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Recognition of $M \times M$ by Its Complex Group Algebra Where M Is a Simple K_3-Group

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Abstract: In this paper we prove that if M is a simple K_3-group, then $M \times M$ is uniquely determined by its order and some information on irreducible character degrees and as a consequence of our results we show that $M \times M$ is uniquely determined by the structure of its complex group algebra.

Keywords: character degree; order; complex group algebra

1. Introduction

Let G be a finite group, $\mathrm{Irr}(G)$ be the set of irreducible characters of G, and denote by $\mathrm{cd}(G)$, the set of irreducible character degrees of G. A finite group G is called a K_3-group if $|G|$ has exactly three distinct prime divisors. By [1], simple K_3-groups are A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ and $U_4(2)$. Chen et al. in [2,3] proved that all simple K_3-groups and the Mathieu groups are uniquely determined by their orders and one or both of their largest and second largest irreducible character degrees. In [4], it is proved that $L_2(q)$ is uniquely determined by its group order and its largest irreducible character degree when q is a prime or when $q = 2^a$ for an integer $a \geq 2$ such that $2^a - 1$ or $2^a + 1$ is a prime.

Let p be an odd prime number. In [5–8], it is proved that the simple groups $L_2(q)$ and some extensions of them, where $q \mid p^3$ are uniquely determined by their orders and some information on irreducible character degrees.

In ([9], Problem 2*)R. Brauer asked: *Let G and H be two finite groups. If for all fields $\mathbb{F}$, two group algebras $\mathbb{F}G$ and $\mathbb{F}H$ are isomorphic can we get that G and H are isomorphic?* This is false in general. In fact, E. C. Dade [10] constructed two nonisomorphic metabelian groups G and H such that $\mathbb{F}G \cong \mathbb{F}H$ for all fields $\mathbb{F}$. In [11], Tong-Viet posed the following question:

Question. *Which groups can be uniquely determined by the structure of their complex group algebras?*

In general, the complex group algebras do not uniquely determine the groups, for example, $\mathbb{C}D_8 \cong \mathbb{C}Q_8$. It is proved that nonabelian simple groups, quasi-simple groups and symmetric groups are uniquely determined up to isomorphism by the structure of their complex group algebras (see [12–18]). Khosravi et al. proved that $L_2(p) \times L_2(p)$ is uniquely determined by its complex group algebra, where $p \geq 5$ is a prime number (see [19]). In [20], Khosravi and Khademi proved that the characteristically simple group $A_5 \times A_5$ is uniquely determined by its order and its character degree graph (vertices are the prime divisors of the irreducible character degrees of G and two vertices p and q are joined by an edge if pq divides some irreducible character degree of G). In this paper, we prove that if M is a simple K_3-group, then $M \times M$ is uniquely determined by its order and some information about its irreducible character degrees. In particular, this result is the generalization of ([19], Theorem 2.4) for $p = 5, 7$ and 17. Also as a consequence of our results we show that $M \times M$ is uniquely determined by the structure of its complex group algebra.

2. Preliminaries

If $\chi = \sum_{i=1}^{k} e_i\chi_i$, where for each $1 \le i \le k$, $\chi_i \in \mathrm{Irr}(G)$ and e_i is a natural number, then each χ_i is called an irreducible constituent of χ.

Lemma 1. (Itô's Theorem) ([21], Theorem 6.15) *Let $A \trianglelefteq G$ be abelian. Then $\chi(1)$ divides $|G : A|$, for all $\chi \in \mathrm{Irr}(G)$.*

Lemma 2. ([21], Corollary 11.29) *Let $N \trianglelefteq G$ and $\chi \in \mathrm{Irr(G)}$. If θ is an irreducible constituent of χ_N, then $\chi(1)/\theta(1) \mid |G : N|$.*

Lemma 3. ([2], Lemma 1) *Let G be a nonsolvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$.*

Lemma 4. (Itô-Michler Theorem) [22] *Let $\rho(G)$ be the set of all prime divisors of the elements of $\mathrm{cd}(G)$. Then $p \notin \rho(G) = \{p : p \text{ is a prime number}, p \mid \chi(1), \chi \in \mathrm{Irr}(G)\}$ if and only if G has a normal abelian Sylow p-subgroup.*

Lemma 5. ([3], Lemma 2) *Let G be a finite solvable group of order $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$, where p_1, p_2, ..., p_n are distinct primes. If $(kp_n + 1) \nmid p_i^{\alpha_i}$, for each $i \le n - 1$ and $k > 0$, then the Sylow p_n-subgroup is normal in G.*

Lemma 6. ([19], Theorem 2.4) *Let $p \ge 5$ be a prime number. If G is a finite group such that (i) $|G| = |L_2(p)|^2$, (ii) $p^2 \in \mathrm{cd}(G)$, (iii) there does not exist any element $a \in \mathrm{cd}(G)$ such that $2p^2 \mid a$, (iv) if p is a Mersenne prime or a Fermat prime, then $(p \pm 1)^2 \in \mathrm{cd}(G)$, then $G \cong L_2(p) \times L_2(p)$.*

3. The Main Results

Lemma 7. *Let S be a simple K_3-group and let G be an extension of S by S. Then $G \cong S \times S$.*

Proof. There exists a normal subgroup of G which is isomorphic to S and we denote it by the same notation. By [23], we know that $|\mathrm{Out}(S)| \leqslant 4$ and $G/C_G(S) \hookrightarrow \mathrm{Aut}(S)$, which implies that $C_G(S) \neq 1$. As S is a nonabelian simple group, $S \cap C_G(S) = 1$ and it follows that $SC_G(S) \cong S \times C_G(S)$. Also $C_G(S) \cong SC_G(S)/S \trianglelefteq G/S \cong S$ which implies that G is isomorphic to $S \times S$. □

Theorem 1. *Let G be a finite group. Then $G \cong A_5 \times A_5$ if and only if $|G| = |A_5|^2$ and $5^2 \in \mathrm{cd}(G)$.*

Proof. Obviously by Itô's theorem, we get that $O_5(G) = 1$. First we show that G is not a solvable group. If G is a solvable group, then let H be a Hall subgroup of G of order $2^4 5^2$. Since $G/H_G \hookrightarrow S_9$, we get that $5 \mid |H_G|$. If $5^2 \mid |H_G|$, then $25 \in \mathrm{cd}(H_G)$. On the other hand, $25^2 < |H_G| \le 2^4 5^2$, a contradiction. If $|H_G| = 2^4 5$, then $|G/H_G| = 45$. Let L/H_G be a Sylow 5-subgroup of G/H_G. Then $L/H_G \trianglelefteq G/H_G$ and so $L \trianglelefteq G$ and $|L| = 5^2 2^4$. Then $25 \in \mathrm{cd}(L)$, which is a contradiction. If $|H_G| \mid 2^3 5$, then P, a Sylow 5-subgroup of H_G is a normal subgroup of G, which is a contradiction by Lemma 4. Therefore G is a nonsolvable group.

Since G is nonsolvable, by Lemma 3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. As $|G| = 2^4 3^2 5^2$, we have $K/H \cong A_5, A_6$ or $A_5 \times A_5$ by [23]. If $K/H \cong A_6$, then $|H| = 5$ or 10. Using Lemma 2, $5 \in \mathrm{cd}(H)$, a contradiction. If $K/H \cong A_5$, then $|H| = 60$ or $|H| = 30$. By Lemma 2, $5 \in \mathrm{cd}(H)$. If H is a solvable group, then by Lemma 5, $P \trianglelefteq H$, where $P \in \mathrm{Syl}_5(H)$, which is a contradiction. Therefore $|H| = 60$ and so $H \cong A_5$. Hence G is an extension of A_5 by A_5 and by Lemma 7, $G \cong A_5 \times A_5$. If $K/H \cong A_5 \times A_5$, then $|H| = 1$ and $G \cong A_5 \times A_5$. □

Theorem 2. *Let G be a finite group. Then $G \cong L_2(17) \times L_2(17)$ if and only if $|G| = |L_2(17)|^2$ and $17^2 \in \mathrm{cd}(G)$.*

Proof. Obviously $O_{17}(G) = 1$. On the contrary let G be a solvable group. First we show that there exists no normal subgroup N of G such that

(a) $|N| = 2^i 3^j 17^k$, where $k \neq 0$ and $i < 8$; or (b) $|N| = 2^8 17^2$; or (c) $|N| = 2^8 17$.

Let N be a normal subgroup of G. If $|N| = 2^i 3^j 17^k$, where $k \neq 0$ and $i < 8$, then by Lemma 5, $P \trianglelefteq G$, where $P \in \mathrm{Syl}_{17}(G)$. Hence $O_{17}(G) \neq 1$, which is a contradiction. If $|N| = 2^8 17^2$, then $17^2 \in \mathrm{cd}(N)$, which is impossible. If $|N| = 2^8 17$, then $|G/N| = 3^4 17$. If $T/N \in \mathrm{Syl}_{17}(G/N)$, then $T/N \trianglelefteq G/N$. Therefore $T \trianglelefteq G$, where $|T| = 17^2 2^8$ and this is a contradiction as we stated above.

Let M be a minimal normal subgroup of G, which is an elementary abelian p-group. Obviously $p \neq 17$. Let $p = 2$. Then $|M| = 2^i$, where $0 < i \leq 8$ and so $|G/M| = 2^{8-i} 3^4 17^2$. Then $T/M \trianglelefteq G/M$, where $T/M \in \mathrm{Syl}_{17}(G/M)$. Therefore $T \trianglelefteq G$ and $|T| = 17^2 2^i$, which is a contradiction. Hence $p = 3$ and $|M| = 3^i$, where $1 \leq i \leq 4$.

If $i = 4$, then $G/C_G(M) \hookrightarrow \mathrm{Aut}(M) \cong \mathrm{GL}(4,3)$ and $|\mathrm{GL}(4,3)| = 2^9 \times 3^6 \times 5 \times 13$. Hence $17^2 \mid |C_G(M)|$. Since M is an abelian subgroup of G, thus $3^4 \mid |C_G(M)|$. If $|C_G(M)| = 17^2 3^4 2^j$, where $j \neq 8$, then by the above discussion we get a contradiction. Otherwise, $C_G(M) = G$ and so by Burnside normal p-complement theorem, G has a normal 3-complement of order $17^2 2^8$, which is a contradiction.

If $i = 3$, then $|G/M| = 2^8 17^2 3$. Let H/M be a Hall subgroup of G/M of order $2^8 17^2$. Then $|H| = 2^8\, 3^3 17^2$. Since $G/H_G \hookrightarrow S_3$, thus $3^3 17^2 \mid |H_G|$. If $2^8 \nmid |H_G|$, then by the above discussion we get a contradiction. Therefore $|H_G| = 2^8 3^3 17^2$, i.e., $H \trianglelefteq G$. Let B be a Hall subgroup of H of order $|B| = 2^8 17^2$. Then similarly to the above $2^8 17 \mid |B_H|$. If $|B_H| = 2^8 17^2$, then we get a contradiction. If $|B_H| = 2^8 17$, then $T/B_H \trianglelefteq B/B_H$ where $T/B_H \in \mathrm{Syl}_{17}(B/B_H)$. Therefore $|T| = 2^8 17^2$, which is a contradiction.

If $i = 2$, then $|G/M| = 2^8 3^2 17^2$. Let H/M be a Hall subgroup of G/M of order $2^8 17^2$. Then $|H| = 2^8 3^2 17^2$. Thus similarly to the above, $17^2 \mid |H_G|$ and $17^2 \in \mathrm{cd}(H_G)$. Then by the same argument as above we get that H_G has a normal subgroup of order $2^i 17^2$, which is a contradiction.

If $i = 1$, then $|G/M| = 2^8 3^3 17^2$. Let H/M be a Hall subgroup of G/M of order $2^8 17^2$. Then $|H| = 2^8 17^2 3$. Since $G/H_G \hookrightarrow S_{27}$ we get that $17 \mid |H_G|$. If $2^8 \nmid |H_G|$ or $|H_G| = 2^8 17^k$, where $k \neq 0$, then we get a contradiction. If $|H_G| = 2^8 17^2 3$, then H_G has a normal subgroup of order $2^i 17^2$, which is a contradiction. If $|H_G| = 2^8 \times 17 \times 3$, then $|G/H_G| = 3^3 17$. Therefore $T/H_G \trianglelefteq G/H_G$, where $T/H_G \in \mathrm{Syl}_{17}(G/H_G)$. Hence $T \trianglelefteq G$ and $|T| = 2^8 17^2 3$, which is a contradiction as we stated above.

Therefore G is nonsolvable and by Lemma 3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong L_2(17)$ or $L_2(17) \times L_2(17)$ and $|G/K| \mid |\mathrm{Out}(K/H)|$.

If $K/H \cong L_2(17)$, then $|H| = 2^3 3^2 17$ or $2^4 3^2 17$ and so $17 \in \mathrm{cd}(H)$. If H is a solvable group, then by Lemma 5, $P \trianglelefteq H$, where $P \in \mathrm{Syl}_{17}(H)$, which is a contradiction by Lemma 4. Otherwise by Lemma 3 and [23] we get that $H \cong L_2(17)$. Therefore G is an extension of $L_2(17)$ by $L_2(17)$ and by Lemma 7, $G \cong L_2(17) \times L_2(17)$.

Obviously if $K/H \cong L_2(17) \times L_2(17)$, then $G \cong L_2(17) \times L_2(17)$. □

In the sequel, we show that if G is a finite group of order $|L_2(7) \times L_2(7)|$, such that G has an irreducible character of order 7^2 or 2^6, then we can not conclude that $G \cong L_2(7) \times L_2(7)$. So we need more assumptions to characterize $L_2(7) \times L_2(7)$.

Remark 1. *Using the notations of* GAP [24], *if* $A = \mathrm{SmallGroup}(56, 11)$ *and* $H = A \times A \times \mathbb{Z}_9$, *then* $|H| = |L_2(7) \times L_2(7)|$ *and* H *has an irreducible character of degree* 7^2.

Similarly if $B = \mathrm{SmallGroup}(784, 160)$ *and* $K = B \times S_3 \times S_3$, *then* $|H| = |L_2(7) \times L_2(7)|$ *and* H *has an irreducible character of degree* 2^6.

Theorem 3. *Let* G *be a finite group. Then* $G \cong L_2(7) \times L_2(7)$ *if and only if* $|G| = 2^6 3^2 7^2$ *and* $2^6, 7^2 \in \mathrm{cd}(G)$.

Proof. If G is a solvable group, then let H be a Hall subgroup of G of order $2^6 7^2$. Since $G/H_G \hookrightarrow S_9$, we have $|H_G| = 2^i 7^j$, where $0 \leq i \leq 6$ and $1 \leq j \leq 2$. Using Lemma 2, $2^i, 7^j \in \mathrm{cd}(H_G)$. If $O_2(H_G) \neq 1$, then by Lemma 2, $|O_2(H_G)| \in \mathrm{cd}(O_2(H_G))$, which is a contradiction. Similarly $O_7(H_G) = 1$, which shows that G is a nonsolvable group.

Therefore G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong L_2(8), L_2(7)$ or $L_2(7) \times L_2(7)$ and $|G/K| \mid |\mathrm{Out}(K/H)|$.

If $K/H \cong L_2(8)$, then $|H| = 56$. Using Lemma 2, $8 \in \mathrm{cd}(H)$ and since $64 > 56$, we get a contradiction.

If $K/H \cong L_2(7)$, then $|H| = 2^2 \times 3 \times 7$ or $2^3 \times 3 \times 7$. If $|H| = 2^2 \times 3 \times 7$, then by Lemma 2, $7 \in \mathrm{cd}(H)$. Since there exists no nonabelian simple group S such that $|S| \mid |H|$, we get that H is a solvable group. then by Lemma 5, $P \trianglelefteq H$ where $P \in \mathrm{Syl}_7(H)$, which is a contradiction by Lemma 4. So $|H| = 2^3 \times 3 \times 7$, by the same argument for the proof of Theorem A in [2], we get that $H \cong L_2(7)$. Therefore G is an extension of $L_2(7)$ by $L_2(7)$ and by Lemma 7, $G \cong L_2(7) \times L_2(7)$.

If $K/H \cong L_2(7) \times L_2(7)$, obviously we have $G \cong L_2(7) \times L_2(7)$. □

Remark 2. *We note that Theorems 1, 2 and 3 are generalizations of Lemma 6 for special cases $p = 5, 7, 17$.*

Lemma 8. *Let G be a finite group. If $|G| = 2^i 3^j 5$, where $i \geq 3$ or $j \geq 1$, and $2^i, 3^j \in \mathrm{cd}(G)$, then G is not solvable. If $|G| = 2^i 3^j 5^2$, where $i \geq 6$ or $j \geq 2$, and $2^i, 3^j \in \mathrm{cd}(G)$, then G is not solvable.*

Proof. On the contrary let G be a solvable group.

Let $O_2(G) \neq 1$ and $|O_2(G)| = 2^t$, where $1 \leq t \leq i$. By the assumption, there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) = 2^i$. If $\sigma \in \mathrm{Irr}(\mathrm{O_2(G)})$ such that $[\chi_{O_2(G)}, \sigma] \neq 0$, then by Lemma 2, $2^i/\sigma(1)$ is a divisor of $|G : O_2(G)| = 2^{i-t}$. Since $\sigma(1) \mid |O_2(G)|$, we get that $\sigma(1) = 2^t$, which is a contradiction. Similarly $O_3(G) = 1$.

Therefore $\mathrm{Fit}(G) = O_5(G) \neq 1$. We know that $G/C_G(\mathrm{Fit}(G)) \hookrightarrow \mathrm{Aut}(\mathrm{Fit}(G))$ and since G is a solvable group, $C_G(\mathrm{Fit}(G)) \leqslant \mathrm{Fit}(G)$. Therefore $|G|$ is a divisor of $|\mathrm{Fit}(G)| \cdot |\mathrm{Aut}(\mathrm{Fit}(G))|$ and easily we can see that in each case we get a contradiction. □

Similarly to the above we have the following result:

Lemma 9. *Let G be a finite group.*

(a) *If $|G| = 2^i 3^j 7$, where $i \geq 2$ or $j \geq 2$, and $2^i, 3^j \in \mathrm{cd}(G)$, then G is not solvable.*
(b) *If $|G| = 2^i 3^j 7^2$, where $i \geq 6$ or $j \geq 3$, and $2^i, 3^j \in \mathrm{cd}(G)$, then G is not solvable.*

Theorem 4. *Let G be a finite group.*

(a) *If $|G| = 2^6 3^4 5^2$ and $2^6, 3^4 \in \mathrm{cd}(G)$, then $G \cong A_6 \times A_6$ or $G \cong \mathbb{Z}_5 \times U_4(2)$;*
(b) *If $|G| = 2^{12} 3^8 5^2$ and $2^{12}, 3^8 \in \mathrm{cd}(G)$, then $G \cong U_4(2) \times U_4(2)$.*

Proof. Lemma 8 gives us that G is not solvable and so G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$.

(**a**) By assumptions K/H is isomorphic to $A_5, A_6, U_4(2), A_5 \times A_5$ or $A_6 \times A_6$.

If $K/H \cong A_5$, then $|H| = 2^4 3^3 5$ or $|H| = 2^3 3^3 5$. By Lemma 8, H is not solvable and H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$ such that B/A is a direct product of m copies of a nonabelian simple group S and $|H/B| \mid |\mathrm{Out}(B/A)|$. If $|H| = 2^4 3^3 5$, we have $B/A \cong A_5$ or A_6. Then $|A| = 36, 18, 6$ or 3, which is a contradiction. If $|H| = 2^3 3^3 5$, then similarly we get a contradiction.

If $K/H \cong A_6$, then $|H| = 2^i 3^2 5$, where $1 \leq i \leq 3$. By Lemma 2, $2^i, 3^2 \in \mathrm{cd}(H)$. Using Lemma 8, H is not a solvable group and so $i \neq 1$. Also H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$ such that B/A is a direct product of m copies of a nonabelian simple group S and $|H/B| \mid |\mathrm{Out}(B/A)|$. If $|H| = 2^3 3^2 5$, by Theorem B in [2], we get that $H \cong A_6$, and so by Lemma 7, $G \cong A_6 \times A_6$. If $|H| = 2^2 3^2 5$, then $|A| = 3$, which is a contradiction.

If $K/H \cong U_4(2)$, then $|H| = 5$ and $G = K$. Therefore G is an extension of $\mathbb{Z}_5$ by $U_4(2)$. We know that $G/C_G(H) \hookrightarrow \mathrm{Aut}(H)$ and $(G/H)/(C_G(H)/H) \cong G/C_G(H)$. So G is a central extension of H by $U_4(2)$. Since the Schur multiplier of $U_4(2)$ is 2, we get that $G \cong \mathbb{Z}_5 \times U_4(2)$.

Let $K/H \cong A_5 \times A_5$. We know that $\mathrm{Out}(K/H) \cong \mathrm{Out}(A_5) \wr S_2$, and so $|G/K| \mid 8$. Thus $|H| = 2^i 3^2$, where $0 \leq i \leq 2$, which is a contradiction.

Finally, if $K/H \cong A_6 \times A_6$, then $G \cong A_6 \times A_6$.

(b) In this case, we have $K/H \cong A_5, A_6, U_4(2), A_5 \times A_5, A_6 \times A_6$ or $U_4(2) \times U_4(2)$.

If $K/H \cong A_5$, then $|H| = 2^{10}3^7 5$ or $2^9 3^7 5$. By Lemma 8, H is not a solvable group and H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$ such that B/A is a nonabelian simple group. Therefore A is a $\{2,3\}$-group such that $O_2(A) = O_3(A) = 1$ and this is a contradiction.

If $K/H \cong A_6$, then similarly to the above we get a contradiction.

If $K/H \cong U_4(2)$, then $|H| = 2^i 3^4 5$, where $5 \leq i \leq 6$. By Lemma 2, $2^i, 3^4 \in \mathrm{cd}(H)$. Therefore H is not a solvable group and H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$ such that B/A is a nonabelian simple group. If $|H| = 2^5 3^4 5$, then A is a $\{2,3\}$-group such that $O_2(A) = O_3(A) = 1$ and this is a contradiction. If $|H| = 2^6 3^4 5$, by Theorem A in [2], we get that $H \cong U_4(2)$ and by Lemma 7, $G \cong U_4(2) \times U_4(2)$.

Let $K/H \cong A_5 \times A_5$. We know that $\mathrm{Out}(K/H) \cong \mathrm{Out}(A_5) \wr S_2$. Therefore $|G/K| \mid 8$ and thus $|H| = 2^i 3^6$, where $5 \leq i \leq 8$, which is a contradiction.

If $K/H \cong A_6 \times A_6$, then $|\mathrm{Out}(K/H)| = 2^5$ and thus $|H| = 2^i 3^4$, where $1 \leq i \leq 6$, which is a contradiction.

Therefore $K/H \cong U_4(2) \times U_4(2)$, and so $G \cong U_4(2) \times U_4(2)$. $\square$

Corollary 1. *If* $|G| = 2^6 3^4 5^2$ *and* $2^6, 3^4 \in \mathrm{cd}(G)$ *and* $6 \notin \mathrm{cd}(G)$, *then* $G \cong A_6 \times A_6$.

Theorem 5. *If* $|G| = 2^{10}3^6 7^2$ *and* $2^{10}, 3^6 \in \mathrm{cd}(G)$, *then* $G \cong U_3(3) \times U_3(3)$.

Proof. By Lemma 9 it follows that G is not solvable and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong L_2(7), L_2(8), U_3(3), L_2(7) \times L_2(7), L_2(8) \times L_2(8)$ or $U_3(3) \times U_3(3)$ and $|G/K| \mid |\mathrm{Out}(K/H)|$.

If $K/H \cong L_2(7)$, then $|H| = 2^7 3^5 7$ or $2^6 3^5 7$. By Lemma 9, H is not solvable and H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$ such that B/A is a nonabelian simple group. Therefore A is a $\{2,3\}$-group such that $O_2(A) = O_3(A) = 1$, which is a contradiction. If $K/H \cong L_2(8)$, then similarly to the above we get a contradiction.

If $K/H \cong L_2(7) \times L_2(7)$ or $K/H \cong L_2(8) \times L_2(8)$, then H is a $\{2,3\}$-group, and we get a contradiction similarly.

If $K/H \cong U_3(3)$, then $|H| = 2^5 3^3 7$ or $2^4 3^3 7$. By Lemma 9, H is not a solvable group and H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$ such that B/A is a nonabelian simple group.

If $|H| = 2^4 3^3 7$, then A is a $\{2,3\}$-group such that $O_2(A) = O_3(A) = 1$, which is a contradiction. If $|H| = 2^5 3^3 7$, by Theorem C in [2], we get that $H \cong U_3(3)$ and by Lemma 7, $G \cong U_3(3) \times U_3(3)$.

Finally, if $K/H \cong U_3(3) \times U_3(3)$, then obviously $G \cong U_3(3) \times U_3(3)$. $\square$

Theorem 6. *If G is a finite group such that*

(i) $|G| = 2^6 3^4 7^2$,
(ii) $2^6, 3^4 \in \mathrm{cd}(G)$,
(iii) $6, 12, 18 \notin \mathrm{cd}(G)$,

then $G \cong L_2(8) \times L_2(8)$.

Proof. By Lemmas 3 and 9, we get that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong L_2(7), L_2(8), U_3(3), L_2(7) \times L_2(7)$ or $L_2(8) \times L_2(8)$, and $|G/K| \mid |\mathrm{Out}(K/H)|$.

If $K/H \cong L_2(7)$, then $|H| = 2^3 3^3 7$ or $2^2 3^3 7$. By Lemma 9, H is not a solvable group and H has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$ such that B/A is a nonabelian simple group and $|H/B| \mid |\mathrm{Out}(B/A)|$.

If $|H| = 2^3 3^3 7$, we have $B/A \cong L_2(7)$ or $L_2(8)$. If $B/A \cong L_2(7)$, then $|A| = 3^2$, a contradiction. If $B/A \cong L_2(8)$, then by Itô's theorem, $|A| = 1$ and $1 \trianglelefteq B \cong L_2(8) \trianglelefteq H$, where $|H : B| = 3$. By the proof of Lemma 1 in [2] (Lemma 3 in the present paper), H/B is isomorphic to a subgroup of $\mathrm{Out}(B/A)$ and by [23] we have $H \cong L_2(8).3$. Using GAP $\mathrm{cd}(H) = \{1, 7, 8, 21, 27\}$, $Z(H) = 1$ and $\mathrm{Aut}(H) \cong H$. Now similarly to the proof of Lemma 7, $G \cong (L_2(8).3) \times L_2(7)$. Then $6 \in \mathrm{cd}(G)$, which is a contradiction by (iii). If $|H| = 2^2 3^3 7$, then by Lemma 9, H is not a solvable group, and this is a contradiction by [23].

If $K/H \cong L_2(8)$, then $|H| = 2^3 \cdot 3^2 \cdot 7$ or $2^3 \cdot 3 \cdot 7$. Using Lemma 9, H is not a solvable group. If $|H| = 2^3 \cdot 3^2 \cdot 7$, by the same argument as Theorem C in [2], we get that $H \cong L_2(8)$ and by Lemma 7, $G \cong L_2(8) \times L_2(8)$. If $|H| = 2^3 \cdot 3 \cdot 7$, then by Theorem A in [2], $H \cong L_2(7)$. Since $K/H \cong L_2(8)$, similarly to the proof of Lemma 7, we get that $K \cong L_2(7) \times L_2(8)$. So G is a an extension of Z_3 by $L_2(7) \times L_2(8)$. Since $6 \in \mathrm{cd}(G)$ or $18 \in \mathrm{cd}(G)$, we get a contradiction by (iii).

If $K/H \cong U_3(3)$, then $|H| = 42$ or $|H| = 21$.

If $|H| = 42$, then H is solvable and H' is a cyclic group, since $|H|$ is square-free. Therefore $|H'| = 7$ and $|H/H'| = 6$. Now easily we see that the equation $\sum_{\varphi \in \mathrm{Irr}(H)} \varphi^2(1) = |H|$, where $\varphi(1) \mid |H|$, has no solution and so we get a contradiction.

If $|H| = 21$, then by Lemma 2, we get that $3 \in \mathrm{cd}(H)$ and so H is a Frobenius group of order 21, which is denoted by $7 : 3$. Also $Z(H) = 1$ and $\mathrm{Aut}(H) \cong H.2$. Now similarly to the proof of Lemma 7, we get that $K \cong (7 : 3) \times U_3(3)$. Since $|G : K| = 2$, we have $G \cong (7 : 3) \times U_3(3)).2$ and so $6 \in \mathrm{cd}(G)$ or $12 \in \mathrm{cd}(G)$, which is a contradiction by (iii).

If $K/H \cong L_2(7) \times L_2(7)$. We know that $\mathrm{Out}(K/H) \cong \mathrm{Out}(L_2(7)) \wr S_2$. Then $|G/K| \mid 8$ and thus $|H| = 3^2$, which is a contradiction.

Finally $K/H \cong L_2(8) \times L_2(8)$, and so $G \cong L_2(8) \times L_2(8)$. $\square$

Theorem 7. *If* $|G| = |L_3(3)|^2$ *and* $2^8, 3^6 \in \mathrm{cd}(G)$, *then* $G \cong L_3(3) \times L_3(3)$.

Proof. First we show that G is not a solvable group. If G is a solvable group, then $O_2(G) = O_3(G) = 1$ and so $\mathrm{Fit}(G) = O_{13}(G) \neq 1$. Since $|\mathrm{Aut}(\mathbb{Z}_{13})| = 2^2 3$, $|\mathrm{Aut}(\mathbb{Z}_{169})| = 2^2 \cdot 3 \cdot 13$ and $|\mathrm{Aut}(\mathbb{Z}_{13} \times \mathbb{Z}_{13})| = 2^5 \cdot 3^2 \cdot 7 \cdot 13$, therefore $|G| \nmid |\mathrm{Fit}(G)| \cdot |\mathrm{Aut}(\mathrm{Fit}(G))|$, which is a contradiction. Therefore G is nonsolvable and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong L_3(3)$ or $L_3(3) \times L_3(3)$, where $|G/K| \mid |\mathrm{Out}(K/H)|$. If $K/H \cong L_3(3) \times L_3(3)$, then $G = L_3(3) \times L_3(3)$. If $K/H \cong L_3(3)$, then $|G/K| = 1$ or 2, and thus $|H| = 2^4 3^3 13$ or $|H| = 2^3 3^3 13$. If H is a solvable group, then $\mathrm{Fit}(H) \cong \mathbb{Z}_{13}$ and $|H| \nmid |\mathrm{Fit}(H)| \cdot |\mathrm{Aut}(\mathrm{Fit}(H))|$, which is a contradiction. Hence H is not a solvable group and so $H \cong L_3(3)$ and by Lemma 7, $G \cong L_3(3) \times L_3(3)$. $\square$

As a consequence of the above theorem, by ([25], Theorem 2.13), we have the following result which is a partial answer to the question arose in [11].

Corollary 2. *Let M be a simple K_3-group and $H = M \times M$. If G is a group such that $\mathbb{C}G \cong \mathbb{C}H$, then $G \cong H$. Thus $M \times M$, where M is a simple K_3-group, is uniquely determined by the structure of its complex group algebra.*

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