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A Generalized Fejér–Hadamard Inequality for Harmonically Convex Functions via Generalized Fractional Integral Operator and Related Results

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Abstract: In this paper, we obtain a version of the Fejér–Hadamard inequality for harmonically convex functions via generalized fractional integral operator. In addition, we establish an integral identity and some Fejér–Hadamard type integral inequalities for harmonically convex functions via a generalized fractional integral operator. Being generalizations, our results reproduce some known results.

Keywords: harmonically convex functions; Hermite–Hadamard inequality; generalized fractional integral operator; Mittag–Leffler function

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1. Introduction

Inequalities for convex functions, for example, the celebrated one is the Hermite–Hadamard inequality, providing a new horizon in the field of mathematical analysis. Many authors have been working on it continuously and several Hermite–Hadamard like integral inequalities have been established for many kinds of functions related to convex functions. Recently, a lot of integral inequalities of the Hermite–Hadamard type for harmonically convex functions via fractional integrals have been published (see [1–5] and references therein). The Hermite–Hadamard inequality for convex functions is stated in the following theorem.

Theorem 1. *Let I be an interval of real numbers and $f : I \to \mathbb{R}$ be a convex function on I. Then, for all $a, b \in I$, the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

Fejér gave a weighted version of the Hermite–Hadamard inequality stated as follows.

Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ be a convex function and $g : [a, b] \to \mathbb{R}$ is non-negative, integrable and symmetric to $\frac{a+b}{2}$. Then, the following inequality holds*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx.$$

It is well known as the Fejér–Hadamard inequality. In the following, we give the definition of harmonically convex functions.

Definition 1. *Reference [3] Let I be an interval of non-zero real numbers. Then, a function $f : I \to \mathbb{R}$ is said to be a harmonically convex function if the inequality*

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leq tf(b) + (1-t)f(a) \tag{1}$$

holds for $a, b \in I$ and $t \in [0, 1]$. If the inequality in Equation (1) is reversed, then f is said to be harmonically concave.

It is interesting to see that a function $f : (a, b) \to \mathbb{R}, (a > 0)$ is harmonically convex iff the function $f \circ h : (\frac{1}{a}, \frac{1}{b}) \to \mathbb{R}$ is convex, where $h : (0, \infty) \to (0, \infty)$ is the hyperbola, i.e., $h(t) = 1/t$.

Definition 2. *Reference [2] A function $h : [a, b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonically symmetric about $\frac{2ab}{a+b}$ if*

$$h(x) = h\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x}}\right)$$

holds for $x \in [a, b]$.

The following definition of the Riemann–Liouville fractional integral is the asset of fractional calculus.

Definition 3. *Reference [6] Let $f \in L[a, b]$. Then, the two sided Riemann–Liouville fractional integral of f of order $\nu > 0$ is defined as:*

$$I^{\nu}_{a^+}f(x) = \frac{1}{\Gamma(\nu)}\int_a^x (x-t)^{\nu-1}f(t)dt, \ x > a$$

and

$$I^{\nu}_{b_-}f(x) = \frac{1}{\Gamma(\nu)}\int_x^b (t-x)^{\nu-1}f(t)dt, \ x < b.$$

In the following, we give the definition of a generalized fractional integral operator which will help us to give a generalized Fejér–Hadamard inequality for harmonically convex functions and related results.

Definition 4. *Reference [7] Let $\mu, \nu, k, l, \gamma, \delta$ be positive real numbers and $\omega \in \mathbb{R}$. Then, the generalized fractional integral operators containing a generalized Mittag–Leffler function for a real valued continuous function f are defined as follows:*

$$\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}f\right)(x) = \int_a^x (x-t)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(x-t)^{\mu})f(t)dt,$$

and

$$\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b_-}f\right)(x) = \int_x^b (t-x)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(t-x)^{\mu})f(t)dt,$$

where the function $E^{\gamma,\delta,k}_{\mu,\nu,l}$ *is a generalized Mittag–Leffler function defined as:*

$$E^{\gamma,\delta,k}_{\mu,\nu,l}(t) = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn} t^n}{\Gamma(\mu n + \nu)(\delta)_{ln}}, \tag{2}$$

and $(\gamma)_{kn} = \frac{\Gamma(\gamma+kn)}{\Gamma(\gamma)}$.

For $\delta = l = 1$, the integral operator $\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}$ reduces to an integral operator $\epsilon^{\gamma,1,k}_{\mu,\nu,1,\omega,a^+}$ containing generalized Mittag–Leffler function $E^{\gamma,1,k}_{\mu,\nu,1}$ introduced by Srivastava and Tomovski in [8]. Along with $\delta = l = 1$, in addition, if $k = 1$, then it reduces to an integral operator defined by Prabhaker in [9] containing Mittag–Leffler function $E^{\gamma}_{\mu,\nu}$. For $\omega = 0$, the integral operator $\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}$ reduces to the Riemann–Liouville fractional integral operator.

In [7,8], properties of the generalized fractional integral operator $\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}$ and the generalized Mittag–Leffler function $E^{\gamma,\delta,k}_{\mu,\nu,l}(t)$ are studied in brief. In [7], it is proved that $E^{\gamma,\delta,k}_{\mu,\nu,l}(t)$ is absolutely convergent for $k < l + \mu$ and $t \in \mathbb{R}$.

Since

$$|E^{\gamma,\delta,k}_{\mu,\nu,l}(t)| \leq \sum_{n=0}^{\infty} \left| \frac{(\gamma)_{kn} t^n}{\Gamma(\mu n + \nu)(\delta)_{ln}} \right|,$$

if we say that $\sum_{n=0}^{\infty} |\frac{(\gamma)_{kn} t^n}{\Gamma(\mu n+\nu)(\delta)_{ln}}| = S$, then

$$|E^{\gamma,\delta,k}_{\mu,\nu,l}(t)| \leq S.$$

We use this property of generalized Mittag–Leffler function subsequently in our results.

In addition, we use the following definitions of special functions known as beta and Euler type form of the hypergeometric functions (see [10]),

$$\beta(\mu,\nu) = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} = \int_0^1 t^{\mu-1}(1-t)^{\nu-1} dt, \quad \mu, \nu > 0,$$

$${}_2F_1(a,b;c;w) = \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-wt)^{-a} dt,$$

where $0 < b < c$ and $|w| < 1$.

In this paper, we give a generalized version of the Fejér–Hadamard inequality for harmonically convex functions via a generalized fractional integral operator. We also obtain bounds of the absolute differences of this generalized Fejér–Hadamard inequality for harmonically convex functions. Being generalizations, we reproduce the results proved in [1–3,5].

2. Main Results

To obtain our main results, we need the following lemmas.

Lemma 1. *Reference [11] For* $0 \leq a < b$ *and* $0 < \mu \leq 1$, *we have*

$$|a^\mu - b^\mu| \leq (b-a)^\mu.$$

Lemma 2. *Let $f : [a, b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}, a < b$, be integrable and a harmonically symmetric function with respect to $\frac{2ab}{a+b}$. Then, for generalized fractional integrals, we have*

$$\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,\frac{1}{b}^+} f \circ g\right)\left(\frac{1}{a}\right) = \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,\frac{1}{a}^-} f \circ g\right)\left(\frac{1}{b}\right)$$
$$= \frac{\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,\frac{1}{b}^+} f \circ g\right)\left(\frac{1}{a}\right) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,\frac{1}{a}^-} f \circ g\right)\left(\frac{1}{b}\right)}{2},$$

where $g(t) = \frac{1}{t}$ for all $t \in [\frac{1}{b}, \frac{1}{a}]$.

Proof. Since f is harmonically symmetric about $\frac{2ab}{a+b}$, we have $f(\frac{1}{x}) = f(\frac{1}{\frac{1}{a}+\frac{1}{b}-x})$. By definition of generalized fractional integral operator

$$\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+} f \circ g\right)\left(\frac{1}{a}\right) = \int_{\frac{1}{b}}^{\frac{1}{a}} \left(\frac{1}{a} - t\right)^{\nu-1} E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(\frac{1}{a} - t\right)^{\mu}\right) f\left(\frac{1}{t}\right) dt, \tag{3}$$

replacing t by $\frac{1}{a} + \frac{1}{b} - x$ in Equation (3), we have

$$\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+} f \circ g\right)\left(\frac{1}{a}\right) = \int_{\frac{1}{b}}^{\frac{1}{a}} \left(x - \frac{1}{b}\right)^{\nu-1} E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(x - \frac{1}{b}\right)^{\mu}\right) f\left(\frac{1}{\frac{1}{a} + \frac{1}{b} - x}\right) dx$$
$$= \int_{\frac{1}{b}}^{\frac{1}{a}} \left(x - \frac{1}{b}\right)^{\nu-1} E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(x - \frac{1}{b}\right)^{\mu}\right) f(x) dx.$$

This implies

$$\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+} f \circ g\right)\left(\frac{1}{a}\right) = \left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-} f \circ g\right)\left(\frac{1}{b}\right). \tag{4}$$

By adding $\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+} f \circ g\right)\left(\frac{1}{a}\right)$ in both sides of Equation (4), we have

$$2\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+} f \circ g\right)\left(\frac{1}{a}\right) = \left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+} f \circ g\right)\left(\frac{1}{a}\right) + \left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}^-} f \circ g\right)\left(\frac{1}{b}\right). \tag{5}$$

Equations (4) and (5) give the required result. □

Theorem 3. *Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a harmonically convex function. Let $a, b \in I, a < b, f \in L[a, b]$ and also let $g : [a, b] \to \mathbb{R}$ be a non-negative, integrable and harmonically symmetric function about $\frac{2ab}{a+b}$. Then, the following inequalities for generalized fractional integrals hold:*

$$f\left(\frac{2ab}{a+b}\right)\left[\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{b}^+} g \circ h\right)\left(\frac{1}{a}\right) + \left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{a}^-} g \circ h\right)\left(\frac{1}{b}\right)\right] \tag{6}$$
$$\leq \left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{b}^+} fg \circ h\right)\left(\frac{1}{a}\right) + \left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{a}^-} fg \circ h\right)\left(\frac{1}{b}\right)$$
$$\leq \frac{f(a) + f(b)}{2}\left[\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{b}^+} g \circ h\right)\left(\frac{1}{a}\right) + \left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{a}^-} g \circ h\right)\left(\frac{1}{b}\right)\right],$$

where $\omega' = \omega\left(\frac{ab}{b-a}\right)^{\mu}$ and $h(t) = \frac{1}{t}$ for all $t \in \left[\frac{1}{b}, \frac{1}{a}\right]$.

Proof. Since f is a harmonically convex function, therefore, for $t \in [0, 1]$, we have

$$2f\left(\frac{2ab}{a+b}\right) \leq f\left(\frac{ab}{ta + (1-t)b}\right) + f\left(\frac{ab}{tb + (1-t)a}\right). \tag{7}$$

Multiplying both sides of Equation (7) by $t^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})g\left(\frac{ab}{tb+(1-t)a}\right)$ and then integrating with respect to t over $[0,1]$, we have

$$\begin{aligned}
&2f\left(\frac{2ab}{a+b}\right)\int_0^1 t^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})g\left(\frac{ab}{tb+(1-t)a}\right)dt\\
&\leq \int_0^1 t^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{tb+(1-t)a}\right)dt\\
&+\int_0^1 t^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{tb+(1-t)a}\right)dt.
\end{aligned} \tag{8}$$

By choosing $x=\frac{tb+(1-t)a}{ab}$ that is $\frac{ab}{ta+(1-t)b}=\frac{1}{\frac{1}{a}+\frac{1}{b}-x}$ in Equation (8), we have

$$\begin{aligned}
&2f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{ab}{b-a}\right)^{\nu}\left(x-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(\frac{ab}{b-a}\right)^{\mu}\left(x-\frac{1}{b}\right)^{\mu}\right)g\left(\frac{1}{x}\right)dx\\
&\leq \int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{ab}{b-a}\right)^{\nu}\left(x-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(\frac{ab}{b-a}\right)^{\mu}\left(x-\frac{1}{b}\right)^{\mu}\right)f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)g\left(\frac{1}{x}\right)dx\\
&+\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{ab}{b-a}\right)^{\nu}\left(x-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(\frac{ab}{b-a}\right)^{\mu}\left(x-\frac{1}{b}\right)^{\mu}\right)f\left(\frac{1}{x}\right)g\left(\frac{1}{x}\right)dx.
\end{aligned} \tag{9}$$

Since f is harmonically symmetric about $\frac{2ab}{a+b}$, therefore, after simplification, Equation (9) becomes

$$\begin{aligned}
&2f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega'\left(x-\frac{1}{b}\right)^{\mu}\right)g\left(\frac{1}{x}\right)dx\\
&\leq \int_{\frac{1}{a}}^{\frac{1}{b}}\left(\frac{1}{a}-x\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega'\left(\frac{1}{a}-x\right)^{\mu}\right)f\left(\frac{1}{x}\right)g\left(\frac{1}{x}\right)dx\\
&+\int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega'\left(x-\frac{1}{b}\right)^{\mu}\right)f\left(\frac{1}{x}\right)g\left(\frac{1}{x}\right)dx.
\end{aligned} \tag{10}$$

This implies

$$\begin{aligned}
&2f\left(\frac{2ab}{a+b}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{a}^-}g\circ h\right)\left(\frac{1}{b}\right)\\
&\leq\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{b}^+}fg\circ h\right)\left(\frac{1}{a}\right)+\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{a}^-}fg\circ h\right)\left(\frac{1}{b}\right).
\end{aligned}$$

Using Lemma 2 in the above inequality, we have

$$\begin{aligned}
&f\left(\frac{2ab}{a+b}\right)\left[\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{a}^-}g\circ h\right)\left(\frac{1}{b}\right)+\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{b}^+}g\circ h\right)\left(\frac{1}{a}\right)\right]\\
&\leq\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{b}^+}fg\circ h\right)\left(\frac{1}{a}\right)+\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{a}^-}fg\circ h\right)\left(\frac{1}{b}\right).
\end{aligned} \tag{11}$$

To prove the second half of the inequality, again from harmonically convexity of f on $[a,b]$ and for $t\in[0,1]$, we have

$$f\left(\frac{ab}{tb+(1-t)a}\right)+f\left(\frac{ab}{ta+(1-t)b}\right)\leq f(a)+f(b). \tag{12}$$

Multiplying both sides of Equation (12) by $t^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})g\left(\frac{ab}{tb+(1-t)a}\right)$, then integrating with respect to t over $[0,1]$, we have

$$\begin{aligned}&\int_0^1 t^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{tb+(1-t)a}\right)dt\\&+\int_0^1 t^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{tb+(1-t)a}\right)dt\\&\leq[f(a)+f(b)]\int_0^1 t^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})g\left(\frac{ab}{tb+(1-t)a}\right)dt.\end{aligned}\tag{13}$$

Setting $x=\frac{tb+(1-t)a}{ab}$ and by using harmonically symmetry of f with respect to $\frac{2ab}{a+b}$ in Equation (13), after simplification, we have

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{b}+}fg\circ h\right)\left(\frac{1}{a}\right)+\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{a}-}fg\circ h\right)\left(\frac{1}{b}\right)\\&\leq[f(a)+f(b)]\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{b}+}g\circ h\right)\left(\frac{1}{a}\right).\end{aligned}\tag{14}$$

Using Lemma 2 in Equation (14), we have

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{b}+}fg\circ h\right)\left(\frac{1}{a}\right)+\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{a}-}fg\circ h\right)\left(\frac{1}{b}\right)\\&\leq\frac{[f(a)+f(b)]}{2}\left[\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{a}-}g\circ h\right)\left(\frac{1}{b}\right)+\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{b}+}g\circ h\right)\left(\frac{1}{a}\right)\right].\end{aligned}\tag{15}$$

By joining Equations (11) and (15), we get Equation (6). □

Remark 1. *In Theorem 3,*

(i) *if we put $\omega=0$ along with $g(x)=1$ and $\nu=1$, then we get [3] [Theorem 2.4].*
(ii) *if we put $\omega=0$ along with $g(x)=1$, then we get [5] [Theorem 4].*
(iii) *if we put $\omega=0$ along with $\nu=1$, then we get [1] [Theorem 8].*

Lemma 3. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on the interior of I and $f'\in L[a,b]$ where $a,b\in I$ and $a<b$. In addition, let $g:I\subset(0,\infty)\to\mathbb{R}$ be an integrable and harmonically symmetric function about $\frac{2ab}{a+b}$. Then, the following equality holds for generalized fractional integrals:*

$$\begin{aligned}&\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}+}(g\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(g\circ h)\left(\frac{1}{b}\right)\right)\\&-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}+}(fg\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(fg\circ h)\left(\frac{1}{b}\right)\right)\\&=\left[\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\int_{\frac{1}{b}}^{t}\left(\frac{1}{a}-s\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(\frac{1}{a}-s\right)^{\mu}\right)(g\circ h)(s)ds\right)(f\circ h)'(t)dt\right.\\&\left.-\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\int_{t}^{\frac{1}{a}}\left(s-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(s-\frac{1}{b}\right)^{\mu}\right)(g\circ h)(s)ds\right)(f\circ h)'(t)dt\right],\end{aligned}$$

where $h(t)=\frac{1}{t}$ for $t\in[\frac{1}{b},\frac{1}{a}]$.

Proof. To prove this lemma, we have

$$\begin{aligned}
&\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\int_{\frac{1}{b}}^{t}\left(\frac{1}{a}-s\right)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega\left(\frac{1}{a}-s\right)^{\mu}\right)(g\circ h)(s)ds\right)(f\circ h)'(t)dt\\
&=\left|\left(\int_{\frac{1}{b}}^{t}\left(\frac{1}{a}-s\right)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega\left(\frac{1}{a}-s\right)^{\mu}\right)(g\circ h)(s)ds\right)(f\circ h)(t)\right|_{\frac{1}{b}}^{\frac{1}{a}}\\
&-\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{1}{a}-t\right)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega\left(\frac{1}{a}-t\right)^{\mu}\right)(g\circ h)(t)(f\circ h)(t)dt\\
&=\left(\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{1}{a}-s\right)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega\left(\frac{1}{a}-s\right)^{\mu}\right)(g\circ h)(s)ds\right)f(a)\\
&-\epsilon_{\mu,\nu,l,\omega,\frac{1}{b}^{+}}^{\gamma,\delta,k}(fg\circ h)\left(\frac{1}{a}\right).
\end{aligned}$$

This implies

$$\begin{aligned}
&\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\int_{\frac{1}{b}}^{t}\left(\frac{1}{a}-s\right)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega\left(\frac{1}{a}-s\right)^{\mu}\right)(g\circ h)(s)ds\right)(f\circ h)'(t)dt\\
&=f(a)\epsilon_{\mu,\nu,l,\omega,\frac{1}{b}+}^{\gamma,\delta,k}(g\circ h)\left(\frac{1}{a}\right)-\epsilon_{\mu,\nu,l,\omega,\frac{1}{b}+}^{\gamma,\delta,k}(fg\circ h)\left(\frac{1}{a}\right).
\end{aligned}\tag{16}$$

Similarly,

$$\begin{aligned}
&\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\int_{t}^{\frac{1}{a}}\left(s-\frac{1}{b}\right)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega\left(s-\frac{1}{b}\right)^{\mu}\right)(g\circ h)(s)ds\right)(f\circ h)'(t)dt\\
&=\left|\left(\int_{t}^{\frac{1}{a}}\left(s-\frac{1}{b}\right)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega\left(s-\frac{1}{b}\right)^{\mu}\right)(g\circ h)(s)ds\right)(f\circ h)(t)\right|_{\frac{1}{b}}^{\frac{1}{a}}\\
&+\int_{\frac{1}{b}}^{\frac{1}{a}}\left(t-\frac{1}{b}\right)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega\left(t-\frac{1}{a}\right)^{\mu}\right)(g\circ h)(f\circ h)(t)dt\\
&=-\left(\int_{\frac{1}{b}}^{\frac{1}{a}}\left(s-\frac{1}{b}\right)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega\left(s-\frac{1}{b}\right)^{\mu}\right)(g\circ h)(s)ds\right)f(b)\\
&+\epsilon_{\mu,\nu,l,\omega,\frac{1}{a}-}^{\gamma,\delta,k}(fg\circ h)\left(\frac{1}{b}\right).
\end{aligned}$$

This implies

$$\begin{aligned}
&\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\int_{t}^{\frac{1}{a}}\left(s-\tfrac{1}{b}\right)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega\left(s-\tfrac{1}{b}\right)^{\mu}\right)(g\circ h)(s)ds\right)(f\circ h)'(t)dt\\
&\quad=-f(b)\epsilon_{\mu,\nu,l,\omega,\frac{1}{a}^{-}}^{\gamma,\delta,k}(g\circ h)\left(\tfrac{1}{b}\right)+\epsilon_{\mu,\nu,l,\omega,\frac{1}{a}^{-}}^{\gamma,\delta,k}(fg\circ h)\left(\tfrac{1}{b}\right).
\end{aligned}\tag{17}$$

Upon subtracting Equation (17) from Equation (16) and, using Lemma 2, we get the result. □

Remark 2. *In Lemma 3, if we take $g(x)=1$ with $\omega=0$, then it gives [5] (Lemma 3).*

Theorem 4. *Let $f: I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on the interior of I and $f'\in L[a,b]$ where $a,b\in I$ and $a<b$. If $|f'|$ is a harmonically convex function on $[a,b]$, $g:[a,b]\subset(0,\infty)\to\mathbb{R}$ is a continuous*

and harmonically symmetric function with respect to $\frac{2ab}{a+b}$*, then the following inequality for generalized fractional integrals holds*

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(g\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-}(g\circ h)\left(\frac{1}{b}\right)\right)\right.\\&\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(fg\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-}(fg\circ h)\left(\frac{1}{b}\right)\right)\right|\\&\leq\frac{\|g\|_\infty\,S\,(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left(C_1(\nu)|f'(a)|+C_2(\nu)|f'(b)|\right),\end{aligned}$$

where $C_1(\nu)=\frac{b^{-2}}{(\nu+2)}{}_2F_1(2,1;\nu+3;1-\frac{a}{b})-\frac{b^{-2}}{(\nu+1)(\nu+2)}{}_2F_1(2,\nu+1;\nu+3;1-\frac{a}{b})+\frac{2(a+b)^{-2}}{(\nu+1)(\nu+2)}{}_2F_1(2,\nu+1;\nu+3;\frac{b-a}{b+a})$ *and* $C_2(\nu)=\frac{b^{-2}}{(\nu+1)(\nu+2)}{}_2F_1(2,2;\nu+3;1-\frac{a}{b})-\frac{b^{-2}}{\nu+2}{}_2F_1(2,\nu+2;\nu+3;1-\frac{a}{b})+\frac{4(a+b)^{-2}}{(\nu+1)}{}_2F_1(2,\nu+1;\nu+2;\frac{b-a}{b+a})-\frac{2(a+b)^{-2}}{(\nu+1)(\nu+2)}{}_2F_1(\nu,\nu+1;\nu+3;\frac{b-a}{b+a})$ *with* $0<\nu\leq 1$*,* $h(t)=\frac{1}{t}$ *for all* $t\in[\frac{1}{b},\frac{1}{a}]$*.*

Proof. By Lemma 3, we have

$$\begin{aligned}&\left|\left(\tfrac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(g\circ h)\left(\tfrac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-}(g\circ h)\left(\tfrac{1}{b}\right)\right)\right.\\&\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(fg\circ h)\left(\tfrac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-}(fg\circ h)\left(\tfrac{1}{b}\right)\right)\right|\\&\leq\int_{\frac{1}{b}}^{\frac{1}{a}}\left|\left(\int_{\frac{1}{b}}^{t}\left(\tfrac{1}{a}-s\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(\tfrac{1}{a}-s\right)^{\mu}\right)(g\circ h)(s)ds\right)\right.\\&\left.-\left(\int_{t}^{\frac{1}{a}}\left(s-\tfrac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(s-\tfrac{1}{b}\right)^{\mu}\right)(g\circ h)(s)ds\right)\right||(f\circ h)'(t)|dt.\end{aligned}\tag{18}$$

Since g is harmonically symmetric with respect to $\frac{2ab}{a+b}$, therefore, $g(\frac{1}{t})=g\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-t}\right)$ for all $t\in[\frac{1}{b},\frac{1}{a}]$, and we have

$$\begin{aligned}&\left|\left(\int_{\frac{1}{b}}^{t}\left(\tfrac{1}{a}-s\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(\tfrac{1}{a}-s\right)^{\mu}\right)(g\circ h)(s)ds\right)\right.\\&\left.-\left(\int_{t}^{\frac{1}{a}}\left(s-\tfrac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(s-\tfrac{1}{b}\right)^{\mu}\right)(g\circ h)(s)ds\right)\right|\\&=\left|\left(\int_{\frac{1}{a}+\frac{1}{b}-t}^{\frac{1}{a}}\left(s-\tfrac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(s-\tfrac{1}{b}\right)^{\mu}\right)(g\circ h)(s)ds\right)\right.\\&\left.+\left(\int_{\frac{1}{a}}^{t}\left(s-\tfrac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(s-\tfrac{1}{b}\right)^{\mu}\right)(g\circ h)(s)ds\right)\right|\\&=\left|\left(\int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left(s-\tfrac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(s-\tfrac{1}{b}\right)^{\mu}\right)(g\circ h)(s)ds\right)\right|\end{aligned}$$

$$\leq\begin{cases}\int_{t}^{\frac{1}{a}+\frac{1}{b}-t}|(s-\frac{1}{b})^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(s-\frac{1}{b})^{\mu})g(s)|ds, & t\in[\frac{1}{a},\frac{a+b}{2ab}]\\ \int_{\frac{1}{a}+\frac{1}{b}-t}^{t}|(s-\frac{1}{b})^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(s-\frac{1}{b})^{\mu})g(s)|ds, & t\in[\frac{a+b}{2ab},\frac{1}{a}].\end{cases}\tag{19}$$

Using Equation (19) in (18), we have

$$\begin{aligned}&\left|\left(\tfrac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(g\circ h)\left(\tfrac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-}(g\circ h)\left(\tfrac{1}{b}\right)\right)\right.\\&\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(fg\circ h)\left(\tfrac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-}(fg\circ h)\left(\tfrac{1}{b}\right)\right)\right|r\\&\leq\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\int_{\frac{1}{b}}^{\frac{1}{a}+\frac{1}{b}-t}\left|\left(s-\tfrac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(s-\tfrac{1}{b}\right)^{\mu}\right)(g\circ h)(s)\right|ds\right)|(f\circ h)'(t)|dt\\&+\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left|\left(s-\tfrac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(s-\tfrac{1}{b}\right)^{\mu}\right)(g\circ h)(s)\right|ds\right)|(f\circ h)'(t)|dt.\end{aligned}\tag{20}$$

Using $\|g\|_\infty = \sup_{t\in[a,b]}|g(t)|$ and absolute convergence of Mittag–Leffer function, the above inequality becomes

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(g\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-}(g\circ h)\left(\frac{1}{b}\right)\right)\right.\\&\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(fg\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-}(fg\circ h)\left(\frac{1}{b}\right)\right)\right|\\&\leq\| g \|_\infty S\left[\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\int_{t}^{\frac{1}{a}+\frac{1}{b}-t}\left(s-\frac{1}{b}\right)^{\nu-1}ds\right)|(f\circ h)'(t)|dt\right.\\&\left.+\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left(\frac{1}{a}-s\right)^{\nu-1}ds\right)|(f\circ h)'(t)|dt\right]\\&=\| g \|_\infty S\left[\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\frac{(\frac{1}{a}-t)^\nu-(t-\frac{1}{b})^\nu}{\nu}\frac{1}{t^2}\right)|f'\left(\frac{1}{t}\right)|dt\right.\\&\left.+\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\frac{\left(t-\frac{1}{b}\right)^\nu-(\frac{1}{a}-t)^\nu}{\nu}\frac{1}{t^2}\right)|f'\left(\frac{1}{t}\right)|dt\right].\end{aligned}\tag{21}$$

Setting $t=\frac{ub+(1-u)a}{ab}$ in Equation (21), we have

$$\begin{aligned}&\left|\left(\tfrac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(g\circ h)\left(\tfrac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-}(g\circ h)\left(\tfrac{1}{b}\right)\right)\right.\\&\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(fg\circ h)\left(\tfrac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-}(fg\circ h)\left(\tfrac{1}{b}\right)\right)\right|\\&\leq\frac{\|g\|_\infty S(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left[\int_0^{\frac{1}{2}}\frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}|f'\left(\frac{ab}{(ub+(1-u)a)}\right)|du\right.\\&\left.+\int_{\frac{1}{2}}^{1}\frac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}|f'\left(\frac{ab}{(ub+(1-u)a)}\right)|du\right].\end{aligned}\tag{22}$$

Since $|f'|$ is harmonically convex on $[a,b]$, it can be written as:

$$\left|f'\left(\frac{ab}{(ub+(1-u)a)^2}\right)\right|\leq u|f'(a)|+(1-u)|f'(b)|.\tag{23}$$

Using Equation (23) in Equation (22), we have

$$
\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(g\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-}(g\circ h)\left(\frac{1}{b}\right)\right)\right.\\
&\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(fg\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-}(fg\circ h)(\frac{1}{b})\right)\right|\\
&\leq\frac{\parallel g\parallel_\infty S\,(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left[\int_0^{\frac{1}{2}}\frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}\left(u|f'(a)|+(1-u)|f'(b)|\right)du\right.\\
&\left.+\int_{\frac{1}{2}}^1\frac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}\left(u|f'(a)|+(1-u)|f'(b)|\right)du\right],
\end{aligned}
$$

which is

$$
\begin{aligned}
&\left|\left(\tfrac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(g\circ h)\left(\tfrac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-}(g\circ h)\left(\tfrac{1}{b}\right)\right)\right.\\
&\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(fg\circ h)\left(\tfrac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-}(fg\circ h)\left(\tfrac{1}{b}\right)\right)\right|\\
&\leq\tfrac{\|g\|_\infty S(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left[\left(\int_0^{\frac{1}{2}}\tfrac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}udu+\int_{\frac{1}{2}}^1\tfrac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}udu\right)|f'(a)|\right.\\
&\left.+\left(\int_0^{\frac{1}{2}}\tfrac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}(1-u)du+\int_{\frac{1}{2}}^1\tfrac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}(1-u)du\right)|f'(b)|\right].
\end{aligned}
\tag{24}
$$

One can have, by using Lemma 1,

$$
\begin{aligned}
&\int_0^{\frac{1}{2}}\tfrac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}udu+\int_{\frac{1}{2}}^1\tfrac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}udu\\
=\ &\int_0^1\tfrac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}udu-\int_{\frac{1}{2}}^1\tfrac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}udu+\int_{\frac{1}{2}}^1\tfrac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}udu\\
=\ &\int_0^1\tfrac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}udu+\int_{\frac{1}{2}}^1\tfrac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}udu+\int_{\frac{1}{2}}^1\tfrac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}udu\\
=\ &\int_0^1\tfrac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}udu+2\int_{\frac{1}{2}}^1\tfrac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}udu.
\end{aligned}
\tag{25}
$$

On simplification, we get

$$
\begin{aligned}
&\int_0^{\frac{1}{2}}\frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}udu+\int_{\frac{1}{2}}^1\frac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}udu\\
&=\frac{b^{-2}}{(\nu+2)}{}_2F_1(2,1;\nu+3;1-\frac{a}{b})-\frac{b^{-2}}{(\nu+1)(\nu+2)}{}_2F_1(2,\nu+1;\nu+3;1-\frac{a}{b})\\
&+\frac{2(a+b)^{-2}}{(\nu+1)(\nu+2)}{}_2F_1(2,\nu+1;\nu+3;\frac{b-a}{b+a})=C_1(\nu).
\end{aligned}
\tag{26}
$$

Similarly,

$$
\begin{aligned}
&\int_0^{\frac{1}{2}}\frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}(1-u)du+\int_{\frac{1}{2}}^1\frac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}(1-u)du\\
&=\frac{b^{-2}}{(\nu+1)(\nu+2)}{}_2F_1(2,2;\nu+3;1-\frac{a}{b})-\frac{b^{-2}}{\nu+2}{}_2F_1(2,\nu+2;\nu+3;1-\frac{a}{b})\\
&+\frac{4(a+b)^{-2}}{(\nu+1)}{}_2F_1(2,\nu+1;\nu+2;\frac{b-a}{b+a})-\frac{2(a+b)^{-2}}{(\nu+1)(\nu+2)}{}_2F_1\left(\nu,\nu+1;\nu+3;\frac{b-a}{b+a}\right)=C_2(\nu).
\end{aligned}
\tag{27}
$$

Using Equations (26) and (27) in Equation (24), we get the result. □

Remark 3. *In Theorem 4,*

(i) *if we put $\omega=0$, then we get [2] (Theorem 6).*
(ii) *if we take $\nu=1$ along with $\omega=0$, then we get [2] (Corollary 1(1)).*

(iii) *if we take $g(x) = 1$ along with $\omega = 0$, then we get [2] (Corollary 1(2)).*
(iv) *if we take $\nu = 1$, $g(x) = 1$ along with $\omega = 0$, then we get [2] (Corollary 1(3)).*

Theorem 5. *Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on the interior of I and $f' \in L[a, b]$ where $a, b \in I$ and $a < b$. If $|f'|^q$, $q > 1$ is a harmonically convex function on $[a, b]$, and $g : [a, b] \subset (0, \infty) \to \mathbb{R}$ be a continuous and harmonically symmetric function about $\frac{2ab}{a+b}$, then the following inequality for generalized fractional integrals holds*

$$\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(g\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(g\circ h)\left(\frac{1}{b}\right)\right)\right.\\
&\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(fg\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(fg\circ h)\left(\frac{1}{b}\right)\right)\right|\\
&\leq \frac{\| g \|_\infty\, S\,(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left[(C_3^{1-\frac{1}{q}}(\nu)\left(C_4(\nu)|f'(a)|^q+C_4(\nu)|f'(b)|^q\right)^{\frac{1}{q}}\right.\\
&\left.+(C_6^{1-\frac{1}{q}}(\nu)\left(C_7(\nu)|f'(a)|^q+C_8(\nu)|f'(b)|^q\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where $C_3(\nu) = \frac{2(a+b)^{-2}}{\nu+1}{}_2F_1\left(2;\nu+1;\nu+3;\frac{b-a}{b+a}\right)$, $C_4(\nu) = \frac{(a+b)^{-2}}{(\nu+1)(\nu+2)}{}_2F_1\left(2;\nu+1;\nu+3;\frac{b-a}{b+a}\right)$, $C_5(\nu) = C_3(\nu) - C_4(\nu)$, $C_6(\nu) = \frac{b^{-2}}{\nu+1}{}_2F_1(2;1;\nu+1;(1-\frac{a}{b})) - \frac{b^{-2}}{\nu+1}{}_2F_1(2;\nu+1;\nu+2;(1-\frac{a}{b})) + C_3(\nu)$, $C_7(\nu) = \frac{b^{-2}}{\nu+1}{}_2F_1(2;1;\nu+2;(1-\frac{a}{b})) - \frac{b^{-2}}{\nu+1}{}_2F_1(2;\nu+1;\nu+2;(1-\frac{a}{b})) + C_4(\nu)$, $C_8(\nu) = C_6(\nu) - C_7(\nu)$ with $0 < \nu \leq 1$, $h(t) = \frac{1}{t}$ for all $t \in [\frac{1}{b}, \frac{1}{a}]$.

Proof. By inequality Equation (22) of Theorem 4, we have

$$\begin{aligned}
&\left|\left(\tfrac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(g\circ h)\left(\tfrac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(g\circ h)\left(\tfrac{1}{b}\right)\right)\right.\\
&\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(fg\circ h)\left(\tfrac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(fg\circ h)\left(\tfrac{1}{b}\right)\right)\right|\\
&\leq \tfrac{\|g\|_\infty S(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left[\int_0^{\frac{1}{2}}\tfrac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}|f'\left(\tfrac{ab}{(ub+(1-u)a)}\right)|du\right.\\
&\left.+\int_{\frac{1}{2}}^1\tfrac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}\left|f'\left(\tfrac{ab}{(ub+(1-u)a)}\right)\right|du\right].
\end{aligned}\tag{28}$$

Using power means, inequality Equation (28) becomes

$$\begin{aligned}
&\left|\left(\tfrac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(g\circ h)\left(\tfrac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(g\circ h)\left(\tfrac{1}{b}\right)\right)\right.\\
&\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(fg\circ h)\left(\tfrac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(fg\circ h)\left(\tfrac{1}{b}\right)\right)\right|\\
&\leq \tfrac{\|g\|_\infty S(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left[\left(\int_0^{\frac{1}{2}}\tfrac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}}\right.\\
&\times\left(\int_0^{\frac{1}{2}}\tfrac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}\left|f'\left(\tfrac{ab}{(ub+(1-u)a)^2}\right)\right|^q du\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^1\tfrac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}}\\
&\left.\times\left(\int_{\frac{1}{2}}^1\tfrac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}\left|f'\left(\tfrac{ab}{(ub+(1-u)a)^2}\right)\right|^q du\right)^{\frac{1}{q}}\right].
\end{aligned}\tag{29}$$

By using the harmonically convexity of $|f'|^q$ in Equation (29), we have

$$\begin{aligned} &\left|\left(\tfrac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(g\circ h)\left(\tfrac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-}(g\circ h)\left(\tfrac{1}{b}\right)\right)\right.\\ &\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(fg\circ h)\left(\tfrac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-}(fg\circ h)\left(\tfrac{1}{b}\right)\right)\right|\\ &\leq \tfrac{\|g\|_\infty S(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left[\left(\int_0^{\frac{1}{2}}\tfrac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}}\right.\\ &\times\left(\int_0^{\frac{1}{2}}\tfrac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}(u|f'(a)|^q+(1-u)|f'(b)|^q)\right)^{\frac{1}{q}}\\ &+\left(\int_{\frac{1}{2}}^1\tfrac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}}\\ &\left.\times\left(\int_{\frac{1}{2}}^1\tfrac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}(u|f'(a)|^q+(1-u)|f'(b)|^q)\right)^{\frac{1}{q}}\right]. \end{aligned} \tag{30}$$

That is,

$$\begin{aligned} &\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(g\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-}(g\circ h)\left(\frac{1}{b}\right)\right)\right. \qquad (31)\\ &\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(fg\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-}(fg\circ h)\left(\frac{1}{b}\right)\right)\right|\\ &\leq\| g \|_\infty S\frac{(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left[\left(\int_0^{\frac{1}{2}}\frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}}\right.\\ &\times\left(\int_0^{\frac{1}{2}}\frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}udu|f'(a)|^q+\int_0^{\frac{1}{2}}\frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}(1-u)du|f'(b)|^q)\right)^{\frac{1}{q}}\\ &+\left(\int_{\frac{1}{2}}^1\frac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\frac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}udu|f'(a)|^q\right. \qquad (32)\\ &\left.\left.+\int_{\frac{1}{2}}^1\frac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}(1-u)du|f'(b)|^q)\right)^{\frac{1}{q}}\right]. \end{aligned}$$

Now, we evaluate the integrals of Equation (31) by using Lemma 1:

$$\int_0^{\frac{1}{2}}\frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}du\leq\int_0^{\frac{1}{2}}\frac{(1-2u)^\nu}{(ub+(1-u)a)^2}du=\frac{1}{2}\int_0^1\frac{(1-u)^\nu}{(\frac{ub}{2}+(1-\frac{u}{2})a)^2}du. \tag{33}$$

Substituting $u=1-w$ in Equation (33), we have

$$\begin{aligned} &\int_0^{\frac{1}{2}}\frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}du\leq 2(a+b)^{-2}\int_0^1 w^\nu\left(1-w(\frac{b-a}{b+a})\right)^{-2}dw\\ &=2\frac{(a+b)^{-2}}{\nu+1}{}_2F_1\left(2;\nu+1;\nu+2;\frac{b-a}{b+a}\right)=C_3(\nu). \end{aligned}$$

Similarly,

$$\int_0^{\frac{1}{2}}\frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}udu\leq\int_0^{\frac{1}{2}}\frac{(1-2u)^\nu}{(ub+(1-u)a)^2}udu=\frac{1}{4}\int_0^1\frac{u(1-u)^\nu}{(\frac{ub}{2}+(1-\frac{u}{2})a)^2}du. \tag{34}$$

Substituting $u = 1 - w$ in Equation (34), we have

$$\int_0^{\frac{1}{2}} \frac{(1-u)^\nu - u^\nu}{(ub+(1-u)a)^2} udu \leq (a+b)^{-2} \int_0^1 (1-w)w^\nu \left(1 - w(\frac{b-a}{b+a})\right)^{-2} dw$$
$$= \frac{(a+b)^{-2}}{(\nu+1)(\nu+2)} {}_2F_1\left(2; \nu+1; \nu+3; \frac{b-a}{b+a}\right) = C_4(\nu).$$

$$\int_0^{\frac{1}{2}} \frac{(1-u)^\nu - u^\nu(1-u)}{(ub+(1-u)a)^2}(1-u)du \leq C_3(\nu) - C_4(\nu) = C_5(\nu). \tag{35}$$

$$\begin{aligned}&\int_{\frac{1}{2}}^1 \frac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}du = \int_0^1 \frac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}du + \int_0^{\frac{1}{2}} \frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}du \\ &\leq \frac{b^{-2}}{\nu+1} {}_2F_1\left(2;1;\nu+2;(1-\tfrac{a}{b})\right) - \frac{b^{-2}}{\nu+1} {}_2F_1\left(2;\nu+1;\nu+2;(1-\tfrac{a}{b})\right) + C_3(\nu) = C_6(\nu).\end{aligned} \tag{36}$$

$$\int_{\frac{1}{2}}^1 \frac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}udu = \int_0^1 \frac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}udu + \int_0^{\frac{1}{2}} \frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}udu$$
$$\leq \frac{b^{-2}}{\nu+2} {}_2F_1\left(2;1;\nu+3;(1-\frac{a}{b})\right) - \frac{b^{-2}}{(\nu+1)(\nu+2)} {}_2F_1\left(2;\nu+1;\nu+3;(1-\frac{a}{b})\right) + C_4(\nu) = C_7(\nu)$$

and

$$\int_{\frac{1}{2}}^1 \frac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}(1-u)du \leq C_6(\nu) - C_7(\nu) = C_8(\nu). \tag{37}$$

Using Equation (31), we get the result. □

Remark 4. *The following remarks can be obtained by giving specific values to parameters in Theorem 5:*

(i) *If we take $\omega = 0$, then we get [2] (Theorem 7).*
(ii) *If we take $\nu = 1$ along with $\omega = 0$, then we get [2] (Corollary 2(1)).*
(iii) *If we take $g(x) = 1$ along with $\omega = 0$, then we get [2] (Corollary 2(2)).*
(iv) *If we take $\nu = 1$, $g(x) = 1$ along with $\omega = 0$, then we get [2] (Corollary 2(3)).*

Theorem 6. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on the interior of I such that $f' \in L[a,b]$, where $a, b \in I$ and $a < b$. If $|f'|^q$, $q > 1$ is a harmonically convex function on $[a,b]$, and $g : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a continuous and harmonically symmetric function about $\frac{2ab}{a+b}$. Then, the following inequality for generalized fractional integrals holds*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}+}(g\circ h)\left(\frac{1}{a}\right) + \epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(g\circ h)\left(\frac{1}{b}\right)\right)\right.$$
$$\left. - \left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}+}(fg\circ h)\left(\frac{1}{a}\right) + \epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(fg\circ h)\left(\frac{1}{b}\right)\right)\right|$$
$$\leq \frac{\|g\|_\infty S\,(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left(C_9^{\frac{1}{p}}(\nu)\left(\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right)^{\frac{1}{q}} + C_{10}^{\frac{1}{p}}(\nu)\left(\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right)^{\frac{1}{q}}\right),$$

where $C_9(\nu) = \frac{(a+b)^{-2p}}{2^{-2p+1}(\nu p+1)} {}_2F_1(2p, \nu p+1; \nu p+2; \frac{b-a}{b+a})$ and $C_{10}(\nu) = \frac{b^{-2p}}{2(\nu p+1)} {}_2F_1(2p, 1; \nu p+2; \frac{1}{2}(1-\frac{a}{b}))$ with $0 \leq \nu < 1$, $h(t) = \frac{1}{t}$ for all $t \in [\frac{1}{b}, \frac{1}{a}]$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By inequality Equation (22) of Theorem 4, we have

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^{+}}(g\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_{-}}(g\circ h)\left(\frac{1}{b}\right)\right)\right.\\&\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^{+}}(fg\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_{-}}(fg\circ h)\left(\frac{1}{b}\right)\right)\right|\\&\leq\frac{\|g\|_{\infty}S(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left[\int_0^{\frac{1}{2}}\frac{(1-u)^{\nu}-u^{\nu}}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{(ub+(1-u)a)}\right)\right|du\right.\\&\left.+\int_{\frac{1}{2}}^{1}\frac{u^{\nu}-(1-u)^{\nu}}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{(ub+(1-u)a)}\right)\right|du\right].\end{aligned}\tag{38}$$

By using Hölder inequality and harmonically convexity of $|f'|^q$, Equation (38) follows:

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^{+}}(g\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_{-}}(g\circ h)\left(\frac{1}{b}\right)\right)\right.\\&\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^{+}}(fg\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_{-}}(fg\circ h)\left(\frac{1}{b}\right)\right)\right|\\&\leq\frac{\parallel g\parallel_{\infty}S\,(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left[\left(\int_0^{\frac{1}{2}}\frac{((1-u)^{\nu}-u^{\nu})^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}\right.\\&\times\left(\int_0^{\frac{1}{2}}(u|f'(a)|^q+(1-u)|f'(b)|^q)du\right)^{\frac{1}{q}}\\&\left.+\left(\int_{\frac{1}{2}}^{1}\frac{(u^{\nu}-(1-u)^{\nu})^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}(u|f'(a)|^q+(1-u)|f'(b)|^q)du\right)^{\frac{1}{q}}\right].\end{aligned}$$

After simplification, we have

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^{+}}(g\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_{-}}(g\circ h)\left(\frac{1}{b}\right)\right)\right.\\&\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^{+}}(fg\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_{-}}(fg\circ h)\left(\frac{1}{b}\right)\right)\right|\\&\leq\frac{\parallel g\parallel_{\infty}S\,(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left[\left(\int_0^{\frac{1}{2}}\frac{((1-u)^{\nu}-u^{\nu})^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}\right.\\&\times\left(\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right)^{\frac{1}{q}}\\&\left.+\left(\int_{\frac{1}{2}}^{1}\frac{(u^{\nu}-(1-u)^{\nu})^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right)^{\frac{1}{q}}\right].\end{aligned}\tag{39}$$

We evaluate the integrals by using Lemma 1

$$\int_0^{\frac{1}{2}}\frac{((1-u)^{\nu}-u^{\nu})^p}{(ub+(1-u)a)^{2p}}du\leq\int_0^{\frac{1}{2}}\frac{(1-2u)^{\nu p}}{(ub+(1-u)a)^{2p}}du=\frac{1}{2}\int_0^{1}\frac{(1-u)^{\nu p}}{(\frac{ub}{2}+(1-\frac{u}{2})a)^{2p}}du.\tag{40}$$

Putting $u=1-w$ in Equation (40), we have

$$\begin{aligned}&\int_0^{\frac{1}{2}}\frac{((1-u)^{\nu}-u^{\nu})^p}{(ub+(1-u)a)^{2p}}du\leq\frac{1}{2}\int_0^{1}w^{\nu p}\left(\frac{a+b}{2}\right)^{-2p}\left(1-w\left(\frac{b-a}{b+a}\right)\right)^{-2p}dw\\&=\frac{(a+b)^{-2p}}{2^{-2p+1}(\nu p+1)}{}_2F_1\left(2p,\nu p+1;\nu p+2;\frac{b-a}{b+a}\right)=C_9(\nu).\end{aligned}\tag{41}$$

Similarly,

$$\int_{\frac{1}{2}}^{1} \frac{(u^{\nu} - (1-u)^{\nu})^{p}}{(ub + (1-u)a)^{2p}} du \leq \int_{\frac{1}{2}}^{1} \frac{(2u-1)^{\nu p}}{(ub + (1-u)a)^{2p}} du \tag{42}$$

and putting $u = 1 - w$ on the right-hand side of inequality Equation (42), we have

$$\begin{aligned} \int_{\frac{1}{2}}^{1} \frac{(u^{\nu}-(1-u)^{\nu})^{p}}{(ub+(1-u)a)^{2p}} du &\leq \int_{0}^{\frac{1}{2}} \frac{(1-2w)^{\nu p}}{((1-w)b+wa)^{2p}} dw \\ &= \frac{1}{2} \int_{0}^{1} \frac{(1-w)^{\nu p}}{(\frac{wa}{2}+(1-\frac{w}{2})b)^{2p}} dw. \\ &= \frac{b^{-2p}}{2(\nu p+1)} {}_2F_1(2p, 1; \nu p + 2; \tfrac{1}{2}(1-\tfrac{a}{b})) = C_{10}(\nu). \end{aligned} \tag{43}$$

Using Equations (41) and (43) in Equation (39), we get the result. □

Remark 5. *On giving particular values to parameter in Theorem 6, we have the following results:*

(i) *If we put $\omega = 0$, then we get [2] (Theorem 8).*
(ii) *If we put $\nu = 1$ along with $\omega = 0$, then we get [2] (Corollary 3(1)).*
(iii) *If we put $g(t) = 1$ along with $\omega = 0$, then we get [2] (Corollary 3(2)).*
(iv) *If we put $\nu = 1$, $g(t) = 1$ along with $\omega = 0$, then we get [2] (Corollary 3(3)).*

3. Conclusions

We have obtained a generalized Fejér–Hadamard inequality for harmonically convex functions via a generalized fractional integral operator. This inequality includes several inclusions—for example, Fejér–Hadamard and Hermite-Hadamarad inequalities for harmonically convex functions via Riemann–Liouville fractional integrals. Taking different specific values of parameters in the generalized Mittag–Leffler function, one can obtain results for some known fractional integral operators—for example, for fractional integral operators defined in [8,9]. In addition, we have established some bounds of the difference of the generalized Fejér–Hadamard inequality, in particular several bounds for particular values of parameters involved in the generalized Mittag–Leffler function.

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