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Optimizing Three-Dimensional Constrained Ordered Weighted Averaging Aggregation Problem with Bounded Variables

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Abstract: A single constrained ordered weighted averaging aggregation (COWA) problem is of considerable importance in many disciplines. Two models are considered: the maximization COWA problem with lower bounded variables and the minimization COWA problem with upper bounded variables. For a three-dimensional case of these models, we present the explicitly optimal solutions theoretically and empirically. The bounds and weights can affect the optimal solution of the three-dimensional COWA problem with bounded variables.

Keywords: constrained ordered weighted averaging aggregation problem; mixed integer linear programming; bounded variables

1. Introduction

An ordered weighted averaging (OWA) operator [1] is a general class of parametric aggregation operators that appears in many research fields such as decision making [2–6], fuzzy system [7,8], statistics [9–11], risk analysis [12] and others [13,14]. For more details, see Carlsson and Fullér [15], Emrouznejad and Marra [16] and Yager et al. [17]. A constrained OWA aggregation problem (COWA) attempts to optimize the OWA aggregation problem with multiple constraints. Yager [18] developed a mixed integer linear programming problem to solve a single COWA problem. Later, Carlsson et al. [19] proposed an algorithm to solve the single constrained OWA optimization problem for any dimensions. Furthermore, Coroianu and Fullér [20] presented an explicitly optimal solution for the single COWA problem with any constraint coefficients. In addition, there are other important references [21–27] dedicated to constrained OWA optimization problem with multiple constraints. However, the decision variables are usually bounded for the most practical problems. Recently, Chen and Tang [28] proposed a three-dimensional COWA problem with lower bounded variables. This paper presents the explicitly optimal solutions for the three-dimensional COWA problem with bounded variables. Two models are considered. One is a maximizing three-dimensional constrained OWA aggregation problem with lower bounded variables (3COWAL). The other is a minimizing three-dimensional constrained OWA aggregation problem with upper bounded variables (3COWAU).

The organization of this paper is as follows. Section 2 briefly reviews the COWA problem. For maximizing 3COWAL, there are two parameters (w_1, w_2, w_3) and (l_1, l_2, l_3) that affect the optimal solution types. We discuss the optimal solution behaviors in Section 3 for $w_1 \geq w_2 \geq w_3$ and Section 4 for $l_1 \geq l_2 \geq l_3$. Section 5 analyzes the optimal solution behaviors of minimizing 3COWAU. Finally, some concluding remarks are presented.

2. Constrained OWA Aggregation Problem

An OWA operator of dimension n is a mapping $F : \mathcal{R}^n \to \mathcal{R}$ that has an associated weighting vector $W = (w_1, w_2, \ldots, w_n)$ satisfying

$$w_1 + w_2 + \ldots + w_n = 1,\ 0 \le w_i \le 1,\ i = 1, 2, \ldots, n$$

and such that

$$G(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} w_i y_i \tag{1}$$

with y_i being the ith largest of $\{x_1, x_2, \ldots, x_n\}$. For simplicity, we will denote this expression as $F(y_1, y_2, \ldots, y_n)$.

Consider the following single COWA problem:

$$\text{Max } W^T Y,\ \text{s.t. } \mathcal{I}^T X \le 1,\ X \ge \mathbf{0}, \tag{2}$$

where the column vectors X, Y, W and $\mathcal{I}$ are

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},\ Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix},\ W = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix},\ \mathcal{I} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

By introducing the $(n-1) \times n$ matrix

$$G = \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & 0 & & 0 & 0 \\ & \vdots & & & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$

and the column binary vectors $Z_i \in \{0,1\}^n$, $i = 1, 2, \ldots, n$, Yager [18] transformed the nonlinear programming single COWA problem (2) to the following mixed integer linear programming problem (MILP):

$$\begin{gathered} \text{Max } W^T Y,\ \text{s.t. } \mathcal{I}^T X \le 1,\ GY \le 0,\ y_i \mathcal{I} - X - MZ_i \le 0,\ i = 1,\ 2, \ldots, n-1, \\ y_n \mathcal{I} - X \le 0,\ \mathcal{I}^T Z_i \le n - i,\ i = 1,\ 2, \ldots, n-1,\ Z_i \in \{0,1\}^n,\ i = 1,\ 2, \ldots, n-1,\ X \ge \mathbf{0}, \end{gathered} \tag{3}$$

where M is a very large positive number. To reduce the multiple solutions of the MILP (3), Chen and Tang [28] introduced the following constraints:

$$Z_{i+1} \le Z_i,\ i = 1,\ 2, \ldots, n-2.$$

Then, the more efficient MILP of a single COWA problem is as follows:

$$\begin{gathered} \text{Max } W^T Y,\ \text{s.t. } \mathcal{I}^T X \le 1,\ GY \le 0,\ y_i \mathcal{I} - X - MZ_i \le 0,\ i = 1,\ 2, \ldots, n-1, y_n \mathcal{I} - X \le 0, \\ \mathcal{I}^T Z_i \le n - i,\ i = 1,\ 2, \ldots, n-1,\ Z_{i+1} \le Z_i,\ i = 1,\ 2, \ldots, n-2, Z_i \in \{0,1\}^n, \\ i = 1,\ 2, \ldots, n-1,\ X \ge \mathbf{0}. \end{gathered} \tag{4}$$

3. Maximizing a Three-Dimensional Constrained OWA Aggregation Problem with Lower Bounded Variables for $w_1 \ge w_2 \ge w_3$

It is fairly common in practical optimization problems that the decision variables are usually bounded. A typical decision variable x_i is bounded from below by l_i and from above by u_i, where $l_i \leq u_i$. Sections 3 and 4 analyze the maximization COWA problem with the lower bound constraints. Section 5 analyzes the minimization COWA problem with the upper bound constraints. Chen and Tang [28] proposed the following COWAL:

$$\begin{gathered}\text{Max } W^T Y,\ \text{s.t. } \mathcal{I}^T X \leq 1,\ GY \leq 0,\ ,\ y_i\mathcal{I} - X - MZ_i \leq 0,\ i = 1,\ 2, \ldots, n-1, \\ y_n\mathcal{I} - X \leq 0,\ \mathcal{I}^T Z_i \leq n - i,\ i = 1,\ 2, \ldots, n-1,\ Z_{i+1} \leq Z_i,\ i = 1,\ 2, \ldots, n-2,\ Z_i \in \{0,1\}^n, \\ i = 1,\ 2, \ldots, n-1,\ X \geq L,\end{gathered} \tag{5}$$

where the lower bounded vector

$$L = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix}.$$

The lower bounded vector can be transformed into the zero vector by using the standard transformations $X' = X - L$. The COWAL is as follows:

$$\begin{gathered}\text{Max } W^T Y,\ \text{s.t. } \mathcal{I}^T X' \leq 1 - \mathcal{I}^T L,\ GY \leq 0,\ y_i\mathcal{I} - X' - MZ_i \leq L,\ i = 1,\ 2, \ldots, n-1, \\ y_n\mathcal{I} - X' \leq L,\ \mathcal{I}^T Z_i \leq n - i,\ i = 1,\ 2, \ldots, n-1,\ Z_{i+1} \leq Z_i,\ i = 1,\ 2, \ldots, n-2,\ Z_i \in \{0,1\}^n, \\ i = 1,\ 2, \ldots, n-1,\ X' \geq \mathbf{0}.\end{gathered} \tag{6}$$

If $1 - \mathcal{I}^T L < 0$, then the COWAL has no feasible solution. If $1 - \mathcal{I}^T L = 0$, then $X' = \mathbf{0}$ is the unique optimal solution, so $X = L$. The following will discuss the case that $1 - \mathcal{I}^T L > 0$.

Consider the 3COWAL for the case of

$$l_1 + l_2 + l_3 \leq 1. \tag{7}$$

Two parameters (w_1, w_2, w_3) and (l_1, l_2, l_3) are considered in 3COWAL. This section discusses 3COWAL with

$$w_1 \geq w_2 \geq w_3. \tag{8}$$

There are six permutations of (l_1, l_2, l_3). First, consider the case of

$$l_1 \geq l_2 \geq l_3. \tag{9}$$

At optimality, the first constraint of model (6) becomes

$$x_1' + x_2' + x_3' = 1 - l_1 - l_2 - l_3. \tag{10}$$

There are three types (A, B, C) of (x_1', x_2', x_3') according to the number of zero components.
The number of zero components of first type A is two. The possible values of (x_1', x_2', x_3') are

$$(1 - l_1 - l_2 - l_3, 0,\ 0),\ (0, 1 - l_1 - l_2 - l_3, 0) \text{ and } (0,\ 0,\ 1 - l_1 - l_2 - l_3).$$

For the case of $(x_1', x_2', x_3') = (1 - l_1 - l_2 - l_3, 0,\ 0)$, we have

$$(x_1, x_2, x_3) = (1 - l_2 - l_3, l_2, l_3)$$

and

$$\begin{gathered}(y_1, y_2, y_3) = (1 - l_2 - l_3, l_2, l_3),\ (l_2, 1 - l_2 - l_3, l_3),\ (l_2, l_3, 1 - l_2 - l_3), \\ (1 - l_2 - l_3, l_3, l_2),\ (l_3, 1 - l_2 - l_3, l_2) \text{ or } (l_3, l_2, 1 - l_2 - l_3).\end{gathered}$$

Consider $(y_1, y_2, y_3) = (l_2, 1 - l_2 - l_3, l_3)$, so $l_2 \geq 1 - l_2 - l_3 \geq l_3$, implying $2l_2 + l_3 \geq 1$. From Label (7), it follows that $l_2 \geq l_1$, in contradiction to assumption (9). The same contradiction is also for $(y_1, y_2, y_3) = (l_2, l_3, 1 - l_2 - l_3)$, $(1 - l_2 - l_3, l_3, l_2)$, $(l_3, 1 - l_2 - l_3, l_2)$ and $(l_3, l_2, 1 - l_2 - l_3)$. Therefore, for $(x'_1, x'_2, x'_3) = (1 - l_1 - l_2 - l_3, 0, 0)$, the reasonable candidate for the optimal solution is $(y_1, y_2, y_3) = (1 - l_2 - l_3, l_2, l_3)$. For cases of $(0, 1 - l_1 - l_2 - l_3, 0)$ and $(0, 0, 1 - l_1 - l_2 - l_3)$, the reasonable candidates of (y_1, y_2, y_3) are shown in Table 1.

Table 1. The possible values of (x'_1, x'_2, x'_3), (x_1, x_2, x_3) and (y_1, y_2, y_3) for 3COWAL with $l_1 \geq l_2 \geq l_3$.

	(x'_1, x'_2, x'_3)	(x_1, x_2, x_3)	(y_1, y_2, y_3)
A-1	$(1 - l_1 - l_2 - l_3, 0, 0)$	$(1 - l_2 - l_3, l_2, l_3)$	$(1 - l_2 - l_3, l_2, l_3)$
A-2	$(0, 1 - l_1 - l_2 - l_3, 0)$	$(l_1, 1 - l_1 - l_3, l_3)$	$(1 - l_1 - l_3, l_1, l_3)$, $(l_1, 1 - l_1 - l_3, l_3)$
A-3	$(0, 0, 1 - l_1 - l_2 - l_3)$	$(l_1, l_2, 1 - l_1 - l_2)$	$(1 - l_1 - l_2, l_1, l_2)$, $(l_1, 1 - l_1 - l_2, l_2)$, $(l_1, l_2, 1 - l_1 - l_2)$
B1-1	$(\frac{1-2l_1-l_3}{2}, \frac{1-2l_2-l_3}{2}, 0)$	$(\frac{1-l_3}{2}, \frac{1-l_3}{2}, l_3)$	$(\frac{1-l_3}{2}, \frac{1-l_3}{2}, l_3)$
B1-2	$(\frac{1-2l_1-l_2}{2}, 0, \frac{1-l_2-2l_3}{2})$	$(\frac{1-l_2}{2}, l_2, \frac{1-l_2}{2})$	$(\frac{1-l_2}{2}, \frac{1-l_2}{2}, l_2)$
B1-3	$(0, \frac{1-l_1-2l_2}{2}, \frac{1-l_1-2l_3}{2})$	$(l_1, \frac{1-l_1}{2}, \frac{1-l_1}{2})$	$(l_1, \frac{1-l_1}{2}, \frac{1-l_1}{2})$
B2-1	$(l_3 - l_1, 1 - l_2 - 2l_3, 0)$	$(l_3, 1 - 2l_3, l_3)$	
B2-2	$(1 - l_1 - 2l_3, l_3 - l_2, 0)$	$(1 - 2l_3, l_3, l_3)$	
B2-3	$(l_2 - l_1, 0, 1 - 2l_2 - l_3)$	$(l_2, l_2, 1 - 2l_2)$	
B2-4	$(1 - l_1 - 2l_2, 0, l_2 - l_3)$	$(1 - 2l_2, l_2, l_2)$	$(1 - 2l_2, l_2, l_2)$
B2-5	$(0, l_1 - l_2, 1 - 2l_1 - l_3)$	$(l_1, l_1, 1 - 2l_1)$	$(l_1, l_1, 1 - 2l_1)$
B2-6	$(0, 1 - 2l_1 - l_2, l_1 - l_3)$	$(l_1, 1 - 2l_1, l_1)$	$(l_1, l_1, 1 - 2l_1)$
C	$(1/3 - l_1, 1/3 - l_2, 1/3 - l_3)$	$(1/3, 1/3, 1/3)$	$(1/3, 1/3, 1/3)$

In Table 1, there are six candidates for optimal solution (y_1, y_2, y_3) for type A. Among these six candidates, we will show that the largest objective function $F(Y) = w_1 y_1 + w_2 y_2 + w_3 y_3$ is that of $(y_1, y_2, y_3) = (1 - l_2 - l_3, l_2, l_3)$. Before we prove this result in detail, we present a well-known fact.

Theorem 1. *For $(x_1, x_2 \ldots, x_n)$, $(x'_1, x'_2 \ldots, x'_n)$, $s_k = \sum_{i=1}^{k} x_i$ and $s'_k = \sum_{i=1}^{k} x'_i$, $k = 1, 2, \ldots, n$, if $s_k \geq s'_k$, $k = 1, 2, \ldots, n$, then for all $(w_1, w_2, \ldots, w_n)$ with $w_k \geq w_{k+1}$, $k = 1, 2, \ldots, n-1$, we have*

$$\sum_{i=1}^{n} w_i x_i \geq \sum_{i=1}^{n} w_i x'_i.$$

Comparing the objective function value of $(y_1, y_2, y_3) = (1 - l_2 - l_3, l_2, l_3)$ with that of $(1 - l_1 - l_3, l_1, l_3)$, we get that

$$s_1 = 1 - l_2 - l_3 \geq s'_1 = 1 - l_1 - l_3,$$

$$s_2 = 1 - l_3 \geq s'_2 = 1 - l_3,$$

$$s_3 = 1 \geq s'_3 = 1.$$

It implies that the most favorable value of the objective function is that with $(1 - l_2 - l_3, l_2, l_3)$. A similar argument shows that $F(1 - l_2 - l_3, l_2, l_3)$ is larger than those of $(l_1, 1 - l_1 - l_3, l_3)$, $(1 - l_1 - l_2, l_1, l_2)$, $(l_1, 1 - l_1 - l_2, l_2)$ and $(l_1, l_2, 1 - l_1 - l_2)$. Therefore, the optimal solution for type A is $(1 - l_2 - l_3, l_2, l_3)$.

For the one zero component of (x'_1, x'_2, x'_3), the possible values are

$$(x'_1, x'_2, 0),\ (x'_1, 0, x'_3) \text{ and } (0, x'_2, x'_3).$$

At optimal, the possible values of x'_1, x'_2 and x'_3 with at least one $x'_i = 0$, $i = 1, 2, 3$ satisfy

$$x'_1 + l_1 = x'_2 + l_2,\ x'_1 + l_1 = x'_3 + l_3 \text{ or } x'_2 + l_2 = x'_3 + l_3.$$

We choose $x'_i + l_i = x'_j + l_j$ for type B1, and $x'_i + l_i = l_k$ or $x'_j + l_j = l_k$, $i \neq j \neq k$, $i, j, k = 1, 2, 3$, for type B2. A similar argument shows that all possible candidates for optimal solutions (y_1, y_2, y_3) are

$$(\frac{1-l_3}{2}, \frac{1-l_3}{2}, l_3),\ (\frac{1-l_2}{2}, \frac{1-l_2}{2}, l_2),\ (l_1, \frac{1-l_1}{2}, \frac{1-l_1}{2})$$

for type B1, and

$$(1-2l_2, l_2, l_2),\ (l_1, l_1, 1-2l_1)$$

for type B2. The largest objective function value is that with $(\frac{1-l_3}{2}, \frac{1-l_3}{2}, l_3)$ for type B1 and $(l_1, l_1, 1-2l_1)$ for type B2. Furthermore, $F(\frac{1-l_3}{2}, \frac{1-l_3}{2}, l_3) \geq F(l_1, l_1, 1-2l_1)$. Therefore, the optimal solution for type B is $(\frac{1-l_3}{2}, \frac{1-l_3}{2}, l_3)$.

Type C is the nonzero components. From Label (10), it follows that

$$(x'_1, x'_2, x'_3) = (1/3 - l_1, 1/3 - l_2, 1/3 - l_3),\ (x_1, x_2, x_3) = (1/3, 1/3, 1/3) \text{ and}$$
$$(y_1, y_2, y_3) = (1/3, 1/3, 1/3).$$

Therefore, there are six candidate optimal solutions for type A, five candidate optimal solutions for type B and one candidate optimal solution for type C. Detailed results of (x'_1, x'_2, x'_3), (x_1, x_2, x_3), (y_1, y_2, y_3), $F(y_1, y_2, y_3)$ and condition for 3COWAL with $l_1 \geq l_2 \geq l_3$ are presented in Table 2.

Table 2. The candidate optimal solutions of (x'_1, x'_2, x'_3), (x_1, x_2, x_3) and (y_1, y_2, y_3), $F(y_1, y_2, y_3)$ and condition for 3COWAL with $l_1 \geq l_2 \geq l_3$.

	(x'_1, x'_2, x'_3)	(x_1, x_2, x_3)	(y_1, y_2, y_3)	$F(y_1, y_2, y_3)$	**Condition**
A1	$(1-l_1-l_2-l_3, 0, 0)$	$(1-l_2-l_3,\ l_2,\ l_3)$	$(1-l_2-l_3,\ l_2,\ l_3)$	$w_1+l_2(-w_1+w_2)+l_3(-w_1+w_3)$	
A2	$(0, 1-l_1-l_2-l_3, 0)$	$(l_1, 1-l_1-l_3,\ l_3)$	$(1-l_1-l_3,\ l_1,\ l_3)$	$w_1+l_1(-w_1+w_2)+l_3(-w_1+w_3)$	$2l_1+l_3 \leq 1$
A3	$(0, 1-l_1-l_2-l_3, 0)$	$(l_1, 1-l_1-l_3,\ l_3)$	$(l_1, 1-l_1-l_3,\ l_3)$	$w_2+l_1(w_1-w_2)+l_3(-w_2+w_3)$	$2l_1+l_3 \geq 1$
A4	$(0,\ 0,\ 1-l_1-l_2-l_3)$	$(l_1,\ l_2, 1-l_1-l_2)$	$(1-l_1-l_2,\ l_1,\ l_2)$	$w_1+l_1(-w_1+w_2)+l_2(-w_1+w_3)$	$2l_1+l_2 \leq 1$
A5	$(0,\ 0,\ 1-l_1-l_2-l_3)$	$(l_1,\ l_2, 1-l_1-l_2)$	$(l_1, 1-l_1-l_2,\ l_2)$	$w_2+l_1(w_1-w_2)+l_2(-w_2+w_3)$	$2l_1+l_2 \geq 1, l_1+2l_2 \leq 1$
A6	$(0,\ 0,\ 1-l_1-l_2-l_3)$	$(l_1,\ l_2, 1-l_1-l_2)$	$(l_1,\ l_2, 1-l_1-l_2)$	$w_3+l_1(w_1-w_3)+l_2(w_2-w_3)$	$l_1+2l_2 \geq 1$
B1	$(\frac{1-2l_1-l_3}{2}, \frac{1-2l_2-l_3}{2}, 0)$	$(\frac{1-l_3}{2}, \frac{1-l_3}{2},\ l_3)$	$(\frac{1-l_3}{2}, \frac{1-l_3}{2},\ l_3)$	$\frac{1-w_3-l_3+3l_3w_3}{2}$	$l_3 \leq 1/3, 2l_1+l_3 \leq 1$
B2	$(\frac{1-2l_1-l_2}{2}, 0, \frac{1-l_2-2l_3}{2})$	$(\frac{1-l_2}{2},\ l_2, \frac{1-l_2}{2})$	$(\frac{1-l_2}{2}, \frac{1-l_2}{2},\ l_2)$	$\frac{1-w_3-l_2+3l_2w_3}{2}$	$l_2 \leq 1/3, 2l_1+l_2 \leq 1$
B3	$(0, \frac{1-l_1-2l_2}{2}, \frac{1-l_1-2l_3}{2})$	$(l_1, \frac{1-l_1}{2}, \frac{1-l_1}{2})$	$(l_1, \frac{1-l_1}{2}, \frac{1-l_1}{2})$	$\frac{1-w_1-l_1+3l_1w_1}{2}$	$l_1 \geq 1/3, l_1+2l_2 \leq 1$
B4	$(1-l_1-2l_2, 0, l_2-l_3)$	$(1-2l_2,\ l_2,\ l_2)$	$(1-2l_2,\ l_2,\ l_2)$	$w_1+l_2-3l_2w_1$	$l_2 \leq 1/3, l_1+2l_2 \leq 1$
B5	$(0,\ l_1-l_2,\ 1-2l_1-l_3)$	$(l_1,\ l_1, 1-2l_1)$	$(l_1,\ l_1, 1-2l_1)$	$w_3+l_1-3l_1w_3$	$l_1 \geq 1/3, 2l_1+l_3 \leq 1$
C	$(1/3-l_1,\ 1/3-l_2,\ 1/3-l_3)$	$(1/3,\ 1/3,\ 1/3)$	$(1/3,\ 1/3,\ 1/3)$	$1/3$	$l_1 \leq 1/3, l_2 \leq 1/3, l_3 \leq 1/3$

The largest objective function value is that of $\mathrm{A1}(1-l_2-l_3, l_2, l_3)$ for type A, $\mathrm{B1}(\frac{1-l_3}{2}, \frac{1-l_3}{2}, l_3)$ for type B and $\mathrm{C}(1/3, 1/3, 1/3)$ for type C. A similar argument shows that

$$F(\mathrm{A1}) \geq F(\mathrm{B1}) \text{ and } F(\mathrm{A1}) \geq F(\mathrm{C}).$$

Therefore, for the case of $w_1 \geq w_2 \geq w_3$, the optimal solution for $l_1 \geq l_2 \geq l_3$ is A1. Similarly, the optimal solutions of the remaining five permutations $l_1 \geq l_3 \geq l_2$, $l_2 \geq l_1 \geq l_3$, $l_2 \geq l_3 \geq l_1$, $l_3 \geq l_1 \geq l_2$ and $l_3 \geq l_2 \geq l_1$ can be derived. Detailed optimal solutions are described as follows.

Theorem 2. *For $w_1 \geq w_2 \geq w_3$ and $l_1 + l_2 + l_3 \leq 1$, the optimal solution of 3COWAL is as follows:*

$$(y_1^*, y_2^*, y_3^*) = \begin{cases} (1-l_2-l_3, l_2, l_3), \text{ if } l_1 \geq l_2 \geq l_3 \\ (1-l_2-l_3, l_3, l_2), \text{ if } l_1 \geq l_3 \geq l_2 \\ (1-l_1-l_3, l_1, l_3), \text{ if } l_2 \geq l_1 \geq l_3 \\ (1-l_1-l_3, l_3, l_1), \text{ if } l_2 \geq l_3 \geq l_1 \\ (1-l_1-l_2, l_1, l_2), \text{ if } l_3 \geq l_1 \geq l_2 \\ (1-l_1-l_2, l_2, l_1), \text{ if } l_3 \geq l_2 \geq l_1. \end{cases} \tag{11}$$

4. Maximizing Three-Dimensional Constrained OWA Aggregation Problem with Lower Bounded Variables for $l_1 \geq l_2 \geq l_3$

This section considers the optimal solution behaviors for 3COWAL with $l_1 \geq l_2 \geq l_3$. The main result is described as follows.

Theorem 3. *For $l_1 \geq l_2 \geq l_3$ and $l_1 + l_2 + l_3 \leq 1$, the optimal solution Y^* of 3COWAL is as follows.*

(1) *For $w_1 \geq w_2 \geq w_3$ or $w_1 \geq w_3 \geq w_2$, the optimal solution is $A1(1-l_2-l_3, l_2, l_3)$.*

(2) *For $w_2 \geq w_1 \geq w_3$, the optimal solution Y^* is*

$$if\ 2l_1 + l_3 \geq 1,\ then\ Y^* = A3(l_1, 1-l_1-l_3, l_3)\ else\ Y^* = B1\ (\frac{1-l_3}{2}, \frac{1-l_3}{2}, l_3).$$

(3) *For $w_2 \geq w_3 \geq w_1$, the optimal solution Y^* is*

$$if\ 2l_1 + l_3 \geq 1,\ then\ Y^* = A3(l_1, 1-l_1-l_3, l_3).$$

$$else\ if\ w_3 \leq 1/3\ then\ Y^* = B1\ (\frac{1-l_3}{2}, \frac{1-l_3}{2}, l_3);$$

$$else\ if\ l_1 \geq 1/3,\ then\ Y^* = B5(l_1, l_1, 1-2l_1)\ else\ Y^* = C(1/3, 1/3, 1/3).$$

(4) *For $w_3 \geq w_1 \geq w_2$, the optimal solution Y^* is*

$$if\ l_1 + 2l_2 \geq 1,\ then\ Y^* = A6(l_1, l_2, 1-l_1-l_2).$$

$$else\ if\ w_1 \geq 1/3\ then\ Y^* = B4\ (1-2l_2, l_2, l_2);$$

$$else\ if\ l_1 \geq 1/3\ then\ Y^* = B3(l_1, \frac{1-l_1}{2}, \frac{1-l_1}{2})\ else\ Y^* = C(1/3, 1/3, 1/3).$$

(5) *For $w_3 \geq w_2 \geq w_1$, the optimal solution Y^* is*

$$if\ l_1 + 2l_2 \geq 1,\ then\ Y^* = A6(l_1, l_2, 1-l_1-l_2).$$

$$else\ if\ l_1 \geq 1/3,\ then\ Y^* = B3(l_1, \frac{1-l_1}{2}, \frac{1-l_1}{2})\ else\ Y^* = C(1/3, 1/3, 1/3).$$

Proof. There are six permutations of (w_1, w_2, w_3). From Theorem 2, for $w_1 \geq w_2 \geq w_3$, the optimal solution is A1$(1 - l_2 - l_3, l_2, l_3)$. Similarly, the optimal solution also is A1 for $w_1 \geq w_3 \geq w_2$.

We now consider the case that $w_2 \geq w_1 \geq w_3$. From Table 2, all of the twelve candidates are divided into two categories to obtain optimal solution: (I) A1, A3, A4, A5, A6 and C; (II) A2, B1, B2, B3, B4 and B5. We will show that the largest objective function value is that of A3 $(l_1, 1 - l_1 - l_3, l_3)$ for category I.

Comparing the objective function value of A3 with that of A1, from Theorem 1, since $w_2 \geq w_1 \geq w_3$, we have to compare $s_1 = y_2$, $s_2 = y_1 + y_2$ and $s_3 = y_1 + y_2 + y_3$ with those of $s'_1 = y'_2$, $s'_2 = y'_1 + y'_2$ and $s'_3 = y'_1 + y'_2 + y'_3$. From

$$s_1 = 1 - l_1 - l_3 \geq s'_1 = l_2,$$

$$s_2 = 1 - l_3 \geq s'_2 = 1 - l_3,$$

$$s_3 = 1 \geq s'_3 = 1,$$

the comparison results imply that A3 is superior to A1. Similarly, $F(\text{A3})$ is larger than those of A2, A5, A6 and C. Therefore, the optimal solution for category I is A3. A similar argument shows that the optimal solution for category II is B1 $(\frac{1-l_3}{2}, \frac{1-l_3}{2}, l_3)$. From the conditions of A3 and B1 displayed in Table 2, the optimal solution for $w_2 \geq w_1 \geq w_3$ is A3 for $2l_1 + l_3 \geq 1$, and B1 for $2l_1 + l_3 \leq 1$.

Similarly, for $w_2 \geq w_3 \geq w_1$, $w_3 \geq w_1 \geq w_2$ and $w_3 \geq w_2 \geq w_1$, the optimal solutions can be derived. □

We next present two numerical experiments to evaluate the optimal solutions of 3COWAL with $l_1 \geq l_2 \geq l_3$.

Tables 3 and 4 entries correspond to a pair (S, W) and give the number of different instances of $(l_1, l_2, l_3, w_1, w_2, w_3)$ satisfying type of candidate solution (S) and weight (W). The adopted measure is the number of instances. From Table 2, we adopt twelve types of candidate solutions and six permutations of weight. More precisely, $S \in$ {A1, A2, A3, A4, A5, A6, B1, B2, B3, B4, B5, C }, $W \in \{ w_1 > w_2 > w_3,\ w_1 > w_3 > w_2, w_2 > w_1 > w_3,\ w_2 > w_3 > w_1,\ w_3 > w_1 > w_2,\ w_3 > w_2 > w_1\}$, $l_i \in \{-lb, -0.9, -0.8, \ldots, lb\}$ and $w_i \in \{0, 0.1, 0.2, \ldots, 1\}$, $i = 1, 2, 3$.

The value of bound lb is lb =1 for Table 3 and lb =2 Table 4. For each weight, the instances $(l_1, l_2, l_3, w_1, w_2, w_3)$ of 3COWAL are 8744 for $lb = 1$ and 57,464 for $lb = 2$. The total instances of 3COWAL are 397,248. An examination of Tables 3 and 4 reveals that the largest number of instances is A1 for $w_1 > w_2 > w_3$ and $w_1 > w_3 > w_2$, B1 for $w_2 > w_1 > w_3$ and $w_2 > w_3 > w_1$, and B3 for $w_3 > w_1 > w_2$ and $w_3 > w_2 > w_1$. Among all the instances of 3COWAL, the zero number is A2, A4, A5 and B2. Therefore, A1, B1 and B3 are superior in the number of the instances, while A2, A4, A5 and B2 are inferior ones for 3COWAL with $l_1 \geq l_2 \geq l_3$.

Table 3. The number of instances satisfying solution type and weight for 3COWAL with $-1 \le l_3 < l_2 < l_1 \le 1$.

	(y_1, y_2, y_3)	$w_1 > w_2 > w_3$	$w_1 > w_3 > w_2$	$w_2 > w_1 > w_3$	$w_2 > w_3 > w_1$	$w_3 > w_1 > w_2$	$w_3 > w_2 > w_1$	**Total**
A1	$(1 - l_2 - l_3,\ l_2,\ l_3)$	8744	8744	0	0	0	0	17,488
A2	$(1 - l_1 - l_3,\ l_1,\ l_3)$	0	0	0	0	0	0	0
A3	$(l_1, 1 - l_1 - l_3,\ l_3)$	0	0	1632	1632	0	0	3264
A4	$(1 - l_1 - l_2,\ l_1,\ l_2)$	0	0	0	0	0	0	0
A5	$(l_1, 1 - l_1 - l_2,\ l_2)$	0	0	0	0	0	0	0
A6	$(l_1,\ l_2, 1 - l_1 -\ l_2)$	0	0	0	0	1920	1920	3840
B1	$(\frac{1-l_3}{2}, \frac{1-l_3}{2},\ l_3)$	0	0	7112	5474	0	0	12,586
B2	$(\frac{1-l_2}{2}, \frac{1-l_2}{2},\ l_2)$	0	0	0	0	0	0	0
B3	$(l_1, \frac{1-l_1}{2}, \frac{1-l_1}{2})$	0	0	0	0	3026	3912	6938
B4	$(1 - 2l_2,\ l_2,\ l_2)$	0	0	0	0	1614	0	1614
B5	$(l_1,\ l_1, 1 - 2l_1)$	0	0	0	910	0	0	910
C	$(1/3,\ 1/3,\ 1/3)$	0	0	0	728	2184	2912	5824

Table 4. The number of instances satisfying solution type and weight for 3COWAL with $-2 \le l_3 < l_2 < l_1 \le 2$.

	(y_1, y_2, y_3)	$w_1 > w_2 > w_3$	$w_1 > w_3 > w_2$	$w_2 > w_1 > w_3$	$w_2 > w_3 > w_1$	$w_3 > w_1 > w_2$	$w_3 > w_2 > w_1$	**Total**
A1	$(1 - l_2 - l_3,\ l_2,\ l_3)$	57,464	57,464	0	0	0	0	114,928
A2	$(1 - l_1 - l_3,\ l_1,\ l_3)$	0	0	0	0	0	0	0
A3	$(l_1, 1 - l_1 - l_3,\ l_3)$	0	0	19,352	19,352	0	0	38,704
A4	$(1 - l_1 - l_2,\ l_1,\ l_2)$	0	0	0	0	0	0	0
A5	$(l_1, 1 - l_1 - l_2,\ l_2)$	0	0	0	0	0	0	0
A6	$(l_1,\ l_2, 1 - l_1 - l_2)$	0	0	0	0	15,640	17,008	32,648
B1	$(\frac{1-l_3}{2}, \frac{1-l_3}{2},\ l_3)$	0	0	38,112	29,004	0	0	67,116
B2	$(\frac{1-l_2}{2}, \frac{1-l_2}{2},\ l_2)$	0	0	0	0	0	0	0
B3	$(l_1, \frac{1-l_1}{2}, \frac{1-l_1}{2})$	0	0	0	0	19,566	28,312	47,878
B4	$(1 - 2l_2,\ l_2,\ l_2)$	0	0	0	0	10,114	0	10,114
B5	$(l_1,\ l_1, 1 - 2l_1)$	0	0	0	5060	0	0	5060
C	$(1/3,\ 1/3,\ 1/3)$	0	0	0	4048	12,144	12,144	28,336

5. Minimizing Three-Dimensional Constrained OWA Aggregation Problem with Upper Bounded Variables

Consider the minimizing COWAU problem described as follows:

$$\begin{gathered}\text{Min } W^T Y,\ \text{s.t. } \mathcal{I}^T X \geq 1,\ GY \leq 0,\ y_i\mathcal{I} - X - MZ_i \leq 0,\ i = 1,2,\ldots,n-1,\ y_n\mathcal{I} - X \leq 0,\\ \mathcal{I}^T Z_i \leq n-i,\ i = 1,2,\ldots,n-1,\ Z_{i+1} \leq Z_i,\ i = 1,2,\ldots,n-2,\ Z_i \in \{0,1\}^n,\\ i = 1,2,\ldots,n-1,\ X \leq U\end{gathered} \tag{12}$$

where the column vector

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

Using the standard transformations $X' = U - X$, these lead to the following model

$$\begin{gathered}\text{Min } W^T Y,\ \text{s.t. } \mathcal{I}^T X' \leq \mathcal{I}^T U - 1,\ GY \leq 0,\ y_i\mathcal{I} + X' - MZ_i \leq U,\ i = 1,2,\ldots,n-1,\\ y_n\mathcal{I} + X' \leq U,\ \mathcal{I}^T Z_i \leq n-i,\ i = 1,2,\ldots,n-1,\ Z_{i+1} \leq Z_i,\ i = 1,2,\ldots,n-2,\\ Z_i \in \{0,1\}^n,\ i = 1,2,\ldots,n-1,\ X' \geq \mathbf{0}.\end{gathered} \tag{13}$$

If $\mathcal{I}^T U - 1 < 0$, we conclude that the COWAU has no feasible solutions. If $\mathcal{I}^T U - 1 = 0$, then $X' = \mathbf{0}$ is the unique optimal solution, so $X = U$.

This section considers 3COWAU for

$$\mathcal{I}^T U - 1 = u_1 + u_2 + u_3 \geq 1. \tag{14}$$

Similar analyses to Sections 3 and 4 can be derived. The results are described as follows.

At optimality, the first constraint of the model (13) becomes

$$x'_1 + x'_2 + x'_3 = u_1 + u_2 + u_3 - 1. \tag{15}$$

There are three types (A′, B′, C′) of (x'_1, x'_2, x'_3) according to the number of zero components.

For 3COWAU with

$$u_1 \geq u_2 \geq u_3, \tag{16}$$

there are six candidate optimal solutions for type A′, seven candidate optimal solutions for type B′ and one candidate optimal solution for type C′. Detailed results of (x'_1, x'_2, x'_3), (x_1, x_2, x_3), (y_1, y_2, y_3), $F(y_1, y_2, y_3)$ and condition are presented in Table 5.

Table 5. The candidate optimal solutions of (x'_1, x'_2, x'_3), (x_1, x_2, x_3) and (y_1, y_2, y_3), $F(y_1, y_2, y_3)$ and condition for 3COWAU with $u_1 \geq u_2 \geq u_3$.

	(x'_1, x'_2, x'_3)	(x_1, x_2, x_3)	(y_1, y_2, y_3)	$F(y_1, y_2, y_3)$	**Condition**
A′1	$(u_1 + u_2 + u_3 - 1,\ 0,\ 0)$	$(1 - u_2 - u_3,\ u_2,\ u_3)$	$(1 - u_2 - u_3,\ u_2,\ u_3)$	$w_1 + u_2(-w_1 + w_2) + u_3(-w_1 + w_3)$	$2u_2 + u_3 \leq 1$
A′2	$(u_1 + u_2 + u_3 - 1,\ 0,\ 0)$	$(1 - u_2 - u_3,\ u_2,\ u_3)$	$(u_2, 1 - u_2 - u_3, u_3)$	$w_2 + u_2(w_1 - w_2) + u_3(-w_2 + w_3)$	$2u_2 + u_3 \geq 1$, $u_2 + 2u_3 \leq 1$
A′3	$(u_1 + u_2 + u_3 - 1,\ 0,\ 0)$	$(1 - u_2 - u_3,\ u_2,\ u_3)$	$(u_2, u_3, 1 - u_2 - u_3)$	$w_3 + u_2(w_1 - w_3) + u_3(w_2 - w_3)$	$u_2 + 2u_3 \geq 1 u_2 \geq 1/3$
A′4	$(0,\ u_1 + u_2 + u_3 - 1,\ 0)$	$(u_1, 1 - u_1 - u_3,\ u_3)$	$(u_1, 1 - u_1 - u_3, u_3)$	$w_2 + u_1(w_1 - w_2) + u_3(-w_2 + w_3)$	$u_1 + 2u_3 \leq 1$,$u_3 \leq 1/3$
A′5	$(0,\ u_1 + u_2 + u_3 - 1,\ 0)$	$(u_1, 1 - u_1 - u_3,\ u_3)$	$(u_1,\ u_3, 1 - u_1 - u_3)$	$w_3 + u_1(w_1 - w_3) + u_3(w_2 - w_3)$	$u_1 + 2u_3 \geq 1$
A′6	$(0,\ 0,\ u_1 + u_2 + u_3 - 1)$	$(u_1, u_2, 1 - u_1 - u_2)$	$(u_1,\ u_2, 1 - u_1 - u_2)$	$w_3 + u_1(w_1 - w_3) + u_2(w_2 - w_3)$	
B′1	$(\frac{2u_1+u_3-1}{2},\ \frac{2u_2+u_3-1}{2}, 0)$	$(\frac{1-u_3}{2}, \frac{1-u_3}{2},\ u_3)$	$(\frac{1-u_3}{2}, \frac{1-u_3}{2},\ u_3)$	$\frac{1-w_3-u_3+3u_3w_3}{2}$	$u_3 \leq 1/3$,$2u_2 + u_3 \geq 1$
B′2	$(\frac{2u_1+u_3-1}{2}, \frac{2u_2+u_3-1}{2}, 0)$	$(\frac{1-u_3}{2}, \frac{1-u_3}{2},\ u_3)$	$(u_3, \frac{1-u_3}{2}, \frac{1-u_3}{2})$	$\frac{1-w_1-u_3+3u_3w_1}{2}$	$u_3 \geq 1/3$
B′3	$(\frac{2u_1+u_2-1}{2}, 0, \frac{2u_3+u_2-1}{2})$	$(\frac{1-u_2}{2},\ u_2, \frac{1-u_2}{2})$	$(u_2, \frac{1-u_2}{2}, \frac{1-u_2}{2})$	$\frac{1-w_1-u_2+3u_2w_1}{2}$	$u_2 \geq 1/3$,$u_2 + 2u_3 \geq 1$
B′4	$(0, \frac{2u_2+u_1-1}{2}, \frac{2u_3+u_1-1}{2})$	$(u_1, \frac{1-u_1}{2}, \frac{1-u_1}{2})$	$(u_1, \frac{1-u_1}{2}, \frac{1-u_1}{2})$	$\frac{1-w_1-u_1+3u_1w_1}{2}$	$u_1 + 2u_3 \geq 1$
B′5	$(u_1 - u_3, u_2 + 2u_3 - 1,\ 0)$	$(u_3, 1 - 2u_3,\ u_3)$	$(u_3,\ u_3, 1 - 2u_3)$	$w_3 + u_3 - 3u_3w_3$	$u_3 \geq 1/3$
B′6	$(u_1 - u_3, u_2 + 2u_3 - 1,\ 0)$	$(u_3, 1 - 2u_3,\ u_3)$	$(1 - 2u_3,\ u_3,\ u_3)$	$w_1 + u_3 - 3u_3w_1$	$u_3 \leq 1/3$, $u_2 + 2u_3 \geq 1$
B′7	$(u_1 - u_2, 0, 2u_2 + u_3 - 1)$	$(u_2, u_2,\ 1 - 2u_2)$	$(u_2,\ u_2, 1 - 2u_2)$	$w_3 + u_2 - 3u_2w_3$	$u_2 \geq 1/3$,$2u_2 + u_3 \geq 1$
C′	$(u_1 - 1/3, u_2 - 1/3, u_3 - 1/3)$	$(1/3,\ 1/3,\ 1/3)$	$(1/3,\ 1/3,\ 1/3)$	$1/3$	$u_1 \geq 1/3$,$u_2 \geq 1/3$,$u_3 \geq 1/3$

Theorem 4. *For 3COWAU with $u_1 \geq u_2 \geq u_3$ and $w_1 \geq w_2 \geq w_3$, the smallest objective function value is that of $(y_1, y_2, y_3) = A'1(1 - u_2 - u_3, u_2, u_3)$ for type A', $B'1(\frac{1-u_3}{2}, \frac{1-u_3}{2}, u_3)$ for type B' and $C'(1/3, 1/3, 1/3)$ for type C'.*

For 3COWAU with $u_1 \geq u_2 \geq u_3$, the optimal solutions of different permutations of weight are described as follows.

Theorem 5. *For 3COWAU with $u_1 \geq u_2 \geq u_3$ and $u_1 + u_2 + u_3 \geq 1$, the optimal solution Y^* is described as follows.*

(1) For $w_1 \geq w_2 \geq w_3$, the optimal solution Y^* is

$$\begin{gathered} if\ u_3 \geq 1/3,\ then\ Y^* = C'(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3})\ else\ if\ 2u_2 + u_3 \leq 1 \\ then\ Y^* = A'1(1 - u_2 - u_3, u_2, u_3)\ else\ Y^* = B'1(\tfrac{1-u_3}{2}, \tfrac{1-u_3}{2}, u_3). \end{gathered}$$

(2) For $w_1 \geq w_3 \geq w_2$, the optimal solution Y^* is

$$if\ 2u_2 + u_3 \leq 1,\ then\ Y^* = A'1\ (1 - u_2 - u_3, u_2, u_3).$$

$$else\ if\ w_3 \geq 1/3,\ then\ Y^* = B'7\ (u_2, u_2, 1 - 2u_2);$$

$$else\ if\ u_3 \leq 1/3,\ then\ Y^* = B'1(\frac{1-u_3}{2}, \frac{1-u_3}{2}, u_3)\ else\ Y^* = C'(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}).$$

(3) For $w_2 \geq w_1 \geq w_3$, the optimal solution Y^* is

$$if\ u_1 + 2u_3 \leq 1,\ then\ Y^* = A'4\ (u_1, 1 - u_1 - u_3, u_3).$$

$$else\ if\ w_1 \leq 1/3,\ then\ Y^* = B'4\ (u_1, \frac{1-u_1}{2}, \frac{1-u_1}{2});$$

$$else\ if\ u_3 \leq 1/3,\ then\ Y^* = B'6(1 - 2u_3, u_3, u_3)\ else\ Y^* = C'(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}).$$

(4) For $w_2 \geq w_3 \geq w_1$, the optimal solution Y^* is

$$if\ u_1 + 2u_3 \leq 1,\ then\ Y^* = A'4(u_1, 1 - u_1 - u_3, u_3)\ else\ Y^* = B'4(u_1, \frac{1-u_1}{2}, \frac{1-u_1}{2}).$$

(5) For $w_3 \geq w_1 \geq w_2$ and $w_3 \geq w_2 \geq w_1$, the optimal solution is $A'6(u_1, u_2, 1 - u_1 - u_2)$.

To compare the optimal solution behaviors of maximizing 3COWAL with those of minimizing 3COWAU, the performance is shown in Figure 1. The first bar corresponds to the number of optimal solutions of maximizing 3COWAL and second to the minimizing 3COWAU, where the weight type W$[i]$ denotes the ith component of $W = \{w_1 > w_2 > w_3,\ w_1 > w_3 > w_2, w_2 > w_1 > w_3,\ w_2 > w_3 > w_1,\ w_3 > w_1 > w_2,\ w_3 > w_2 > w_1\}$. The number of optimal solutions of W$[i]$, $i = 1,\ 2,\ \ldots, 6$, for maximizing 3COWAL is the same as that of W$[6\ - i]$ for minimizing 3COWAU. Therefore, the numbers of optimal solutions for maximizing 3COWAL are same as those of minimizing 3COWAU but in reverse order. The correspondence between the optimal solution of these two mathematical models is worthy of future research.

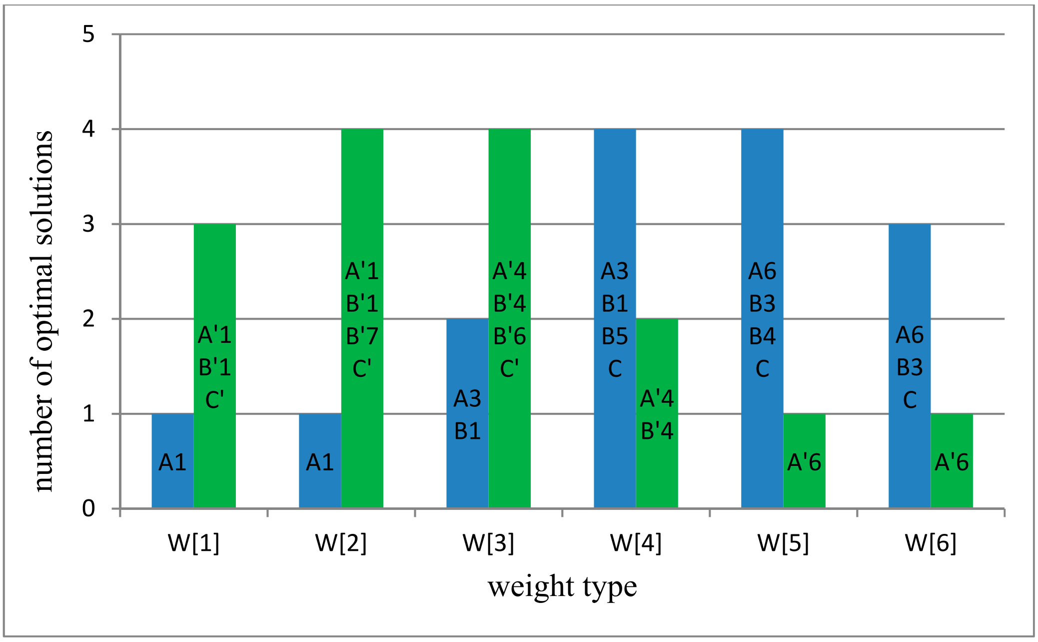

Figure 1. The number of optimal solutions for maximizing 3COWAL and minimizing 3COWAU.

We perform two numerical experiments to evaluate the optimal solution behaviors of 3COWAU with $u_1 \geq u_2 \geq u_3$.

Tables 6 and 7 give the number of different instances $(u_1, u_2, u_3, w_1, w_2, w_3)$ satisfying type of candidate solution (S) and weight (W). More precisely,

$$S \in \{\mathrm{A}'1,\ \mathrm{A}'2,\ \ldots,\ \mathrm{A}'6,\ \mathrm{B}'1, \mathrm{B}'2,\ \ldots,\ \mathrm{B}'7,\ \mathrm{C}'\},$$
$$W \in \{w_1 > w_2 > w_3,\ w_1 > w_3 > w_2,\ w_2 > w_1 > w_3, w_2 > w_3 > w_1,\ w_3 > w_1 > w_2,\ w_3 > w_2 > w_1\},$$
$$u_i \in \{-ub, -0.9, -0.8, \ldots, ub\} \text{ and } w_i \in \{0,\ 0.1,\ 0.2, \ldots,\ 1\} i = 1,\ 2,\ 3.$$

The value of bound ub is $ub = 1$ for Table 6 and $ub = 2$ Table 7. For each weight, the number of the instances of 3COWAU is 8744 for $ub = 1$ and 57,464 for $ub = 2$. The total instances of 3COWAU are 397,248. From Tables 6 and 7, the largest number of instances is $\mathrm{B}'1(\frac{1-u_3}{2}, \frac{1-u_3}{2}, u_3)$ for $w_1 > w_2 > w_3$, $\mathrm{A}'1$ $(1 - u_2 - u_3, u_2, u_3)$ for $w_1 > w_3 > w_2$, $\mathrm{A}'4$ $(u_1, 1 - u_1 - u_3, u_3)$ for $w_2 > w_1 > w_3$ and $w_2 > w_3 > w_1$, and $\mathrm{A}'6(u_1, u_2, 1 - u_1 - u_2)$ for $w_3 > w_1 > w_2$ and $w_3 > w_2 > w_1$. We also notice that the number of instances is zero for A′2, A′3, A′5, B′2, B′3 and B′5. Therefore, A′1, A′4, A′6 and B′1 are the best candidates, while A′2, A′3, A′5, B′2, B′3 and B′5 are inferior ones for 3COWAU with $u_1 \geq u_2 \geq u_3$.

Table 6. The number of instances satisfying solution type and weight for 3COWAU with $-1 \leq u_3 \leq u_2 \leq u_1 \leq 1$.

	(y_1, y_2, y_3)	$w_1 > w_2 > w_3$	$w_1 > w_3 > w_2$	$w_2 > w_1 > w_3$	$w_2 > w_3 > w_1$	$w_3 > w_1 > w_2$	$w_3 > w_2 > w_1$	**Total**
A′1	$(1-u_2-u_3, u_2, u_3)$	624	624	0	0	0	0	1248
A′2	$(u_2, 1-u_2-u_3, u_3)$	0	0	0	0	0	0	0
A′3	$(u_2, u_3, 1-u_2-u_3)$	0	0	0	0	0	0	0
A′4	$(u_1, 1-u_1-u_3, u_3)$	0	0	912	888	0	0	1800
A′5	$(u_1, u_3, 1-u_1-u_3)$	0	0	0	0	0	0	0
A′6	$(u_1, u_2, 1-u_1-u_2)$	0	0	0	0	1632	1632	3264
B′1	$(\frac{1-u_3}{2}, \frac{1-u_3}{2}, u_3)$	728	546	0	0	0	0	1274
B′2	$(u_3, \frac{1-u_3}{2}, \frac{1-u_3}{2})$	0	0	0	0	0	0	0
B′3	$(u_2, \frac{1-u_2}{2}, \frac{1-u_2}{2})$	0	0	0	0	0	0	0
B′4	$(u_1, \frac{1-u_1}{2}, \frac{1-u_1}{2})$	0	0	558	744	0	0	1302
B′5	$(u_3, u_3, 1-2u_3)$	0	0	0	0	0	0	0
B′6	$(1-2u_3, u_3, u_3)$	0	0	92	0	0	0	92
B′7	$(u_2, u_2, 1-2u_2)$	0	252	0	0	0	0	252
C′	(1/3, 1/3, 1/3)	280	210	70	0	0	0	560

Table 7. The number of instances satisfying solution type and weight for 3COWAU with $-2 \leq u_3 \leq u_2 \leq u_1 \leq 2$.

	(y_1, y_2, y_3)	$w_1 > w_2 > w_3$	$w_1 > w_3 > w_2$	$w_2 > w_1 > w_3$	$w_2 > w_3 > w_1$	$w_3 > w_1 > w_2$	$w_3 > w_2 > w_1$	**Total**
A′1	$(1-u_2-u_3, u_2, u_3)$	9184	9184	0	0	0	0	18,368
A′2	$(u_2, 1-u_2-u_3, u_3)$	0	0	0	0	0	0	0
A′3	$(u_2, u_3, 1-u_2-u_3)$	0	0	0	0	0	0	0
A′4	$(u_1, 1-u_1-u_3, u_3)$	0	0	13,472	13,328	0	0	26,800
A′5	$(u_1, u_3, 1-u_1-u_3)$	0	0	0	0	0	0	0
A′6	$(u_1, u_2, 1-u_1-u_2)$	0	0	0	0	26,352	26,352	52,704
B′1	$(\frac{1-u_3}{2}, \frac{1-u_3}{2}, u_3)$	11,728	8796	0	0	0	0	20,524
B′2	$(u_3, \frac{1-u_3}{2}, \frac{1-u_3}{2})$	0	0	0	0	0	0	0
B′3	$(u_2, \frac{1-u_2}{2}, \frac{1-u_2}{2})$	0	0	0	0	0	0	0
B′4	$(u_1, \frac{1-u_1}{2}, \frac{1-u_1}{2})$	0	0	9768	13,024	0	0	22,792
B′5	$(u_3, u_3, 1-2u_3)$	0	0	0	0	0	0	0
B′6	$(1-2u_3, u_3, u_3)$	0	0	1752	0	0	0	1752
B′7	$(u_2, u_2, 1-2u_2)$	0	4292	0	0	0	0	4292
C′	(1/3, 1/3, 1/3)	5440	4080	1360	0	0	0	10,880

6. Conclusions

This paper presents the optimal solutions for both maximizing 3COWAL and minimizing 3COWAU. For maximizing 3COWAL with $l_1 \geq l_2 \geq l_3$, there are six candidate optimal solutions for type A, five candidate optimal solutions for type B and one candidate optimal solution for type C. Theoretically and empirically, the largest number of instances is A1 $(1 - l_2 - l_3, l_2, l_3)$ for $w_1 > w_2 > w_3$ and $w_1 > w_3 > w_2$, B1 $(\frac{1-l_3}{2}, \frac{1-l_3}{2}, l_3)$ for $w_2 > w_1 > w_3$ and $w_2 > w_3 > w_1$, and B3 $(l_1, \frac{1-l_1}{2}, \frac{1-l_1}{2})$ for $w_3 > w_1 > w_2$ and $w_3 > w_2 > w_1$. For minimizing 3COWAU with $u_1 \geq u_2 \geq u_3$, there are six candidate optimal solutions for type A′, seven candidate optimal solutions for type B′ and one candidate optimal solution for type C′. The largest number of instances is B′1 $(\frac{1-u_3}{2}, \frac{1-u_3}{2}, u_3)$ for $w_1 > w_2 > w_3$, A′1 $(1 - u_2 - u_3, u_2, u_3)$ for $w_1 > w_3 > w_2$, A′4 $(u_1, 1 - u_1 - u_3, u_3)$ for $w_2 > w_1 > w_3$ and $w_2 > w_3 > w_1$, and A′6 $(u_1, u_2, 1 - u_1 - u_2)$ for $w_3 > w_1 > w_2$ and $w_3 > w_2 > w_1$. Therefore, the best candidate optimal solutions are A1, B1 and B3 for maximizing 3COWAL with $l_1 \geq l_2 \geq l_3$, and A′1, A′4, A′6 and B′1 for minimizing 3COWAU with $u_1 \geq u_2 \geq u_3$.

Extending the analysis to high dimensions is worthy of future research in addition to analysis of the correspondence between the optimal solution of maximizing 3COWAL and minimizing 3COWAU. Thus, the analysis of maximizing COWAL and minimizing COWAU is a subject of considerable ongoing research.

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References

1. Yager, R.R. On ordered weighted averaging aggregation operators in multicriteria decision making. *IEEE Trans. Syst. Man Cybern.* **1988**, *18*, 183–190. [CrossRef]
2. Chaji, A.; Fukuyama, H.; Shiraz, R.K. Selecting a model for generating OWA operator weights in MAGDM problems by maximum entropy membership function. *Comp. Ind. Eng.* **2018**, *124*, 370–378. [CrossRef]
3. De Miguel, L.; Sesma-Sara, M.; Elkano, M.; Asiain, M.; Bustince, H. An algorithm for group decision making using n-dimensional fuzzy sets, admissible orders and OWA operators. *Inf. Fusion* **2017**, *37*, 126–131. [CrossRef]
4. Liu, P.; Wang, Y. Interval neutrosophic prioritized OWA operator and its application to multiple attribute decision making. *J. Syst. Sci. Complex.* **2016**, *29*, 681–697. [CrossRef]
5. Merigó, J.M.; Gil-Lafuente, A.M.; Zhou, L.G.; Chen, H.Y. Induced and linguistic generalized aggregation operators and their application in linguistic group decision making. *Group Decis. Negot.* **2012**, *21*, 531–549. [CrossRef]
6. Zhou, L.; Chen, H.; Liu, J. Generalized multiple averaging operators and their applications to group decision making. *Group Decis. Negot.* **2013**, *22*, 331–358. [CrossRef]
7. Vigier, H.P.; Scherger, V.; Terceño, A. An application of OWA operators in fuzzy business diagnosis. *Appl. Soft Comput.* **2017**, *54*, 440–448. [CrossRef]
8. Yager, R.R. On the analytic representation of the Leximin ordering and its application to flexible constraint propagation. *Eur. J. Oper. Res.* **1997**, *102*, 176–192. [CrossRef]
9. Garg, B.; Garg, R. Enhanced accuracy of fuzzy time series model using ordered weighted aggregation. *Appl. Soft Comput.* **2016**, *48*, 265–280. [CrossRef]
10. Mohammed, E.A.; Naugler, C.T.; Far, B.H. Breast tumor classification using a new OWA operator. *Exp. Syst. Appl.* **2016**, *61*, 302–313. [CrossRef]
11. Yager, R.R.; Beliakov, G. OWA operators in regression problems. *IEEE Trans. Fuzzy Syst.* **2010**, *18*, 106–113. [CrossRef]
12. Ogryczak, W.; Śliwiński, T. On solving linear programs with the ordered weighted averaging objective. *Eur. J. Oper. Res.* **2003**, *148*, 80–91. [CrossRef]
13. Ameri, F.; Zoej, M.J.V.; Mokhtarzade, M. Multi-Criteria, Graph-Based Road Centerline Vectorization Using Ordered Weighted Averaging Operators. *Photogramm. Eng. Remote Sens.* **2016**, *82*, 107–120. [CrossRef]

14. Chen, S.M.; Kuo, L.W. Autocratic decision making using group recommendations based on interval type-2 fuzzy sets, enhanced Karnik–Mendel algorithms, and the ordered weighted aggregation operator. *Inf. Sci.* **2017**, *412*, 174–193. [CrossRef]
15. Carlsson, C.; Fullér, R. Maximal Entropy and Minimal Variability OWA Operator Weights: A Short Survey of Recent Developments. In *Soft Computing Applications for Group Decision-Making and Consensus Modeling*; Mikael, C., Janusz, K., Eds.; Springer: Cham, Switzerland, 2018; pp. 187–199.
16. Emrouznejad, A.; Marra, M. Ordered weighted averaging operators 1988–2014: A citation-based literature survey. *Int. J. Intell. Syst.* **2014**, *29*, 994–1014. [CrossRef]
17. Yager, R.R.; Kacprzyk, J.; Beliakov, G. *Recent Developments in the Ordered Weighted Averaging Operators: Theory and Practice*; Springer: Berlin, Germany, 2011.
18. Yager, R.R. Constrained OWA aggregation. *Fuzzy Sets Syst.* **1996**, *81*, 89–101. [CrossRef]
19. Carlsson, C.; Fullér, R.; Majlender, P. A note on constrained OWA aggregation. *Fuzzy Sets Syst.* **2003**, *139*, 543–546. [CrossRef]
20. Coroianu, L.; Fullér, R. On the constrained OWA aggregation problem with single constraint. *Fuzzy Sets Syst.* **2018**, *332*, 37–43. [CrossRef]
21. Ahn, B.S. Parameterized OWA operator weights: An extreme point approach. *Int. J. Approx. Reason.* **2010**, *51*, 820–831. [CrossRef]
22. Ahn, B.S. A New Approach to Solve the Constrained OWA Aggregation Problem. *IEEE Trans. Fuzzy Syst.* **2017**, *25*, 1231–1238. [CrossRef]
23. Fernández, E.; Pozo, M.A.; Puerto, J.; Scozzari, A. Ordered Weighted Average optimization in Multiobjective Spanning Tree Problem. *Eur. J. Oper. Res.* **2017**, *260*, 886–903. [CrossRef]
24. Fullér, R.; Majlender, P. On obtaining minimal variability OWA operator weights. *Fuzzy Sets Syst.* **2003**, *136*, 203–215. [CrossRef]
25. Kasperski, A.; Zieliński, P. Using the WOWA operator in robust discrete optimization problems. *Int. J. Approx. Reason.* **2016**, *68*, 54–67. [CrossRef]
26. Ogryczak, W.; Olender, P. Ordered median problem with demand distribution weights. *Optim. Lett.* **2016**, *10*, 1071–1086. [CrossRef]
27. Ogryczak, W.; Śliwiński, T. On efficient WOWA optimization for decision support under risk. *Int. J. Approx. Reason.* **2009**, *50*, 915–928. [CrossRef]
28. Chen, Y.F.; Tang, H.C. A Three Dimensional Constrained OWA Aggregation Problem with Lower Bounded Variables. *Symmetry* **2018**, *10*, 339. [CrossRef]