow \infty$, we find that

$$[d(z, gu)]^2 \leq \psi([d(z, gu)]^2) < [d(z, gu)]^2,$$

a contradiction. Hence, we have $[d(z, gu)]^2 = 0$ which gives that $gu = z = hu$. Using the similar reasoning, supposing that $fu \neq z$ we have

$$[d(fu, gx_{2n+1})]^2 \leq \psi(F(u, x_{2n+1})), \quad (12)$$

where

$$F(u, x_{2n+1}) = \max \{ [d(fu, gx_{2n+1})d(hu, fu)], [d(fu, gx_{2n+1})d(hx_{2n+1}, gx_{2n+1})], \\ [d(hx_{2n+1}, fu)d(hu, gx_{2n+1})], \frac{1}{2s} [d(hx_{2n+1}, gx_{2n+1})d(hu, gx_{2n+1})] \}.$$

Again, by taking Remark 2 into account and by letting $\limsup$ in (12) as $n \rightarrow \infty$,

$$[d(fu, z)]^2 \leq \psi([d(fu, z)]^2) < [d(fu, z)]^2,$$

which is a contradiction. Therefore, $fu = z = hu = gu$, i.e., the maps f, g and h have a coincidence point. If we consider the supplementary assumption, then the pairs g, h and f, h are weakly compatible, we have

$$\begin{aligned} hgu = ghu &\Rightarrow gz = hz \\ hfu = fhu &\Rightarrow fz = hz, \end{aligned}$$

so

$$h(z) = g(z) = f(z). \quad (13)$$

We shall show that z is the common fixed point of f, g and h . Without loss of generality, suppose, on the contrary, that $z \neq gz$. Hence, by (3) we get

$$[d(fx_{2n}, gz)]^2 \leq \psi(F(x_{2n}, z)), \quad (14)$$

where

$$F(x_{2n}, z) = \max \{ [d(fx_{2n}, gz)d(hx_{2n}, fx_{2n})], [d(fx_{2n}, gz)d(hz, gz)], \\ [d(hz, fx_{2n})d(hx_{2n}, gz)], \frac{1}{2s} [d(hz, gz)d(hx_{2n}, gz)] \}.$$

By letting $\limsup$ in (14) as $n \rightarrow \infty$, together with applying Remark 2, we find that

$$[d(z, gz)]^2 \leq \psi([d(z, gz)]^2) < [d(z, gz)]^2,$$

a contradiction. Thus, we have $d(z, gz) = 0$, that is, $z = gz$. By combining with (13) we get $hz = gz = z$ which shows that z is a common fixed point of the mappings f, g and h .

For the uniqueness, we suppose, on the contrary, that f, g and h have two common fixed points z_1 and z_2 such that $z_1 \neq z_2$. Then, by using (3) we get

$$[d(z_1, z_2)]^2 = [d(fz_1, gz_2)]^2 \psi(F(fz_1, gz_2)), \quad (15)$$

where

$$F(fz_1, gz_2) = \max \{ [d(fz_1, gz_2)d(hz_1, fz_1)], [d(fz_1, gz_2)d(hz_2, gz_2)] \\ [d(hz_2, fz_1)d(hz_1, gz_2)], \frac{1}{2s} [d(hz_2, gz_2)d(hz_1, gz_2)] \} \\ \leq \max \{ [d(z_1, z_2)d(z_1, z_1)], [d(z_1, z_2)d(z_2, z_2)] \\ [d(z_2, z_1)d(z_1, z_2)], \frac{1}{2s} [d(z_2, z_2)d(z_1, z_2)] \} \\ \leq [d(z_1, z_2)]^2.$$

Thus, (14) yields that

$$[d(z_1, z_2)]^2 = [d(fz_1, gz_2)]^2 \psi(F(fz_1, gz_2)) = \psi([d(z_1, z_2)]^2) < [d(z_1, z_2)]^2, \quad (16)$$

a contradiction that completes the proof. $\square$

We will now give some immediate consequences of the main result. By replacing the mapping h with the identity mapping on X , in Theorem 3 we deduce the first consequence of the main result.

Corollary 1. Let (X, d) be a complete b -metric space, $\psi \in \Psi$ and f, g be the mappings of X such that for $x, y \in X$ the following inequality is satisfied:

$$[d(fx, gy)]^2 \leq F(x, y) \quad (17)$$

where

$$F(x, y) = \{ (d(fx, gy)d(x, fx)), (d(fx, gy)d(y, gy)) \\ (d(y, fx), d(x, gy)), \frac{1}{2s} (d(y, gy), d(x, gy)) \}$$

If in the Corollary 1 we take $f = g$ we derive the next consequence.

Corollary 2. Let (X, d) be a complete b -metric space, $\psi \in \Psi$ and a mapping $f : X \rightarrow X$ such that for $x, y \in X$ the following inequality is satisfied:

$$[d(fx, gy)]^2 \leq F(x, y) \quad (18)$$

where

$$F(x, y) = \left\{ (d(fx, y)d(x, fx)), (d(fx, y)d(y, y)) \right. \\ \left. (d(y, fx), d(x, y)), \frac{1}{2s}(d(y, y), d(x, y)) \right\}$$

Theorem 4. Let (X, d) be a complete b -metric space, $\psi \in \Psi$ and let f, g, h be mappings from X into itself satisfying the condition:

$$[d(fx, gy)]^3 \leq G(x, y) \quad (19)$$

where

$$G(x, y) = \left\{ d(fx, gy)^2 d(hx, hy), (d(fx, gy)d(hy, gy)d(hx, gy)) \right. \quad (20)$$

$$\left. (d(hx, fx)d(hy, fx), d(hx, gy)) \right\}. \quad (21)$$

Suppose that $h(X)$ is a closed subspace of X and $f(X) \cup g(X) \subseteq h(X)$. Then, f, g and h have a coincidence point. In addition, in case of the pairs $\{f, h\}$ and $\{g, h\}$ are weakly compatible, these maps have a unique common fixed point.

The details of the proof of Theorem 4 are very close to the proof of Theorem 3, with suitable modification, so we skip it.

Example 3. Let $X = [0, 1]$ be a set endowed with a b -metric $d(x, y) = (x - y)^2$ with $s = 2$ and we define three mappings $f, g, h : X \rightarrow X$, by

$$fx = \frac{x}{16}, gx = \begin{cases} \frac{x}{8} & \text{if } x \in [0, 1) \\ 0 & \text{if } x = 1 \end{cases} \quad hx = \begin{cases} \frac{x}{2} & \text{if } x \in [0, 1) \\ 0 & \text{if } x = 1 \end{cases}$$

Clearly, $f(X) \cup g(X) \subset h(X)$, $hf0 = 0 = fh0$, $hg0 = 0 = gh0$ and $hg1 = 0 = gh1$ which shows that the pairs f, h, g, h are weakly compatible. Let $\psi(t) = \frac{t}{4}$.

For any $x \in [0, 1]$ and $y \in [0, 1]$ we have

$$\begin{aligned} [d(fx, gy)]^2 &= \left[\frac{(x-2y)}{16} \right]^4 \leq \frac{1}{4} \max \left\{ \left[\frac{(x-2y)}{16} \right]^2 \cdot \frac{49x^2}{256}, \left[\frac{(x-2y)}{16} \right]^2 \cdot \frac{36y^2}{256} \right\} \\ &= \frac{1}{4} \max \{ d(fx, gy)d(hx, fx), d(fx, gy)d(hy, gy) \} \\ &\leq \frac{F(x, y)}{4} = \psi(F(x, y)). \end{aligned}$$

For $x \in [0, 1]$ and $y = 1$

$$[d(fx, g1)]^2 = \frac{x^4}{16^4} < \frac{1}{4} \cdot \frac{x^2}{16^2} \cdot \frac{49x^2}{16^2} = \frac{1}{4} d(fx, g1) \cdot d(hx, fx) \leq \frac{F(x, 1)}{4} = \psi(F(x, 1))$$

for any $x, y \in X$. Consequently, we deduce that 0 is the unique common fixed point of the maps f, g and h since all assumptions of Theorem 3 are fulfilled.

Example 4. Let the set $X = \{m, n, p, q\}$ and a function $d : X \times X \rightarrow [0, \infty)$ defined as follows:

$d(x, y)$	m	n	p	q
m	0	3	1	4
n	3	0	3	1
p	1	3	0	2
q	4	1	2	0

By a simple calculation, one can verify that the function d is a b -metric, for $s = 2$. We define the self mappings f, g, h on X as

x	m	n	p	q
f	m	m	p	m
g	m	m	m	p
h	m	p	n	n

Since $f(X) = \{m, p\}$, $g(X) = \{m, p\}$ and $h(X) = \{m, n, p\}$, the condition (1) is satisfied. Moreover,

$$hfm = m = fhm, hgm = m = ghm.$$

Let also $\psi(t) = \frac{t}{2}$

Thus, m is the unique common fixed point of the maps f, g and h since all the conditions of Theorem 3 are satisfied.

3. Conclusions

By choosing ψ in a proper way in Theorem 3, Corollaries 1 and 2, we can derive further consequences.

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