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Some Simultaneous Generalizations of Well-Known Fixed Point Theorems and Their Applications to Fixed Point Theory

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Abstract: In this paper, we first establish a new fixed point theorem that generalizes and unifies a number of well-known fixed point results, including the Banach contraction principle, Kannan's fixed point theorem, Chatterjea fixed point theorem, Du-Rassias fixed point theorem and many others. The presented results not only unify and generalize the existing results, but also yield several new fixed point theorems, which are different from the well-known results in the literature.

Keywords: Banach contraction principle; Kannan's fixed point theorem; Chatterjea's fixed point theorem; Du–Rassias's fixed point theorem; simultaneous generalization; $\mathcal{MT}(\lambda)$-function

MSC: 37C25; 47H10; 54H25

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1. Introduction and Preliminaries

Let (X, d) be a metric space and $T : X \to X$ be a self-mapping. A point x in X is said to be a fixed point of T if $Tx = x$. The set of fixed points of T is denoted by $\mathcal{F}(T)$. Throughout this paper, we denote the sets of positive integers and real numbers by $\mathbb{N}$ and $\mathbb{R}$, respectively.

The celebrated Banach contraction principle [1] not only initiated the metric fixed point theory, but also has played an indispensable role in the development of nonlinear functional analysis and applied mathematical analysis. Due to the importance and application potential to several quantitative sciences, the generalizations of the Banach contraction mapping principle have been investigated heavily by several authors in various distinct directions; see, e.g., [2–22] and the related references therein.

Theorem 1 (Banach contraction principle [1])**.** *Let (X, d) be a complete metric space. Suppose that $T : X \to X$ is a Banach type contraction; that is, there exists a nonnegative number $\gamma < 1$ such that*

$$d(Tx, Ty) \le \gamma d(x, y) \quad \textit{for all } x, y \in X.$$

Then, T has a unique fixed point in X.

In 1969, Kannan [23] established the following interesting fixed point theorem, which is different from the Banach contraction principle.

Theorem 2 (Kannan [23]). *Let (X,d) be a complete metric space. Suppose that $T: X \to X$ is a Kannan type contraction; that is, there exists $\lambda \in \left[0, \frac{1}{2}\right)$ such that*

$$d(Tx, Ty) \leq \lambda(d(x, Tx) + d(y, Ty)) \quad \text{for all } x, y \in X.$$

Then, Tadmits a unique fixed point in X.

Chatterjea [24] proved his interesting fixed point theorem in 1972.

Theorem 3 (Chatterjea [24]). *Let (X,d) be a complete metric space. Suppose that $T: X \to X$ is a Chatterjea type contraction; that is, there exists $\lambda \in \left[0, \frac{1}{2}\right)$ such that*

$$d(Tx, Ty) \leq \lambda(d(x, Ty) + d(y, Tx)) \quad \text{for all } x, y \in X.$$

Then, T admits a unique fixed point in X.

Remark 1. *It is worth mentioning that, in a metric space (X,d), Banach's type contraction, the Kannan type contraction and the Chatterjea type contraction are independent and different from each other. For instance, let $X = [-1,1]$ with the metric $d(x,y) = |x-y|$ for $x, y \in X$, and define a mapping $T: X \to X$ by*

$$Tx = -\frac{1}{2}x \quad \text{for all } x \in X.$$

Then, T is not only a Banach type contraction, but also a Kannan type contraction. However, T is not a Chatterjea type contraction. More examples of the difference between the three contractions can be found in [12].

Let (X,d) be a metric space and $T: X \to X$ be a self-mapping. In [12], Du and Rassias defined some new mappings from $X \times X$ to $[0, \infty)$ as follows:

- $K(x,y) = \frac{d(x,Tx)+d(y,Ty)}{2}$ (Kannan type),
- $C(x,y) = \frac{d(x,Ty)+d(y,Tx)}{2}$ (Chatterjea type),
- $I(x,y) = \frac{d(x,Tx)+d(y,Tx)}{2}$,
- $J(x,y) = \frac{d(y,Tx)+d(y,Ty)}{2}$,
- $M(x,y) = \frac{d(x,Tx)+d(y,Ty)+d(y,Tx)}{3}$,
- $P(x,y) = \frac{d(x,Tx)+d(x,Ty)+d(y,Tx)}{3}$,
- $Q(x,y) = \frac{d(y,Ty)+d(x,Ty)+d(y,Tx)}{3}$,
- $U(x,y) = \frac{d(x,Tx)+d(y,Ty)+d(x,Ty)+d(y,Tx)}{4}$,
- $V(x,y) = \frac{d(x,y)+d(x,Tx)+d(y,Ty)+d(x,Ty)+d(y,Tx)}{5}$.

Very recently, Du and Rassias [12] established the following new fixed point theorem for a Meir-Keeler type condition, which is a simultaneous generalization of the Banach contraction principle, Kannan fixed point theorem, Chatterjea fixed point theorem and some known results in the literature.

Theorem 4 (Du and Rassias [12]). *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping. Define a mapping $S : X \times X \to [0, \infty)$ by*

$$S(x, y) = \max\{d(x, y), K(x, y), C(x, y), I(x, y), J(x, y), M(x, y), P(x, y), Q(x, y), U(x, y), V(x, y)\}.$$

Suppose that

(DR) *for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for $x, y \in X$,*

$$\varepsilon \leq S(x, y) < \varepsilon + \delta \quad \textit{implies} \quad d(Tx, Ty) < \varepsilon.$$

Then, T admits a unique fixed point in X.

Theorem 5 (Du and Rassias [12]). *Let (X, d) be a complete metric space, $T : X \to X$ be a self-mapping and $S : X \times X \to [0, \infty)$ be a mapping as in Theorem 4, suppose that there exists a nonnegative real number $\lambda < 1$ such that*

$$d(Tx, Ty) \leq \lambda S(x, y) \ \textit{ for all } x, y \in X.$$

Then, T admits a unique fixed point in X.

In 2016, Du introduced the concept of the $\mathcal{MT}(\lambda)$-function [4] as follows (see also [6–11]).

Definition 1. *Let $\lambda > 0$. A function $\mu : [0, \infty) \to [0, \lambda)$ is said to be an $\mathcal{MT}(\lambda)$-function [4,6–11] if $\limsup_{s \to t^+} \mu(s) < \lambda$ for all $t \in [0, \infty)$. In particular, if $\lambda = 1$, then $\mu : [0, \infty) \to [0, 1)$ is called an $\mathcal{MT}$-function (or $\mathcal{R}$-function) [2–11].*

The following useful characterizations of $\mathcal{MT}(\lambda)$-functions were established by Du in 2016; see [4,6–11].

Theorem 6 (See ([4], Theorem 2.4)). *Let $\lambda > 0$, and let $\mu : [0, \infty) \to [0, \lambda)$ be a function. Then, the following statements are equivalent.*

(1) *μ is an $\mathcal{MT}(\lambda)$-function.*
(2) *$\lambda^{-1}\mu$ is an $\mathcal{MT}$-function.*
(3) *For each $t \in [0, \infty)$, there exist $\xi_t^{(1)} \in [0, \lambda)$ and $\epsilon_t^{(1)} > 0$ such that $\mu(s) \leq \xi_t^{(1)}$ for all $s \in (t, t + \epsilon_t^{(1)})$.*
(4) *For each $t \in [0, \infty)$, there exist $\xi_t^{(2)} \in [0, \lambda)$ and $\epsilon_t^{(2)} > 0$ such that $\mu(s) \leq \xi_t^{(2)}$ for all $s \in [t, t + \epsilon_t^{(2)}]$.*
(5) *For each $t \in [0, \infty)$, there exist $\xi_t^{(3)} \in [0, \lambda)$ and $\epsilon_t^{(3)} > 0$ such that $\mu(s) \leq \xi_t^{(3)}$ for all $s \in (t, t + \epsilon_t^{(3)}]$.*
(6) *For each $t \in [0, \infty)$, there exist $\xi_t^{(4)} \in [0, \lambda)$ and $\epsilon_t^{(4)} > 0$ such that $\mu(s) \leq \xi_t^{(4)}$ for all $s \in [t, t + \epsilon_t^{(4)})$.*
(7) *For any nonincreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n \in \mathbb{N}} \mu(x_n) < \lambda$.*
(8) *For any strictly-decreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n \in \mathbb{N}} \mu(x_n) < \lambda$.*
(9) *For any eventually nonincreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ (i.e., there exists $\ell \in \mathbb{N}$ such that $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$ with $n \geq \ell$) in $[0, \infty)$, we have $0 \leq \sup_{n \in \mathbb{N}} \mu(x_n) < \lambda$.*
(10) *For any eventually strictly-decreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ (i.e., there exists $\ell \in \mathbb{N}$ such that $x_{n+1} < x_n$ for all $n \in \mathbb{N}$ with $n \geq \ell$) in $[0, \infty)$, we have $0 \leq \sup_{n \in \mathbb{N}} \mu(x_n) < \lambda$.*

In this paper, we establish a new fixed point theorem, which is a simultaneous generalization of the Banach contraction principle, Kannan's fixed point theorem, Chatterjea's fixed point theorem, Du–Rassias's fixed point theorem and other new interesting fixed point theorems. Applying our new results, we obtain many new fixed point theorems. The presented results not only unify and generalize the existing results, but also yield several new fixed point theorems, which are different from the well-known results in the literature.

2. New Simultaneous Generalizations with Applications to Fixed Point Theory

Let (X, d) be a metric space and $T : X \to X$ be a self-mapping. In this section, we shall suggest some new mappings from $X \times X$ to $[0, \infty)$ as follows:

- $BC(x,y) = \frac{d(x,y)+d(y,Tx)}{2}$,
- $BK_1(x,y) = \frac{d(x,y)+d(x,Tx)}{2}$,
- $BK_2(x,y) = \frac{d(x,y)+d(y,Ty)}{2}$,
- $BM_1(x,y) = \frac{d(x,y)+d(y,Ty)+d(y,Tx)}{3}$,
- $BM_2(x,y) = \frac{d(x,y)+d(x,Tx)+d(y,Ty)}{3}$,
- $BM_3(x,y) = \frac{d(x,y)+d(x,Tx)+d(y,Tx)}{3}$,
- $BP_1(x,y) = \frac{d(x,y)+d(x,Ty)+d(y,Tx)}{3}$,
- $BP_2(x,y) = \frac{d(x,y)+d(x,Tx)+d(y,Tx)}{3}$,
- $BQ_1(x,y) = \frac{d(x,y)+d(y,Ty)+d(y,Tx)}{3}$,
- $BQ_2(x,y) = \frac{d(x,y)+d(x,Ty)+d(y,Tx)}{3}$,
- $BU_1(x,y) = \frac{d(x,y)+d(y,Ty)+d(x,Ty)+d(y,Tx)}{4}$,
- $BU_2(x,y) = \frac{d(x,y)+d(x,Tx)+d(y,Ty)+d(y,Tx)}{4}$,
- $BU_3(x,y) = \frac{d(x,y)+d(x,Tx)+d(x,Ty)+d(y,Tx)}{4}$.

In what follows, we establish some generalizations of the Du–Rassias fixed point theorem (i.e., Theorem 5). These new results also simultaneously generalize the Banach contraction principle, the Kannan fixed point theorem, the Chatterjea fixed point theorem and some other interesting results in the literature.

Theorem 7. *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping. Suppose that*

(H) *there exists an $\mathcal{MT}$-function $\varphi : [0, \infty) \to [0, 1)$ such that*

$$\min\{d(Tx, Ty), d(y, Ty)\} \le \varphi(d(x, y))S(x, y) \tag{1}$$

for all $x, y \in X$, where

$$\begin{aligned} S(x,y) = \max\{&d(x,y), C(x,y), BC(x,y), K(x,y), BK_1(x,y), BK_2(x,y), \\ &I(x,y), J(x,y), M(x,y), BM_1(x,y), BM_2(x,y), BM_3(x,y), \\ &P(x,y), BP_1(x,y), BP_2(x,y), Q(x,y), BQ_1(x,y), BQ_2(x,y), \\ &U(x,y), BU_1(x,y), BU_2(x,y), BU_3(x,y), V(x,y)\}. \end{aligned}$$

Then, T posses a fixed point in X. Moreover, the sequence $\{T^n x\}$ converges to a fixed point of T for any $x \in X$.

Proof. Let $z \in X$ be given. If $Tz = z$, then $z \in \mathcal{F}(T)$, and we are done. Otherwise, if $Tz \neq z$, we define a sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_1 = z$ and $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If there exists $k \in \mathbb{N}$ such that $x_{k+1} = x_k$, then $x_k \in \mathcal{F}(T)$, and the desired conclusion is proven. For this reason, we henceforth will assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, we have

- $C(x_n, x_{n+1}) = \frac{d(x_n,x_{n+2})}{2} \leq \frac{d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})}{2}$,
- $BC(x_n, x_{n+1}) = \frac{d(x_n,x_{n+1})}{2}$,
- $K(x_n, x_{n+1}) = \frac{d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})}{2}$,
- $BK_1(x_n, x_{n+1}) = d(x_n, x_{n+1})$,
- $BK_2(x_n, x_{n+1}) = \frac{d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})}{2}$,
- $I(x_n, x_{n+1}) = \frac{d(x_n,x_{n+1})}{2}$,
- $J(x_n, x_{n+1}) = \frac{d(x_{n+1},x_{n+2})}{2}$,
- $M(x_n, x_{n+1}) = \frac{d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})}{3}$,
- $BM_1(x_n, x_{n+1}) = \frac{d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})}{3}$,
- $BM_2(x_n, x_{n+1}) = \frac{2d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})}{3}$,
- $BM_3(x_n, x_{n+1}) = \frac{2d(x_n,x_{n+1})}{3}$,
- $P(x_n, x_{n+1}) = \frac{d(x_n,x_{n+1})+d(x_n,x_{n+2})}{3} \leq \frac{2d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})}{3}$,
- $BP_1(x_n, x_{n+1}) = \frac{d(x_n,x_{n+1})+d(x_n,x_{n+2})}{3} \leq \frac{2d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})}{3}$,
- $BP_2(x_n, x_{n+1}) = \frac{2d(x_n,x_{n+1})}{3}$,
- $Q(x_n, x_{n+1}) = \frac{d(x_{n+1},x_{n+2})+d(x_n,x_{n+2})}{3} \leq \frac{d(x_n,x_{n+1})+2d(x_{n+1},x_{n+2})}{3}$,
- $BQ_1(x_n, x_{n+1}) = \frac{d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})}{3}$,
- $BQ_2(x_n, x_{n+1}) = \frac{d(x_n,x_{n+1})+d(x_n,x_{n+2})}{3} \leq \frac{2d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})}{3}$,
- $U(x_n, x_{n+1}) = \frac{d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+d(x_n,x_{n+2})}{4} \leq \frac{d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})}{2}$,
- $BU_1(x_n, x_{n+1}) = \frac{d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+d(x_n,x_{n+2})}{4} \leq \frac{d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})}{2}$,
- $BU_2(x_n, x_{n+1}) = \frac{2d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})}{4}$,
- $BU_3(x_n, x_{n+1}) = \frac{2d(x_n,x_{n+1})+d(x_n,x_{n+2})}{4} \leq \frac{3d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})}{4}$,
- $V(x_n, x_{n+1}) = \frac{2d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+d(x_n,x_{n+2})}{5} \leq \frac{3d(x_n,x_{n+1})+2d(x_{n+1},x_{n+2})}{5}$.

Assume that there exists $j \in \mathbb{N}$ such that $d(x_j, x_{j+1}) < d(x_{j+1}, x_{j+2})$. Then, by the definition of S and the above, we have $S(x_j, x_{j+1}) < d(x_{j+1}, x_{j+2})$. By condition (H), we have

$$\begin{aligned} d(x_{j+1}, x_{j+2}) &= \min\{d(Tx_j, Tx_{j+1}), d(x_{j+1}, Tx_{j+1})\} \\ &\leq \varphi(d(x_j, x_{j+1}))S(x_j, x_{j+1}) \\ &< d(x_{j+1}, x_{j+2}), \end{aligned}$$

a contradiction. Therefore, it must be $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1})$, and hence

$$S(x_n, x_{n+1}) \leq d(x_n, x_{n+1}), \tag{2}$$

for all $n \in \mathbb{N}$. By condition (H) and (2), we find that

$$d(x_{n+1}, x_{n+2}) \leq \varphi(d(x_n, x_{n+1}))d(x_n, x_{n+1}) \quad \text{for all } n \in \mathbb{N}. \tag{3}$$

By (3), we know that the sequence $\{d(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ is strictly decreasing in $[0,\infty)$. Since φ is an $\mathcal{MT}$-function, by applying Theorem 6 with $\lambda = 1$, we obtain $0 \le \sup\limits_{n\in\mathbb{N}} \varphi(d(x_n, x_{n+1})) < 1$. Let:

$$\gamma := \sup_{n\in\mathbb{N}} \varphi(d(x_n, x_{n+1})). \tag{4}$$

Then, we conclude that $\gamma \in [0,1)$. As a next step, we claim that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Indeed, for any $n \in \mathbb{N}$, by (3) and (4), we get

$$d(x_{n+1}, x_{n+2}) \le \varphi(d(x_n, x_{n+1}))d(x_n, x_{n+1}) \ \le \gamma d(x_n, x_{n+1})$$

and hence

$$d(x_{n+1}, x_{n+2}) \le \gamma d(x_n, x_{n+1}) \le \cdots \le \gamma^n d(x_1, x_2). \tag{5}$$

Let $\alpha_n = \frac{\gamma^{n-1}}{1-\gamma} d(x_1, x_2)$, $n \in \mathbb{N}$. For $m, n \in \mathbb{N}$ with $m > n$, from (5), we obtain:

$$d(x_m, x_n) \le \sum_{j=n}^{m-1} d(x_{j+1}, x_j) \le \alpha_n. \tag{6}$$

Since $0 \le \gamma < 1$, we have $\lim\limits_{n\to\infty} \alpha_n = 0$. Hence, by (6), we find:

$$\lim_{n\to\infty} \sup\{d(x_m, x_n) : m > n\} = 0,$$

which shows that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. By the completeness of X, there exists $v \in X$ such that $x_n \to v$ as $n \to \infty$. In order to finish the proof, it is sufficient to show $v \in \mathcal{F}(T)$. For any $n \in \mathbb{N}$, we have

- $C(x_n, v) = \frac{d(x_n, Tv) + d(v, x_{n+1})}{2}$,
- $BC(x_n, v) = \frac{d(x_n, v) + d(v, x_{n+1})}{2}$,
- $K(x_n, v) = \frac{d(x_n, x_{n+1}) + d(v, Tv)}{2}$,
- $BK_1(x_n, v) = \frac{d(x_n, v) + d(x_n, x_{n+1})}{2}$,
- $BK_2(x_n, v) = \frac{d(x_n, v) + d(v, Tv)}{2}$,
- $I(x_n, v) = \frac{d(x_n, x_{n+1}) + d(v, x_{n+1})}{2}$,
- $J(x_n, v) = \frac{d(v, x_{n+1}) + d(v, Tv)}{2}$,
- $M(x_n, v) = \frac{d(x_n, x_{n+1}) + d(v, Tv) + d(v, x_{n+1})}{3}$,
- $BM_1(x_n, v) = \frac{d(x_n, v) + d(v, Tv) + d(v, x_{n+1})}{3}$,
- $BM_2(x_n, v) = \frac{d(x_n, v) + d(x_n, x_{n+1}) + d(v, Tv)}{3}$,
- $BM_3(x_n, v) = \frac{d(x_n, v) + d(x_n, x_{n+1}) + d(v, x_{n+1})}{3}$,
- $P(x_n, v) = \frac{d(x_n, x_{n+1}) + d(x_n, Tv) + d(v, x_{n+1})}{3}$,
- $BP_1(x_n, v) = \frac{d(x_n, v) + d(x_n, Tv) + d(v, x_{n+1})}{3}$,
- $BP_2(x_n, v) = \frac{d(x_n, v) + d(x_n, x_{n+1}) + d(v, x_{n+1})}{3}$,
- $Q(x_n, v) = \frac{d(v, Tv) + d(x_n, Tv) + d(v, x_{n+1})}{3}$,
- $BQ_1(x_n, v) = \frac{d(x_n, v) + d(v, Tv) + d(v, x_{n+1})}{3}$,

- $BQ_2(x_n, v) = \frac{d(x_n,v)+d(x_n,Tv)+d(v,x_{n+1})}{3}$,
- $U(x_n, v) = \frac{d(x_n,x_{n+1})+d(v,Tv)+d(x_n,Tv)+d(v,x_{n+1})}{4}$,
- $BU_1(x_n, v) = \frac{d(x_n,v)+d(v,Tv)+d(x_n,Tv)+d(v,x_{n+1})}{4}$,
- $BU_2(x_n, v) = \frac{d(x_n,v)+d(x_n,x_{n+1})+d(v,Tv)+d(v,x_{n+1})}{4}$,
- $BU_3(x_n, v) = \frac{d(x_n,v)+d(x_n,x_{n+1})+d(x_n,Tv)+d(v,x_{n+1})}{4}$,
- $V(x_n, v) = \frac{d(x_n,v)+d(x_n,x_{n+1})+d(v,Tv)+d(x_n,Tv)+d(v,x_{n+1})}{5}$,

which yields that

$$\lim_{n\to\infty} S(x_n, v) = \frac{2}{3}d(v, Tv). \tag{7}$$

Since

$$\min\{d(Tx_n, Tv), d(v, Tv)\} \leq \varphi(d(x_n, v))S(x_n, v) \quad \text{for all } n \in \mathbb{N},$$

by taking the limit as $n \to \infty$ on the last inequality and using (7), we get

$$d(v, Tv) \leq \frac{2}{3}d(v, Tv)$$

which implies $d(v, Tv) = 0$, and hence, $v \in \mathcal{F}(T)$. The proof is completed. □

By applying Theorem 7, we obtain the following new unique fixed point theorem.

Theorem 8. *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping on X. Let $S : X \times X \to [0, \infty)$ be a mapping as in Theorem 7. Suppose that:*

$(h1)$ *there exists an $\mathcal{MT}$-function $\varphi : [0, \infty) \to [0, 1)$ such that:*

$$d(Tx, Ty) \leq \varphi(d(x, y))S(x, y)$$

for all $x, y \in X$.

Then, T admits a unique fixed point v in X. Moreover, the sequence $\{T^n x\}$ converges to v for any $x \in X$.

Proof. Clearly, $(h1)$ implies (H). Applying Theorem 7, $\mathcal{F}(T) \neq \emptyset$, and the sequence $\{T^n x\}$ converges to a fixed point of T for any $x \in X$. We claim that $\mathcal{F}(T)$ is a singleton set. Suppose there exist $u, v \in \mathcal{F}(T)$ with $u \neq v$. Thus, $d(u, v) > 0$. Since

$$K(u, v) = 0,$$

$$C(u, v) = BC(u, v) = BP_1(u, v) = BQ_2(u, v) = d(u, v),$$

$$BK_1(u, v) = BK_2(u, v) = I(u, v) = J(u, v) = U(u, v) = BU_2(u, v) = \frac{1}{2}d(u, v),$$

$$M(u, v) = BM_2(u, v) = \frac{1}{3}d(u, v)$$

$$BM_1(u, v) = BM_3(u, v) = P(u, v) = BP_2(u, v) = Q(u, v) = BQ_1(u, v) = \frac{2}{3}d(u, v),$$

$$BU_1(u, v) = BU_3(u, v) = \frac{3}{4}d(u, v)$$

and

$$V(u, v) = \frac{3}{5}d(u, v),$$

by $(h1)$, we obtain:

$$d(u,v) = d(Tu,Tv) < S(u,v) = d(u,v),$$

a contradiction. Therefore, $\mathcal{F}(T)$ is a singleton set, and T has a unique fixed point in X. The proof is completed. $\square$

Remark 2. *[7, Theorem 2.1], [8, Theorem 2.1] and Theorems 1–3 and 5 are all special cases of Theorem 8.*

The following generalized Banach contraction principle, generalized Kannan fixed point theorem and generalized Chatterjea fixed point theorem are immediate consequences of Theorems 7 and 8.

Corollary 1 (Generalized Banach contraction principle). *Let (X,d) be a complete metric space and $T : X \to X$ be a self-mapping. Suppose that there exist a mapping $W : X \times X \to [0,\infty)$ and an $\mathcal{MT}$-function $\kappa : [0,\infty) \to [0,1)$ such that*

$$W(x,y) \le \kappa(d(x,y))d(x,y) \quad \textit{for all } x,y \in X.$$

Then, the following conclusions hold:

- (a) *If $W(x,y) = \min\{d(Tx,Ty), d(y,Ty)\}$ for all $x,y \in X$, then T admits a fixed point in X. Moreover, the sequence $\{T^n x\}$ converges to a fixed point of T for any $x \in X$.*
- (b) *If $W(x,y) = d(y,Ty)$ for all $x,y \in X$, then T admits a fixed point in X. Moreover, the sequence $\{T^n x\}$ converges to a fixed point of T for any $x \in X$.*
- (c) *If $W(x,y) = d(Tx,Ty)$ for all $x,y \in X$, then T admits a unique fixed point v in X. Moreover, the sequence $\{T^n x\}$ converges to v for any $x \in X$.*

Corollary 2 (Generalized Kannan fixed point theorem). *Let (X,d) be a complete metric space and $T : X \to X$ be a self-mapping. Suppose that there exist a mapping $W : X \times X \to [0,\infty)$ and an $\mathcal{MT}\left(\frac{1}{2}\right)$-function $\alpha : [0,\infty) \to \left[0,\frac{1}{2}\right)$ such that*

$$W(x,y) \le \alpha(d(x,y))[d(x,Tx) + d(y,Ty)] \quad \textit{for all } x,y \in X.$$

Then, the following conclusions hold:

- (a) *If $W(x,y) = \min\{d(Tx,Ty), d(y,Ty)\}$ for all $x,y \in X$, then T admits a fixed point in X. Moreover, the sequence $\{T^n x\}$ converges to a fixed point of T for any $x \in X$.*
- (b) *If $W(x,y) = d(y,Ty)$ for all $x,y \in X$, then T admits a fixed point in X. Moreover, the sequence $\{T^n x\}$ converges to a fixed point of T for any $x \in X$.*
- (c) *If $W(x,y) = d(Tx,Ty)$ for all $x,y \in X$, then T admits a unique fixed point v in X. Moreover, the sequence $\{T^n x\}$ converges to v for any $x \in X$.*

Corollary 3 (Generalized Chatterjea fixed point theorem). *Let (X,d) be a complete metric space and $T : X \to X$ be a self-mapping. Suppose that there exist a mapping $W : X \times X \to [0,\infty)$ and an $\mathcal{MT}\left(\frac{1}{2}\right)$-function $\alpha : [0,\infty) \to \left[0,\frac{1}{2}\right)$ such that*

$$W(x,y) \le \alpha(d(x,y))[d(x,Ty) + d(y,Tx)] \quad \textit{for all } x,y \in X.$$

Then, the following conclusions hold:

(a) *If* $W(x,y) = \min\{d(Tx,Ty), d(y,Ty)\}$ *for all* $x, y \in X$*, then* T *admits a fixed point in* X*. Moreover, the sequence* $\{T^n x\}$ *converges to a fixed point of* T *for any* $x \in X$*.*
(b) *If* $W(x,y) = d(y,Ty)$ *for all* $x, y \in X$*, then* T *admits a fixed point in* X*. Moreover, the sequence* $\{T^n x\}$ *converges to a fixed point of* T *for any* $x \in X$*.*
(c) *If* $W(x,y) = d(Tx,Ty)$ *for all* $x, y \in X$*, then* T *admits a unique fixed point* v *in* X*. Moreover, the sequence* $\{T^n x\}$ *converges to* v *for any* $x \in X$*.*

Finally, by applying Theorems 6–8, we can obtain the following new fixed point theorems immediately.

Theorem 9. *Let* (X,d) *be a complete metric space and* $T : X \to X$ *be a self-mapping. Suppose that there exist a mapping* $W : X \times X \to [0,\infty)$ *and an* $\mathcal{MT}\left(\frac{1}{2}\right)$*-function* $\alpha : [0,\infty) \to \left[0, \frac{1}{2}\right)$ *such that one of the following conditions holds:*

(i) $W(x,y) \le \alpha(d(x,y))[d(x,Tx) + d(y,Tx)]$ *for all* $x, y \in X$*,*
(ii) $W(x,y) \le \alpha(d(x,y))[d(y,Tx) + d(y,Ty)]$ *for all* $x, y \in X$*,*
(iii) $W(x,y) \le \alpha(d(x,y))[d(x,y) + d(y,Tx)]$ *for all* $x, y \in X$*,*
(iv) $W(x,y) \le \alpha(d(x,y))[d(x,y) + d(x,Tx)]$ *for all* $x, y \in X$*,*
(v) $W(x,y) \le \alpha(d(x,y))[d(x,y) + d(y,Ty)]$ *for all* $x, y \in X$*.*

Then, the following conclusions hold:

(a) *If* $W(x,y) = \min\{d(Tx,Ty), d(y,Ty)\}$ *for all* $x, y \in X$*, then* T *admits a fixed point in* X*. Moreover, the sequence* $\{T^n x\}$ *converges to a fixed point of* T *for any* $x \in X$*.*
(b) *If* $W(x,y) = d(y,Ty)$ *for all* $x, y \in X$*, then* T *admits a fixed point in* X*. Moreover, the sequence* $\{T^n x\}$ *converges to a fixed point of* T *for any* $x \in X$*.*
(c) *If* $W(x,y) = d(Tx,Ty)$ *for all* $x, y \in X$*, then* T *admits a unique fixed point* v *in* X*. Moreover, the sequence* $\{T^n x\}$ *converges to* v *for any* $x \in X$*.*

Theorem 10. *Let* (X,d) *be a complete metric space and* $T : X \to X$ *be a self-mapping. Suppose that there exist a mapping* $W : X \times X \to [0,\infty)$ *and an* $\mathcal{MT}\left(\frac{1}{3}\right)$*-function* $\beta : [0,\infty) \to \left[0, \frac{1}{3}\right)$ *such that one of the following conditions holds:*

(i) $W(x,y) \le \beta(d(x,y))[d(x,Tx) + d(y,Ty) + d(y,Tx)]$ *for all* $x, y \in X$*,*
(ii) $W(x,y) \le \beta(d(x,y))[d(x,Tx) + d(x,Ty) + d(y,Tx)]$ *for all* $x, y \in X$*,*
(iii) $W(x,y) \le \beta(d(x,y))[d(y,Ty) + d(x,Ty) + d(y,Tx)]$ *for all* $x, y \in X$*,*
(iv) $W(x,y) \le \beta(d(x,y))[d(x,y) + d(y,Ty) + d(y,Tx)]$ *for all* $x, y \in X$*,*
(v) $W(x,y) \le \beta(d(x,y))[d(x,y) + d(x,Tx) + d(y,Ty)]$ *for all* $x, y \in X$*,*
(vi) $W(x,y) \le \beta(d(x,y))[d(x,y) + d(x,Tx) + d(y,Tx)]$ *for all* $x, y \in X$*,*
(vii) $W(x,y) \le \beta(d(x,y))[d(x,y) + d(x,Ty) + d(y,Tx)]$ *for all* $x, y \in X$*,*
(viii) $W(x,y) \le \beta(d(x,y))[d(x,y) + d(x,Tx) + d(y,Tx)]$ *for all* $x, y \in X$*,*
(ix) $W(x,y) \le \beta(d(x,y))[d(x,y) + d(y,Ty) + d(y,Tx)]$ *for all* $x, y \in X$*,*
(x) $W(x,y) \le \beta(d(x,y))[d(x,y) + d(x,Ty) + d(y,Tx)]$ *for all* $x, y \in X$*.*

Then, the following conclusions hold:

(a) *If* $W(x,y) = \min\{d(Tx,Ty), d(y,Ty)\}$ *for all* $x, y \in X$*, then* T *admits a fixed point in* X*. Moreover, the sequence* $\{T^n x\}$ *converges to a fixed point of* T *for any* $x \in X$*.*
(b) *If* $W(x,y) = d(y,Ty)$ *for all* $x, y \in X$*, then* T *admits a fixed point in* X*. Moreover, the sequence* $\{T^n x\}$ *converges to a fixed point of* T *for any* $x \in X$*.*

(c) *If $W(x,y) = d(Tx,Ty)$ for all $x,y \in X$, then T admits a unique fixed point v in X. Moreover, the sequence $\{T^n x\}$ converges to v for any $x \in X$.*

Theorem 11. *Let (X,d) be a complete metric space and $T : X \to X$ be a self-mapping. Suppose that there exist a mapping $W : X \times X \to [0,\infty)$ and an $\mathcal{MT}\left(\frac{1}{4}\right)$-function $\mu : [0,\infty) \to \left[0,\frac{1}{4}\right)$ such that one of the following conditions holds:*

(i) $W(x,y) \le \mu(d(x,y))[d(x,Tx) + d(y,Ty) + d(x,Ty) + d(y,Tx)]$ *for all $x,y \in X$,*
(ii) $W(x,y) \le \mu(d(x,y))[d(x,y) + d(y,Ty) + d(x,Ty) + d(y,Tx)]$ *for all $x,y \in X$,*
(iii) $W(x,y) \le \mu(d(x,y))[d(x,y) + d(x,Tx) + d(y,Ty) + d(y,Tx)]$ *for all $x,y \in X$,*
(iv) $W(x,y) \le \mu(d(x,y))[d(x,y) + d(x,Tx) + d(x,Ty) + d(y,Tx)]$ *for all $x,y \in X$.*

Then, the following conclusions hold:

(a) *If $W(x,y) = \min\{d(Tx,Ty), d(y,Ty)\}$ for all $x,y \in X$, then T admits a fixed point in X. Moreover, the sequence $\{T^n x\}$ converges to a fixed point of T for any $x \in X$.*
(b) *If $W(x,y) = d(y,Ty)$ for all $x,y \in X$, then T admits a fixed point in X. Moreover, the sequence $\{T^n x\}$ converges to a fixed point of T for any $x \in X$.*
(c) *If $W(x,y) = d(Tx,Ty)$ for all $x,y \in X$, then T admits a unique fixed point v in X. Moreover, the sequence $\{T^n x\}$ converges to v for any $x \in X$.*

Theorem 12. *Let (X,d) be a complete metric space and $T : X \to X$ be a self-mapping. Suppose that there exists a mapping $W : X \times X \to [0,\infty)$ and an $\mathcal{MT}\left(\frac{1}{5}\right)$-function $\tau : [0,\infty) \to \left[0,\frac{1}{5}\right)$ such that:*

$$W(x,y) \le \tau(d(x,y))[d(x,y) + d(x,Tx) + d(y,Ty) + d(x,Ty) + d(y,Tx)] \quad \text{for all } x,y \in X.$$

Then, the following conclusions hold:

(a) *If $W(x,y) = \min\{d(Tx,Ty), d(y,Ty)\}$ for all $x,y \in X$, then T admits a fixed point in X. Moreover, the sequence $\{T^n x\}$ converges to a fixed point of T for any $x \in X$.*
(b) *If $W(x,y) = d(y,Ty)$ for all $x,y \in X$, then T admits a fixed point in X. Moreover, the sequence $\{T^n x\}$ converges to a fixed point of T for any $x \in X$.*
(c) *If $W(x,y) = d(Tx,Ty)$ for all $x,y \in X$, then T admits a unique fixed point v in X. Moreover, the sequence $\{T^n x\}$ converges to v for any $x \in X$.*

3. Conclusions

The famous Banach contraction principle, Kannan's fixed point theorem and Chatterjea's fixed point theorem are forceful tools in various fields of nonlinear analysis and applied mathematical analysis. In the present paper, by using $\mathcal{MT}(\lambda)$-functions, we establish some new fixed point theorems, which not only unify a number of well-known fixed point results, including the Banach contraction principle, Kannan's fixed point theorem, Chatterjea fixed point theorem, Du-Rassias fixed point theorem and many others, but also yield several new fixed point theorems, which are different from the well-known results in the literature.

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References

1. Banach, S. Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrales. *Fund. Math.* **1922**, *3*, 133–181.
2. Du, W.-S. Some new results and generalizations in metric fixed point theory. *Nonlinear Anal. Theory Methods Appl.* **2010**, *73*, 1439–1446.
3. Du, W.-S. On coincidence point and fixed point theorems for nonlinear multivalued maps. *Topol. Appl.* **2012**, *159*, 49–56.
4. Du, W.-S. New existence results of best proximity points and fixed points for $\mathcal{MT}(\lambda)$-functions with applications to differential equations. *Linear Nonlinear Anal.* **2016**, *2*, 199–213.
5. Du, W.-S.; Hung, Y.-L. A generalization of Mizoguchi-Takahashi's fixed point theorem and its applications to fixed point theory. *Int. J. Math. Anal.* **2017**, *11*, 151–161.
6. Du, W.-S. New fixed point theorems for hybrid Kannan type mappings and $\mathcal{MT}(\lambda)$-functions. *Nonlinear Anal. Differ. Equ.* **2017**, *5*, 89–98.
7. Du, W.-S. On hybrid Chatterjea type fixed point theorem with $\mathcal{MT}(\lambda)$-functions. *Appl. Math. Sci.* **2017**, *11*, 397–406.
8. Du, W.-S. On simultaneous generalizations of well-known fixed point theorems and others. *Int. J. Math. Anal.* **2017**, *11*, 225–232.
9. Du, W.-S. Simultaneous generalizations of fixed point theorems of Mizoguchi-Takahashi type, Nadler type Banach type, Kannan type and Chatterjea type. *Nonlinear Anal. Differ. Equ.* **2017**, *5*, 171–180.
10. Du, W.-S. New simultaneous generalizations of common fixed point theorems of Kannan type, Chatterjea type and Mizoguchi-Takahashi type. *Appl. Math. Sci.* **2017**, *11*, 995–1005.
11. Du, W.-S. New best proximity point theorems with applications to simultaneous generalizations of common fixed point theorems. *Appl. Math. Sci.* **2017**, *11*, 1143–1154.
12. Du, W.-S.; Rassias, T.M. Simultaneous generalizations of known fixed point theorems for a Meir–Keeler type condition with applications. *Int. J. Nonlinear Anal. Appl.* in press.
13. Goebel, K.; Kirk, W.A. *Topics in Metric Fixed Point Theory*; Cambridge University Press: Cambridge, UK, 1990.
14. Hyers, D.H.; Isac, G.; Rassias, T.M. *Topics in Nonlinear Analysis and Applications*; World Scientific Publ. Co.: Singapore, 1997.
15. Karapınar, E.; Erhan, I.M. Fixed point theorems for operators on partial metric spaces. *Appl. Math. Lett.* **2011**, *24*, 1894–1899.
16. Karapınar, E.; Tas, K. Generalized (C)-conditions and related fixed point theorems. *Comput. Math. Appl.* **2011**, *61*, 3370–3380.
17. Karapınar, E. Fixed point theory for cyclic weak ϕ-contraction. *Appl. Math. Lett.* **2011**, *24*, 822–825 .
18. Karapınar, E.; Agarwal, R.P. A note on Ćirić type nonunique fixed point theorems. *Fixed Point Theory Appl.* **2017**, *2017*, 20.
19. Karapınar, E. A note on common fixed point theorems in partial metric spaces. *Miskolc Math. Notes* **2011**, *12*, 185–191.
20. Khamsi, M.A.; Kirk, W.A. *An Introduction to Metric Spaces and Fixed Point Theory*; Pure and Applied Mathematics, Wiley-Interscience: New York, NY, USA, 2001.
21. Kirk, W.A.; Shahzad, N. *Fixed Point Theory in Distance Spaces*; Springer: Cham, Switzerland, 2014.
22. Reich, S.; Zaslavski, A.J. *Genericity in Nonlinear Analysis*; Springer: New York, NY, USA, 2014.
23. Kannan, R. Some results on fixed point - II. *Amer. Math. Monthly* **1969**, *76*, 405–408.
24. Chatterjea, S.K. Fixed-point theorems. *C.R. Acad. Bulgare Sci.* **1972**, *25*, 727–730.