

# On Generalized Roughness in LA-Semigroups

Noor Rehman <sup>1</sup>, Choonkil Park <sup>2,\*</sup>, Syed Inayat Ali Shah <sup>3</sup> and Abbas Ali <sup>1</sup>

<sup>1</sup> Department of Mathematics and Statistics, Riphah International University, Hajj Complex I-14, Islamabad 44000, Pakistan; noorrehman82@yahoo.com (N.R.); abbasali5068@gmail.com (A.A.)

<sup>2</sup> Department of Mathematics, Research Institute of Natural Sciences, Hanyang University, Seoul 04763, Korea

<sup>3</sup> Department of Mathematics, Islamia College University, Peshawar 25120, Pakistan; inayat64@gmail.com

\* Correspondence: baak@hanyang.ac.kr; Tel.: +82-2-2220-0892; Fax: +82-2-2281-0019

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**Abstract:** The generalized roughness in LA-semigroups is introduced, and several properties of lower and upper approximations are discussed. We provide examples to show that the lower approximation of a subset of an LA-semigroup may not be an LA-subsemigroup/ideal of LA-semigroup under a set valued homomorphism.

**Keywords:** roughness; generalized roughness; LA-semigroup

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## 1. Introduction

The algebraic structure of a left almost semigroup, abbreviated as an LA-semigroup, has been introduced by Naseerudin and Kazim in [1]. Later, Mushtaq and others investigated the structure of LA-semigroups and added some important results related to LA-semigroups (see [2–7]). LA-semigroups are also called AG-groupoids. Ideal theory, which was introduced in [8], plays a basic role in the study of LA-semigroups. Pawlak was the first to discuss rough sets with the help of equivalence relation among the elements of a set, which is a key point in discussing the uncertainty [9]. There are at least two methods for the development of rough set theory, the constructive and axiomatic approaches. In constructive methods, lower and upper approximations are constructed from the primitive notions, such as equivalence relations on a universe and neighborhood system. In rough sets, equivalence classes play an important role in the construction of both lower and upper approximations (see [10]). But sometimes in algebraic structures, as is the case in LA-semigroups, finding equivalence relations is too difficult. Many authors have worked on this to initiate rough sets without equivalence relations. Couso and Dubois in [11] initiated a generalized rough set or a  $T$ -rough set with the help of a set valued mapping. It is a more generalized rough set compared with the Pawlak rough set.

In this paper, we initiate the study of generalized roughness in LA-semigroups and of generalized rough sets applied in the crisp form of LA-semigroups. Approximations of LA-subsemigroups and approximations of ideals in LA-semigroups are given.

## 2. Preliminaries

A groupoid  $(S, *)$  is called an LA-semigroup if it satisfies the left invertive law

$$(a * b) * c = (c * b) * a \text{ for all } a, b, c \in S.$$

Throughout the paper,  $S$  and  $R$  will denote LA-semigroups unless stated otherwise. Let  $S$  be an LA-semigroup and  $A$  be a subset of  $S$ . Then  $A$  is called an LA-subsemigroup of  $S$  if  $A^2 \subseteq A$ , that is,  $ab \in A$  for all  $a, b \in A$ . A subset  $A$  of  $S$  is called left ideal (or right ideal) of  $S$  if  $SA \subseteq A$  (or  $AS \subseteq A$ ).

An LA-subsemigroup  $A$  of  $S$  is called bi-ideal of  $S$  if  $(AS)A \subseteq A$ . An LA-subsemigroup  $A$  of  $S$  is called an interior ideal of  $S$  if  $(SA)S \subseteq A$ . An element  $a$  of  $S$  is called idempotent, if  $a^2 = a$  for all  $a \in S$ . If every element of  $S$  is an idempotent, then  $S$  is idempotent.

### 3. Rough Sets

In this section, we study Pawlak roughness and generalized roughness in LA-semigroups.

#### 3.1. Pawlak Approximations in LA-Semigroups

The concept of a rough set was introduced by Pawlak in [9]. According to Pawlak, rough set theory is based on the approximations of a set by a pair of sets called lower approximation and upper approximation of that set. Let  $U$  be a nonempty finite set with an equivalence relation  $R$ . We say  $(U, R)$  is the approximation space. If  $A \subseteq U$  can be written as the union of some classes obtained from  $R$ , then  $A$  is called definable; otherwise, it is not definable. Therefore, the approximations of  $A$  are as follows:

$$\underline{R}(A) = \{x \in U : [x]_R \subseteq A\}$$

$$\overline{R}(A) = \{x \in U : [x]_R \cap A \neq \emptyset\}.$$

The pair  $(\underline{R}(A), \overline{R}(A))$  is a rough set, where  $\underline{R}(A) \neq \overline{R}(A)$ .

**Definition 1.** [5] Let  $\rho$  be an equivalence relation on  $S$ . Then  $\rho$  is called a congruence relation on  $S$  if

$$(a, b) \in \rho \text{ implies that } (ay, by) \in \rho \text{ and } (ya, yb) \in \rho \text{ for all } a, b, y \in S.$$

**Definition 2.** [8] Let  $\rho$  be a congruence relation on  $S$ . Then the approximation of  $S$  is defined by  $\rho(A) = (\underline{\rho}(A), \overline{\rho}(A))$  for every  $A \in \mathcal{P}(S)$ , where  $\mathcal{P}(S)$  is the power set of  $S$ , and

$$\underline{\rho}(A) = \{x \in U : [x]_{\rho} \subseteq A\}$$

and

$$\overline{\rho}(A) = \{x \in U : [x]_{\rho} \cap A \neq \emptyset\}.$$

#### 3.2. Generalized Roughness or T-Roughness in LA-Semigroups

A generalized rough set is the generalization of Pawlak's rough set. In this case, we use set valued mappings instead of congruence classes.

**Definition 3.** [11] Let  $X$  and  $Y$  be two nonempty sets and  $B \subseteq Y$ . Let  $T : X \rightarrow \mathcal{P}(Y)$  be a set valued (SV) mapping, where  $\mathcal{P}(Y)$  denotes the set of all nonempty subsets of  $Y$ . The upper approximation and the lower approximation of  $B$  with respect to  $T$  are defined by

$$\overline{T}(B) = \{x \in X \mid T(x) \cap B \neq \emptyset\}$$

and

$$\underline{T}(B) = \{x \in X \mid T(x) \subseteq B\}.$$

**Definition 4.** [12] Let  $X$  and  $Y$  be two nonempty sets and  $B \subseteq Y$ . Let  $T : X \rightarrow \mathcal{P}(Y)$  be an SV mapping, where  $\mathcal{P}(Y)$  denotes the set of all nonempty subsets of  $Y$ . Then  $(\underline{T}(B), \overline{T}(B))$  is called a  $T$ -rough set.

**Definition 5.** Let  $R$  and  $S$  be two LA-semigroups and  $T : R \rightarrow \mathcal{P}(S)$  be an SV mapping. Then  $T$  is called an SV homomorphism if  $T(a)T(b) \subseteq T(ab)$  for all  $a, b \in R$ .

**Example 1.** Let  $R = \{a, b, c\}$  with the following multiplication table:

$\cdot$	$a$	$b$	$c$
$a$	$a$	$a$	$a$
$b$	$c$	$c$	$c$
$c$	$a$	$a$	$c$

Then  $R$  is an LA-semigroup. Define an SV mapping  $T : R \rightarrow \mathcal{P}(R)$  by  $T(a) = T(c) = \{a, b, c\}$  and  $T(b) = \{b, c\}$ . Then clearly  $T$  is an SV homomorphism.

**Example 2.** Let  $S = \{a, b, c, d, e\}$  with the following multiplication table:

$\cdot$	$a$	$b$	$c$	$d$	$e$
$a$	$e$	$b$	$a$	$b$	$c$
$b$	$b$	$b$	$b$	$b$	$b$
$c$	$c$	$b$	$e$	$b$	$a$
$d$	$b$	$b$	$b$	$b$	$b$
$e$	$a$	$b$	$c$	$b$	$e$

Then  $S$  is an LA-semigroup. Define an SV mapping  $T : S \rightarrow \mathcal{P}(S)$  by  $T(a) = T(b) = T(c) = T(e) = \{a, b, c, d, e\}$  and  $T(d) = \{b, d\}$ . Clearly  $T$  is an SV homomorphism.

**Definition 6.** Let  $R$  and  $S$  be two LA-semigroups and  $T : R \rightarrow \mathcal{P}(S)$  be an SV mapping. Then  $T$  is called a strong set valued (SSV) homomorphism if  $T(a)T(b) = T(ab)$  for all  $a, b \in R$ .

**Example 3.** Let  $R = \{a, b, c\}$  with the following multiplication table:

$\cdot$	$a$	$b$	$c$
$a$	$a$	$a$	$a$
$b$	$c$	$c$	$c$
$c$	$a$	$a$	$c$

Then  $R$  is an LA-semigroup and  $S = \{a, b, c\}$  with the following multiplication table:

$\cdot$	$a$	$b$	$c$
$a$	$b$	$c$	$b$
$b$	$c$	$c$	$c$
$c$	$c$	$c$	$c$

Then  $S$  is an LA-semigroup. Define an SV mapping  $T : R \rightarrow \mathcal{P}(S)$  by  $T(a) = T(c) = \{c\}$  and  $T(b) = \{b, c\}$ . Then  $T$  is an SSV homomorphism.

**Proposition 1.** Let  $T : R \rightarrow \mathcal{P}(S)$  be an SV homomorphism. If  $\emptyset \neq A, B \subseteq S$ , then  $\overline{T}(A)\overline{T}(B) \subseteq \overline{T}(AB)$ .

**Proof.** Let  $x \in \overline{T}(A)\overline{T}(B)$ . Then  $x = ab$ , where  $a \in \overline{T}(A)$  and  $b \in \overline{T}(B)$ . Then  $T(a) \cap A \neq \emptyset$  and  $T(b) \cap B \neq \emptyset$ . Therefore, there exist  $y, z \in S$  such that  $y \in T(a) \cap A$  and  $z \in T(b) \cap B$ , which implies that  $y \in T(a)$ ,  $y \in A$ ,  $z \in T(b)$ , and  $z \in B$ . It follows that  $yz \in T(a)T(b) \subseteq T(ab)$  and  $yz \in AB$ . Thus,  $yz \in T(ab) \cap AB$ , so  $T(ab) \cap AB \neq \emptyset$ . It follows that  $ab \in \overline{T}(AB)$ . Hence,  $x \in \overline{T}(AB)$ ; therefore,  $\overline{T}(A)\overline{T}(B) \subseteq \overline{T}(AB)$ .  $\square$

The following example shows that equality in Proposition 1 may not hold.

**Example 4.** Consider the LA-semigroup  $R$  of Example 1.

Define an SV mapping  $T : R \rightarrow \mathcal{P}(R)$  by  $T(a) = T(c) = \{a, b, c\}$  and  $T(b) = \{b, c\}$ . Then  $T$  is an SV homomorphism. Let  $A = \{a, b\}$  and  $B = \{b\}$ . Then  $\bar{T}(A) = \{a, b, c\}$  and  $\bar{T}(B) = \{a, b, c\}$ . Therefore,  $\bar{T}(A) \bar{T}(B) = \{a, b, c\} \{a, b, c\} = \{a, c\}$ , and  $AB = \{a, b\} \{b\} = \{a, c\}$ . Thus,  $\bar{T}(AB) = \{a, b, c\}$ . Hence,  $\bar{T}(AB) \not\subseteq \bar{T}(A) \bar{T}(B)$ .

**Proposition 2.** Let  $T : R \rightarrow \mathcal{P}(S)$  be an SSV homomorphism. If  $\emptyset \neq A, B \subseteq S$ , then  $\underline{T}(A) \underline{T}(B) \subseteq \underline{T}(AB)$ .

**Proof.** Let  $x \in \underline{T}(A) \underline{T}(B)$ . Then  $x = ab$ , where  $a \in \underline{T}(A)$  and  $b \in \underline{T}(B)$ . Therefore,  $T(a) \subseteq A$  and  $T(b) \subseteq B$ . Thus,  $T(a) T(b) \subseteq AB$ . Therefore,  $T(ab) \subseteq AB$ , which implies  $ab \in \underline{T}(AB)$ . It follows that  $x \in \underline{T}(AB)$ . Hence  $\underline{T}(A) \underline{T}(B) \subseteq \underline{T}(AB)$ .  $\square$

The following example shows that equality in Proposition 2 may not hold.

**Example 5.** Consider the LA-semigroups  $R$  and  $S$  of Example 3. Define an SV mapping  $T : R \rightarrow \mathcal{P}(S)$  by  $T(a) = T(c) = \{c\}$  and  $T(b) = \{b, c\}$ . Then,  $T$  is an SSV homomorphism. Let  $A = \{a, c\}$  and  $B = \{b, c\} \subseteq S$ . Then  $\underline{T}(A) = \{a, c\}$  and  $\underline{T}(B) = \{a, b, c\}$ . Thus,  $\underline{T}(A) \underline{T}(B) = \{a, c\} \{a, b, c\} = \{a, c\}$ , and  $AB = \{a, c\} \{b, c\} = \{b, c\}$ . Thus,  $\underline{T}(AB) = \{a, b, c\}$ . Hence,  $\underline{T}(AB) \not\subseteq \underline{T}(A) \underline{T}(B)$ .

The fact that considered groupoids are LA-semigroups is important in Propositions 3 and 4 and examples.

**Proposition 3.** Let  $T : R \rightarrow \mathcal{P}(S)$  be an SV homomorphism. If  $H$  is an LA-subsemigroup of  $S$ , then  $\bar{T}(H)$  is an LA-subsemigroup of  $R$ .

**Proof.** Let  $x, y \in \bar{T}(H)$ . Then  $\bar{T}(x) \cap H \neq \emptyset$  and  $\bar{T}(y) \cap H \neq \emptyset$ . Thus, there exist  $a, b \in S$  such that  $a \in T(x) \cap H$  and  $b \in T(y) \cap H$ . Thus,  $a \in T(x)$ ,  $a \in H$  and  $b \in T(y)$ ,  $b \in H$ . Therefore,  $ab \in T(x) T(y) \subseteq T(xy)$  and  $ab \in H$ . Hence,  $ab \in T(xy) \cap H$ , and  $T(xy) \cap H \neq \emptyset$ . Therefore,  $xy \in \bar{T}(H)$ . Hence,  $\bar{T}(H)$  is an LA-subsemigroup of  $R$ .  $\square$

**Proposition 4.** Let  $T : R \rightarrow \mathcal{P}(S)$  be an SSV homomorphism. If  $H$  is an LA-subsemigroup of  $S$ , then  $\underline{T}(H)$  is an LA-subsemigroup of  $R$ .

**Proof.** Let  $x, y \in \underline{T}(H)$ . Then  $T(x) \subseteq H$  and  $T(y) \subseteq H$ . Therefore,  $T(x) T(y) \subseteq HH = H^2$ . Thus,  $T(xy) \subseteq H^2$ , so  $T(xy) \subseteq H$ , which implies  $xy \in \underline{T}(H)$ . Hence,  $\underline{T}(H)$  is an LA-subsemigroup of  $R$ .  $\square$

The following example shows that, in the case of an SV homomorphism,  $\underline{T}(A)$  may not be an LA-subsemigroup.

**Example 6.** Consider the LA-semigroup  $S$  of Example 3.

Define an SV mapping  $T : S \rightarrow \mathcal{P}(S)$  by  $T(b) = T(c) = \{a, b, c\}$  and  $T(a) = \{b, c\}$ . Then  $T$  is an SV homomorphism. Let  $A = \{b, c\} \subseteq S$ . Then  $A$  is an LA-subsemigroup of  $S$ , and  $\underline{T}(A) = \{a\}$ . It follows that  $\underline{T}(A) \underline{T}(A) = \{a\} \{a\} = \{b\} \not\subseteq \underline{T}(A)$ . Hence,  $\underline{T}(A)$  is not an LA-subsemigroup of  $S$ .

**Proposition 5.** Let  $T : R \rightarrow \mathcal{P}(S)$  be an SV homomorphism. If  $A$  is a left ideal of  $S$ , then  $\bar{T}(A)$  is a left ideal of  $R$ .

**Proof.** Let  $x$  and  $r$  be elements of  $\bar{T}(A)$  and  $R$ , respectively. Then  $T(x) \cap A \neq \emptyset$ , so there exists  $a \in S$  such that  $a \in T(x) \cap A$ . Thus,  $a \in T(x)$  and  $a \in A$ . Since  $r \in R$ , there exists a  $y \in S$  such that  $y \in T(r)$ . Hence,  $ya \in T(r) a \subseteq SA \subseteq A$ . Thus,  $ya \in A$  and  $ya \in T(r) T(x) \subseteq T(rx)$ . Hence,  $ya \in T(rx) \cap A$ . It follows that  $T(rx) \cap A \neq \emptyset$ . Therefore,  $rx \in \bar{T}(A)$ . Therefore,  $\bar{T}(A)$  is a left ideal of  $R$ .  $\square$

**Corollary 1.** Let  $T : R \rightarrow \mathcal{P}(S)$  be an SV homomorphism. If  $A$  is a right ideal of  $S$ , then  $\overline{T}(A)$  is a right ideal of  $R$ .

**Corollary 2.** Let  $T : R \rightarrow \mathcal{P}(S)$  be an SV homomorphism. If  $A$  is an ideal of  $S$ , then  $\overline{T}(A)$  is an ideal of  $R$ .

**Proposition 6.** Let  $T : R \rightarrow \mathcal{P}(S)$  be an SSV homomorphism. If  $A$  is a left ideal of  $S$ , then  $\underline{T}(A)$  is a left ideal of  $R$ .

**Proof.** Let  $x \in \underline{T}(A)$  and  $r \in R$ . Then  $T(x) \subseteq A$ . Since  $r \in R$ ,  $T(r) \subseteq S$ . Thus,  $T(r)T(x) \subseteq SA \subseteq A$ . Thus,  $T(r)T(x) \subseteq A$ , and  $T(rx) \subseteq A$ . It follows that  $rx \in \underline{T}(A)$ . Hence,  $\underline{T}(A)$  is a left ideal of  $R$ .  $\square$

The following example shows that, in the case of an SV homomorphism,  $\underline{T}(A)$  may not be a left ideal.

**Example 7.** Consider the LA-semigroup  $S$  of Example 2.

Define an SV mapping  $T : S \rightarrow \mathcal{P}(S)$  by  $T(a) = T(b) = T(c) = T(e) = \{a, b, c, d, e\}$  and  $T(d) = \{b, d\}$ . Clearly  $T$  is an SV homomorphism. Let  $A = \{b, d\}$  be a subset of  $S$ . Then  $A$  is a left ideal of  $S$ , and  $\underline{T}(A) = \{d\}$ . Hence,  $S\underline{T}(A) = \{b\} \not\subseteq \underline{T}(A)$ . Therefore,  $\underline{T}(A)$  is not a left ideal of  $R$ .

**Corollary 3.** Let  $T : R \rightarrow \mathcal{P}(S)$  be an SSV homomorphism. If  $A$  is a right ideal of  $S$ , then  $\underline{T}(A)$  is a right ideal of  $R$ .

**Corollary 4.** Let  $T : R \rightarrow \mathcal{P}(S)$  be an SSV homomorphism. If  $A$  is an ideal of  $S$ , then  $\underline{T}(A)$  is an ideal of  $R$ .

**Proposition 7.** Let  $R$  and  $S$  be two idempotent LA-semigroups and  $T : R \rightarrow \mathcal{P}(S)$  be an SV homomorphism. If  $A, B$  are ideals of  $S$ , then

$$\overline{T}(A) \cap \overline{T}(B) = \overline{T}(AB).$$

**Proof.** Since  $AB \subseteq AS \subseteq A$ ,  $AB \subseteq A$ . Thus,  $\overline{T}(AB) \subseteq \overline{T}(A)$ , and  $AB \subseteq SB \subseteq B$ . It follows that  $AB \subseteq B$ . Thus,  $\overline{T}(AB) \subseteq \overline{T}(B)$ . Hence,  $\overline{T}(AB) \subseteq \overline{T}(A) \cap \overline{T}(B)$ .

Let  $c \in \overline{T}(A) \cap \overline{T}(B)$ . Then  $c \in \overline{T}(A)$  and  $c \in \overline{T}(B)$ . Thus,  $T(c) \cap A \neq \emptyset$ , and  $T(c) \cap B \neq \emptyset$ , so there exist  $x, y \in S$  such that  $x \in T(c) \cap A$  and  $y \in T(c) \cap B$ . It follows that  $x \in T(c)$ ,  $x \in A$ , and  $y \in T(c)$ ,  $y \in B$ . Thus,  $xy \in T(c)T(c) \subseteq T(cc) = T(c)$ , and  $x \in A$  and  $y \in B$ . Hence,  $xy \in AB$ , so  $xy \in T(c) \cap AB$ . Thus,  $T(c) \cap AB \neq \emptyset$ . Hence,  $c \in \overline{T}(AB)$ . Thus,  $\overline{T}(A) \cap \overline{T}(B) \subseteq \overline{T}(AB)$ . Therefore,

$$\overline{T}(A) \cap \overline{T}(B) = \overline{T}(AB),$$

as desired.  $\square$

**Proposition 8.** Let  $R$  and  $S$  be two idempotent LA-semigroups and  $T : R \rightarrow \mathcal{P}(S)$  be an SSV homomorphism. If  $A$  and  $B$  are ideals of  $S$ , then

$$\underline{T}(A) \cap \underline{T}(B) = \underline{T}(AB).$$

**Proof.** Let  $AB \subseteq AS \subseteq A$ . Then  $AB \subseteq A$ . Therefore,  $\underline{T}(AB) \subseteq \underline{T}(A)$ , and  $AB \subseteq SB \subseteq B$ . Hence,  $\underline{T}(AB) \subseteq \underline{T}(B)$ . Therefore,

$$\underline{T}(AB) \subseteq \underline{T}(A) \cap \underline{T}(B).$$

Let  $c \in \underline{T}(A) \cap \underline{T}(B)$ . Then  $c \in \underline{T}(A)$  and  $c \in \underline{T}(B)$ . Hence,  $T(c) \subseteq A$  and  $T(c) \subseteq B$ , so  $T(c)T(c) \subseteq AB$ . Thus,  $T(cc) \subseteq AB$ . Thus,  $T(c) \subseteq AB$ . Hence,  $c \in \underline{T}(AB)$ . This implies that  $\underline{T}(A) \cap \underline{T}(B) \subseteq \underline{T}(AB)$ . Therefore,

$$\underline{T}(A) \cap \underline{T}(B) = \underline{T}(AB),$$

as desired.  $\square$

**Proposition 9.** Let  $T : R \rightarrow \mathcal{P}(S)$  be an SV homomorphism. If  $A$  is a bi-ideal of  $S$ , then  $\overline{T}(A)$  is a bi-ideal of  $R$ .

**Proof.** Let  $x, y \in \overline{T}(A)$  and  $r \in R$ . Then  $T(x) \cap A \neq \emptyset$  and  $T(y) \cap A \neq \emptyset$ . Hence, there exist  $a, b \in S$  such that  $a \in T(x) \cap A$  and  $b \in T(y) \cap A$ , so  $a \in T(x)$ ,  $a \in A$ , and  $b \in T(y)$ ,  $b \in A$ . Since  $r \in R$ , there is a  $c \in S$  such that  $c \in T(r)$ . Now,  $(ac)b \in (T(x)T(r))T(y) \subseteq T(xr)T(y) \subseteq T((xr)y)$ . Thus,  $(ac)b \in T((xr)y)$  and  $(ac)b \in A$ , so  $(ac)b \in T((xr)y) \cap A$ . Hence,  $T((xr)y) \cap A \neq \emptyset$ . Thus,  $(xr)y \in \overline{T}(A)$ . Therefore,  $\overline{T}(A)$  is a bi-ideal of  $R$ .  $\square$

**Proposition 10.** Let  $T : R \rightarrow \mathcal{P}(S)$  be an SSV homomorphism. If  $A$  is a bi-ideal of  $S$ , then  $\underline{T}(A)$  is a bi-ideal of  $R$ .

**Proof.** Let  $x, y \in \underline{T}(A)$  and  $r \in R$ . Then  $T(x) \subseteq A$  and  $T(y) \subseteq A$ . Since  $r \in R$ ,  $T(r) \subseteq S$ . Now,  $T((xr)y) = T(xr)T(y) = (T(x)T(r))T(y) \subseteq (AS)A \subseteq A$ . Therefore,  $T((xr)y) \subseteq A$ . Thus,  $(xr)y \in \underline{T}(A)$ . Hence,  $\underline{T}(A)$  is a bi-ideal of  $R$ .  $\square$

The following example shows that, in the case of an SV homomorphism,  $\underline{T}(A)$  may not be a bi-ideal.

**Example 8.** Consider the LA-semigroup  $S$  of Example 2.

Define an SV mapping  $T : S \rightarrow \mathcal{P}(S)$  by  $T(a) = T(b) = T(c) = T(e) = \{a, b, c, d, e\}$  and  $T(d) = \{b\}$ . Then  $T$  is an SV homomorphism. Let  $A = \{b, d\}$ . Then  $A$  is a bi-ideal of  $S$ , and  $\underline{T}(A) = \{d\}$ . Now,  $(\underline{T}(A)S)\underline{T}(A) = \{b\} \not\subseteq \underline{T}(A)$ . Hence,  $\underline{T}(A)$  is not a bi-ideal of  $S$ .

**Proposition 11.** Let  $T : R \rightarrow \mathcal{P}(S)$  be an SV homomorphism. If  $A$  is an interior ideal of  $S$ , then  $\overline{T}(A)$  is an interior ideal of  $R$ .

**Proof.** Let  $r \in \overline{T}(A)$ , and  $a, b \in R$ . Then  $T(r) \cap A \neq \emptyset$ . Thus, there exists a  $c \in S$  such that  $c \in T(r) \cap A$ . This implies that  $c \in T(r)$  and  $c \in A$ . Since  $a, b \in R$ , there exist  $x, y \in S$  such that  $x \in T(a)$  and  $y \in T(b)$ . It follows that  $(xc)y \in (T(a)T(r))T(b) \subseteq T((ar)b)$ , and  $(xc)y \in A$ . Therefore,  $(xc)y \in T((ar)b) \cap A$ . Thus,  $T((ar)b) \cap A \neq \emptyset$ , so  $(ar)b \in \overline{T}(A)$ . Hence,  $\overline{T}(A)$  is an interior ideal of  $R$ .  $\square$

**Proposition 12.** Let  $T : R \rightarrow \mathcal{P}(S)$  be an SSV homomorphism. If  $A$  is an interior ideal of  $S$ , then  $\underline{T}(A)$  is an interior ideal of  $R$ .

**Proof.** Let  $r \in \underline{T}(A)$  and  $a, b \in R$ . Then  $T(r) \subseteq A$ . Since  $a, b \in R$ ,  $T(a) \subseteq S$ ,  $T(b) \subseteq S$ . It follows that  $T((ar)b) = T(ar)T(b) = (T(a)T(r))T(b) \subseteq (SA)S \subseteq A$ . Therefore,  $T((ar)b) \subseteq A$ . Thus,  $(ar)b \in \underline{T}(A)$ . Hence,  $\underline{T}(A)$  is an interior ideal of  $R$ .  $\square$

**Definition 7.** A subset  $A$  of an LA-semigroup  $S$  is called a quasi-ideal of  $S$  if  $SA \cap AS \subseteq A$ .

**Proposition 13.** Let  $T : R \rightarrow \mathcal{P}(S)$  be an SSV homomorphism. If  $A$  is a quasi-ideal of  $S$ , then  $\underline{T}(A)$  is a quasi-ideal of  $R$ .

**Proof.** Let  $A$  be a quasi-ideal of  $S$ . We prove  $\underline{T}(AS \cap SA) \subseteq \underline{T}(A)$ . Let  $x \in \underline{T}(AS \cap SA)$ . Then  $T(x) \subseteq AS \cap SA \subseteq A$ . Therefore,  $T(x) \subseteq A$ . Therefore,  $x \in \underline{T}(A)$ . Thus,  $\underline{T}(AS \cap SA) \subseteq \underline{T}(A)$ . Hence,  $\underline{T}(A)$  is a quasi-ideal of  $R$ .  $\square$

**Proposition 14.** Let  $T : R \rightarrow \mathcal{P}(S)$  be an SV homomorphism. If  $A$  is a quasi-ideal of  $S$ , then  $\overline{T}(A)$  is a quasi-ideal of  $R$ .

**Proof.** Let  $A$  be a quasi-ideal of  $S$ . Then we have to show that  $\overline{T}(AS \cap SA) \subseteq \overline{T}(A)$ . Let  $x \in \overline{T}(AS \cap SA)$ . Then  $T(x) \cap (AS \cap SA) \neq \emptyset$ . Thus, there exists a  $y \in S$  such that  $y \in T(x) \cap (AS \cap SA)$ . This implies that  $y \in T(x)$  and  $y \in (AS \cap SA) \subseteq A$ , so  $y \in T(x)$  and  $y \in A$ . Thus,  $y \in T(x) \cap A$ . Therefore,  $x \in \overline{T}(A)$ . Hence,  $\overline{T}(AS \cap SA) \subseteq \overline{T}(A)$ . Therefore,  $\overline{T}(A)$  is a quasi-ideal of  $R$ .  $\square$

**Definition 8.** An ideal  $P$  of an LA-semigroup  $S$  with left identity  $e$  is said to be prime if  $AB \subseteq P$  implies either  $A \subseteq P$  or  $B \subseteq P$  for all ideals  $A, B$  of  $S$ .

**Proposition 15.** Let  $T : R \rightarrow \mathcal{P}(S)$  be an SSV homomorphism. If  $A$  is a prime ideal of  $S$ , then  $\overline{T}(A)$  is a prime ideal of  $R$ .

**Proof.** Since  $A$  is an ideal of  $S$ , by Corollary 2,  $\overline{T}(A)$  is an ideal of  $R$ . Let  $xy \in \overline{T}(A)$ . Then  $T(xy) \cap A \neq \emptyset$ . Thus, there exists a  $z \in S$  such that  $z \in T(xy) \cap A$ , so  $z \in T(xy) = T(x)T(y)$ , and  $z \in A$ . Since  $z = ab \in T(x)T(y)$ ,  $ab \in A$ , and  $A$  is a prime ideal of  $S$ ,  $a \in A$  or  $b \in A$ , which implies that  $a \in T(x)$  and  $a \in A$  or that  $b \in T(y)$  and  $b \in A$ . Therefore,  $a \in T(x) \cap A$  or  $b \in T(y) \cap A$ . Thus,  $T(x) \cap A \neq \emptyset$  or  $T(y) \cap A \neq \emptyset$ . It follows that  $x \in \overline{T}(A)$  or  $y \in \overline{T}(A)$ . Hence,  $\overline{T}(A)$  is a prime ideal of  $R$ .  $\square$

**Proposition 16.** Let  $T : R \rightarrow \mathcal{P}(S)$  be an SSV homomorphism. If  $A$  is a prime ideal of  $S$ , then  $\underline{T}(A)$  is a prime ideal of  $R$ .

**Proof.** Since  $A$  is an ideal of  $S$ , by Corollary 4,  $\underline{T}(A)$  is an ideal of  $R$ . Let  $xy \in \underline{T}(A)$ . Then  $T(xy) \subseteq A$ . Let  $z \in T(xy) = T(x)T(y)$ , where  $z = ab \in T(x)T(y)$ . Then  $a \in T(x)$ ,  $b \in T(y)$ , and  $ab \in A$ . Since  $A$  is a prime ideal of  $S$ ,  $a \in A$  or  $b \in A$ . Thus,  $a \in T(x) \subseteq A$  or  $b \in T(y) \subseteq A$ . Thus,  $x \in \underline{T}(A)$  or  $y \in \underline{T}(A)$ . Hence,  $\underline{T}(A)$  is a prime ideal of  $R$ .  $\square$

**Remark 1.** The algebraic approach—in particular, the semigroup theory—can be introduced in the area of genetic algorithms and to the evolutionary based procedure for optimization and clustering (see [13]).

#### 4. Conclusions

In this paper, we discussed the generalized roughness in (crisp) LA-subsemigroups or ideals of LA-semigroups with the help of set valued/strong set valued homomorphisms. We have provided examples showing that the lower approximations of a subset of an LA-semigroup may not be an LA-subsemigroup or ideal of LA-semigroup, under a set valued homomorphism.

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