

# Fuzzy Semi-Metric Spaces

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**Abstract:** The  $T_1$ -spaces induced by the fuzzy semi-metric spaces endowed with the special kind of triangle inequality are investigated in this paper. The limits in fuzzy semi-metric spaces are also studied to demonstrate the consistency of limit concepts in the induced topologies.

**Keywords:** fuzzy semi-metric space;  $T_1$ -space; triangle inequality; triangular norm

## 1. Introduction

Given a universal set  $X$ , for any  $x, y \in X$ , let  $\tilde{d}(x, y)$  be a fuzzy subset of  $\mathbb{R}_+$  with membership function  $\zeta_{\tilde{d}(x, y)} : \mathbb{R}_+ \rightarrow [0, 1]$ , where the value  $\zeta_{\tilde{d}(x, y)}(t)$  means that the membership degree of the distance between  $x$  and  $y$  is equal to  $t$ . Kaleva and Seikkala [1] proposed the fuzzy metric space by defining a function  $M^* : X \times X \times [0, \infty) \rightarrow [0, 1]$  as follows:

$$M^*(x, y, t) = \zeta_{\tilde{d}(x, y)}(t). \quad (1)$$

On the other hand, inspired by the Menger space that is a special kind of probabilistic metric space (by referring to Schweizer and Sklar [2–4], Hadžić and Pap [5] and Chang et al. [6]), Kramosil and Michalek [7] proposed another concept of fuzzy metric space.

Let  $X$  be a nonempty universal set, let  $*$  be a  $t$ -norm, and let  $M$  be a mapping defined on  $X \times X \times [0, \infty)$  into  $[0, 1]$ . The 3-tuple  $(X, M, *)$  is called a fuzzy metric space if and only if the following conditions are satisfied:

- for any  $x, y \in X$ ,  $M(x, y, t) = 1$  for all  $t > 0$  if and only if  $x = y$ ;
- $M(x, y, 0) = 0$  for all  $x, y \in X$ ;
- $M(x, y, t) = M(y, x, t)$  for all  $x, y \in X$  and  $t \geq 0$ ;
- $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$  for all  $x, y, z \in X$  and  $s, t \geq 0$  (the so-called triangle inequality).

The mapping  $M$  in fuzzy metric space  $(X, M, *)$  can be regarded as a membership function of a fuzzy subset of  $X \times X \times [0, \infty)$ . Sometimes,  $M$  is called a fuzzy metric of the space  $(X, M, *)$ . According to the first and second conditions of fuzzy metric space, the mapping  $M(x, y, t)$  can be interpreted as the membership degree of the distance that is less than  $t$  between  $x$  and  $y$ . Therefore, the meanings of  $M$  and  $M^*$  defined in Equation (1) are different.

George and Veeramani [8,9] studied some properties of fuzzy metric spaces. Gregori and Romaguera [10–12] also extended their research to study the properties of fuzzy metric spaces and fuzzy quasi-metric spaces. In particular, Gregori and Romaguera [11] proposed the fuzzy quasi-metric spaces in which the symmetric condition was not assumed. In this paper, we study the so-called fuzzy semi-metric space without assuming the symmetric condition. The main difference is that four forms of triangle inequalities that were not addressed in Gregori and Romaguera [11] are considered in this paper. Another difference is that the  $t$ -norm in Gregori and Romaguera [11] was assumed to be continuous. However, the assumption of continuity for  $t$ -norm is relaxed in this paper.

The Hausdorff topology induced by the fuzzy metric space was studied in Wu [13], and the concept of fuzzy semi-metric space was considered in Wu [14]. In this paper, we shall extend to study the  $T_1$ -spaces induced by the fuzzy semi-metric spaces that is endowed with special kind of triangle inequality. Roughly speaking, the fuzzy semi-metric space does not assume the symmetric condition  $M(x, y, t) = M(y, x, t)$ . In this case, there are four kinds of triangle inequalities that can be considered, which will be presented in Definition 2. We shall induce the  $T_1$ -spaces from the fuzzy semi-metric space based on a special kind of triangle inequality, which will generalize the results obtained in Wu [13]. On the other hand, since the symmetric condition is not satisfied in the fuzzy semi-metric space, three kinds of limit concepts will also be considered in this paper. Furthermore, we shall prove the consistency of limit concepts in the induced topologies.

This paper is organized as follows. In Section 2, the basic properties of  $t$ -norm are presented that will be used for the further discussion. In Section 3, we propose the fuzzy semi-metric space that is endowed with four kinds of triangle inequalities. In Section 4, we induce the  $T_1$ -space from a given fuzzy semi-metric space endowed with a special kind of triangle inequality. In Section 5, three kinds of limits in fuzzy semi-metric space will be considered. We also present the consistency of limit concepts in the induced topologies.

## 2. Properties of $t$ -Norm

We first recall the concept of triangular norm (i.e.,  $t$ -norm). We consider the function  $*$  :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  from the product space  $[0, 1] \times [0, 1]$  of unit intervals into the unit interval  $[0, 1]$ . The function  $*$  is called a  $t$ -norm if and only if the following conditions are satisfied:

- (boundary condition)  $a * 1 = a$ ;
- (commutativity)  $a * b = b * a$ ;
- (increasing property) if  $b \leq c$ , then  $a * b \leq a * c$ ;
- (associativity)  $(a * b) * c = a * (b * c)$ .

From the third condition, it follows that, for any  $a \in [0, 1]$ , we have  $0 * a \leq 0 * 1$ . From the first condition, we also have  $0 * 1 = 0$ , which implies  $0 * a = 0$ . The following proposition from Wu [13] will be useful for further study

**Proposition 1.** *By the commutativity of  $t$ -norm, if the  $t$ -norm is continuous with respect to the first component (resp. second component), then it is also continuous with respect to the second component (resp. first component). In other words, for any fixed  $a \in [0, 1]$ , if the function  $f(x) = a * x$  (resp.  $f(x) = x * a$ ) is continuous, then the function  $g(x) = x * a$  (resp.  $g(x) = a * x$ ) is continuous. Similarly, if the  $t$ -norm is left-continuous (resp. right-continuous) with respect to the first or second component, then it is also left-continuous (resp. right-continuous) with respect to each component.*

We first provide some properties that will be used in the subsequent discussion.

**Proposition 2.** *We have the following properties:*

- (i) *Given any fixed  $a, b \in [0, 1]$ , suppose that the  $t$ -norm  $*$  is continuous at  $a$  and  $b$  with respect to the first or second component. If  $\{a_n\}_{n=1}^{\infty}$  and  $\{b_n\}_{n=1}^{\infty}$  are two sequences in  $[0, 1]$  such that  $a_n \rightarrow a$  and  $b_n \rightarrow b$  as  $n \rightarrow \infty$ , then  $a_n * b_n \rightarrow a * b$  as  $n \rightarrow \infty$ .*
- (ii) *Given any fixed  $a, b \in (0, 1]$ , suppose that the  $t$ -norm  $*$  is left-continuous at  $a$  and  $b$  with respect to the first or second component. If  $\{a_n\}_{n=1}^{\infty}$  and  $\{b_n\}_{n=1}^{\infty}$  are two sequences in  $[0, 1]$  such that  $a_n \rightarrow a-$  and  $b_n \rightarrow b-$  as  $n \rightarrow \infty$ , then  $a_n * b_n \rightarrow a * b$  as  $n \rightarrow \infty$ .*
- (iii) *Given any fixed  $a, b \in [0, 1)$ , suppose that the  $t$ -norm  $*$  is right-continuous at  $a$  and  $b$  with respect to the first or second component. If  $\{a_n\}_{n=1}^{\infty}$  and  $\{b_n\}_{n=1}^{\infty}$  are two sequences in  $[0, 1]$  such that  $a_n \rightarrow a+$  and  $b_n \rightarrow b+$  as  $n \rightarrow \infty$ , then  $a_n * b_n \rightarrow a * b$  as  $n \rightarrow \infty$ .*

**Proof.** To prove part (i), since  $a_n \rightarrow a$  as  $n \rightarrow \infty$ , there exist an increasing sequence  $\{p_n\}_{n=1}^\infty$  and a decreasing sequence  $\{q_n\}_{n=1}^\infty$  such that  $p_n \uparrow a$  and  $q_n \downarrow a$  satisfying  $p_n \leq a_n \leq q_n$ . In addition, there exists an increasing sequence  $\{r_n\}_{n=1}^\infty$  and a decreasing sequence  $\{s_n\}_{n=1}^\infty$  such that  $r_n \uparrow b$  and  $s_n \downarrow b$  satisfying  $r_n \leq b_n \leq s_n$ . By Remark 1, we see that the  $t$ -norm is continuous with respect to each component. Given any  $\epsilon > 0$ , using the continuity of  $t$ -norm at  $b$  with respect to the second component, there exists  $n_0 \in \mathbb{N}$  such that

$$a * b - \frac{\epsilon}{2} < a * r_{n_0} \text{ and } a * s_{n_0} < a * b + \frac{\epsilon}{2}. \quad (2)$$

In addition, using the continuity of  $t$ -norm at  $a$  with respect to the first component, there exists  $n_1 \in \mathbb{N}$  such that

$$a * r_{n_0} - \frac{\epsilon}{2} < p_{n_1} * r_{n_0} \text{ and } q_{n_1} * s_{n_0} < a * s_{n_0} + \frac{\epsilon}{2}. \quad (3)$$

According to Equation (2) and using the increasing property of  $t$ -norm, for  $n \geq n_0$ , we have

$$a * b - \frac{\epsilon}{2} < a * r_{n_0} \quad (4)$$

$$\leq a * r_n \leq a * b_n \leq a * s_n \leq a * s_{n_0} < a * b + \frac{\epsilon}{2}. \quad (5)$$

In addition, according to Equation (3), for  $m \geq n_1$  and  $n \geq n_0$ , we have

$$a * r_{n_0} - \frac{\epsilon}{2} < p_{n_1} * r_{n_0} \leq p_m * r_n \leq a_m * b_n \leq q_m * s_n \leq q_{n_1} * s_{n_0} < a * s_{n_0} + \frac{\epsilon}{2}. \quad (6)$$

By taking  $n_2 = \max\{n_0, n_1\}$ , from Equations (4) and (6), we obtain that  $n \geq n_2$  implies

$$\begin{aligned} a * b - \epsilon &< a * r_{n_0} - \frac{\epsilon}{2} \text{ (by Equation (4))} \\ &< a_n * b_n < a * s_{n_0} + \frac{\epsilon}{2} \text{ (from Equation (6) by taking } m = n) \\ &< a * b + \epsilon, \text{ (by Equation (5)),} \end{aligned}$$

which says that  $|a * b - a_n * b_n| < \epsilon$ . This shows the desired convergence.

To prove part (ii), we note that there exist two increasing sequences  $\{p_n\}_{n=1}^\infty$  and  $\{r_n\}_{n=1}^\infty$  such that  $p_n \uparrow a$  and  $r_n \uparrow b$  satisfying  $p_n \leq a_n$  and  $r_n \leq b_n$ . By Remark 1, we see that the  $t$ -norm is left continuous with respect to each component. Given any  $\epsilon > 0$ , using the left-continuity of  $t$ -norm at  $b$  with respect to the second component, there exists  $n_0 \in \mathbb{N}$  such that

$$a * b - \frac{\epsilon}{2} < a * r_{n_0}.$$

In addition, using the left-continuity of  $t$ -norm at  $a$  with respect to the first component, there exists  $n_1 \in \mathbb{N}$  such that

$$a * r_{n_0} - \frac{\epsilon}{2} < p_{n_1} * r_{n_0}.$$

Using the increasing property of  $t$ -norm, for  $m \geq n_1$  and  $n \geq n_0$ , we have

$$a * r_{n_0} - \frac{\epsilon}{2} < p_{n_1} * r_{n_0} \leq p_m * r_n \leq a_m * b_n.$$

Since  $a_n \rightarrow a-$  and  $b_n \rightarrow b-$ , we see that  $a_n \leq a$  and  $b_n \leq b$  for all  $n$ . By taking  $n_2 = \max\{n_0, n_1\}$ , for  $n \geq n_2$ , we obtain

$$a * b - \epsilon < a * r_{n_0} - \frac{\epsilon}{2} < a_n * b_n \leq a * b < a * b + \epsilon,$$

which says that  $|a * b - a_n * b_n| < \epsilon$ . This shows the desired convergence. Part (iii) can be similarly proved, and the proof is complete.  $\square$

The associativity of  $t$ -norm says that the operation  $a_1 * a_2 * \dots * a_p$  is well-defined for  $p \geq 2$ . The following proposition from Wu [13] will be useful for further study.

**Proposition 3.** Suppose that the  $t$ -norm  $*$  is left-continuous at 1 with respect to the first or second component. We have the following properties:

- (i) For any  $a, b \in (0, 1)$  with  $a > b$ , there exists  $r \in (0, 1)$  such that  $a * r \geq b$ .
- (ii) For any  $a \in (0, 1)$  and any  $n \in \mathbb{N}$  with  $n \geq 2$ , there exists  $r \in (0, 1)$  such that  $r * r * \dots * r > a$  for  $n$ -times.

### 3. Fuzzy Semi-Metric Space

In the sequel, we shall define the concept of fuzzy semi-metric space without considering the symmetric condition. Because of lacking symmetry, the concept of triangle inequality should be carefully interpreted. Therefore, we propose four kinds of triangle inequalities.

**Definition 1.** Let  $X$  be a nonempty universal set, and let  $M$  be a mapping defined on  $X \times X \times [0, \infty)$  into  $[0, 1]$ . Then,  $(X, M)$  is called a fuzzy semi-metric space if and only if the following conditions are satisfied:

- for any  $x, y \in X$ ,  $M(x, y, t) = 1$  for all  $t > 0$  if and only if  $x = y$ ;
- $M(x, y, 0) = 0$  for all  $x, y \in X$  with  $x \neq y$ .

We say that  $M$  satisfies the symmetric condition if and only if  $M(x, y, t) = M(y, x, t)$  for all  $x, y \in X$  and  $t > 0$ . We say that  $M$  satisfies the strongly symmetric condition if and only if  $M(x, y, t) = M(y, x, t)$  for all  $x, y \in X$  and  $t \geq 0$ .

We remark that the first condition says that  $M(x, x, t) = 1$  for all  $t > 0$ . However, the value of  $M(x, x, 0)$  is free. Recall that the mapping  $M(x, y, t)$  is interpreted as the membership degree of the distance that is less than  $t$  between  $x$  and  $y$ . Therefore,  $M(x, x, t) = 1$  for all  $t > 0$  means that the distance that is less than  $t > 0$  between  $x$  and  $x$  is always true. The second condition says that  $M(x, y, 0) = 0$  for  $x \neq y$ , which can be similarly realized that the distance that is less than 0 between two distinct elements  $x$  and  $y$  is impossible.

**Definition 2.** Let  $X$  be a nonempty universal set, let  $*$  be a  $t$ -norm, and let  $M$  be a mapping defined on  $X \times X \times [0, \infty)$  into  $[0, 1]$ .

- We say that  $M$  satisfies the  $\bowtie$ -triangle inequality if and only if the following inequality is satisfied:

$$M(x, y, t) * M(y, z, s) \leq M(x, z, t + s) \text{ for all } x, y, z \in X \text{ and } s, t > 0.$$

- We say that  $M$  satisfies the  $\triangleright$ -triangle inequality if and only if the following inequality is satisfied:

$$M(x, y, t) * M(z, y, s) \leq M(x, z, t + s) \text{ for all } x, y, z \in X \text{ and } s, t > 0.$$

- We say that  $M$  satisfies the  $\triangleleft$ -triangle inequality if and only if the following inequality is satisfied:

$$M(y, x, t) * M(y, z, s) \leq M(x, z, t + s) \text{ for all } x, y, z \in X \text{ and } s, t > 0.$$

- We say that  $M$  satisfies the  $\diamond$ -triangle inequality if and only if the following inequality is satisfied:

$$M(y, x, t) * M(z, y, s) \leq M(x, z, t + s) \text{ for all } x, y, z \in X \text{ and } s, t > 0.$$

We say that  $M$  satisfies the strong  $\circ$ -triangle inequality for  $\circ \in \{\bowtie, \triangleright, \triangleleft, \diamond\}$  when  $s, t > 0$  is replaced by  $s, t \geq 0$ .

**Remark 1.** It is obvious that if the mapping  $M$  satisfies the symmetric condition, then the concepts of  $\bowtie$ -triangle inequality,  $\triangleright$ -triangle inequality,  $\triangleleft$ -triangle inequality and  $\diamond$ -triangle inequality are all equivalent.

**Example 1.** Let  $X$  be a universal set, and let  $d : X \times X \rightarrow \mathbb{R}_+$  satisfy the following conditions:

- $d(x, y) \geq 0$  for any  $x, y \in X$ ;
- $d(x, y) = 0$  if and only if  $x = y$  for any  $x, y \in X$ ;
- $d(x, y) + d(y, z) \geq d(x, z)$  for any  $x, y, z \in X$ .

Note that we do not assume  $d(x, y) = d(y, x)$ . For example, let  $X = [0, 1]$ . We define

$$d(x, y) = \begin{cases} y - x, & \text{if } y \geq x, \\ 1, & \text{otherwise.} \end{cases}$$

Then,  $d(x, y) \neq d(y, x)$  and the above three conditions are satisfied. Now, we take  $t$ -norm  $*$  as  $a * b = ab$  and define

$$M(x, y, t) = \begin{cases} \frac{t}{t + d(x, y)}, & \text{if } t > 0, \\ 1, & \text{if } t = 0 \text{ and } d(x, y) = 0, \\ 0, & \text{if } t = 0 \text{ and } d(x, y) > 0, \end{cases} = \begin{cases} \frac{t}{t + d(x, y)} & \text{if } t > 0, \\ 1, & \text{if } t = 0 \text{ and } x = y, \\ 0, & \text{if } t = 0 \text{ and } x \neq y. \end{cases}$$

It is clear to see that  $M(x, y, t) \neq M(y, x, t)$  for  $t > 0$ , since  $d(x, y) \neq d(y, x)$ . We are going to claim that  $(X, M, *)$  is a fuzzy semi-metric space satisfying the  $\bowtie$ -triangle inequality. For  $t > 0$  and  $M(x, y, t) = 1$ , we have  $t = t + d(x, y)$ , which says that  $d(x, y) = 0$ , i.e.,  $x = y$ . Next, we are going to check the  $\bowtie$ -triangle inequality. For  $s > 0$  and  $t > 0$ , we first have

$$\frac{1}{t}d(x, y) + \frac{1}{s}d(y, z) \geq \frac{1}{s+t} [d(x, y) + d(y, z)] \geq \frac{1}{t+s}d(x, z).$$

Then, we obtain

$$\begin{aligned} M(x, y, t) * M(y, z, s) &= \frac{t}{t + d(x, y)} \cdot \frac{s}{s + d(y, z)} = \frac{ts}{ts + td(y, z) + sd(x, y) + d(x, y)d(y, z)} \\ &\leq \frac{ts}{ts + td(y, z) + sd(x, y)} = \frac{1}{1 + \frac{1}{s}d(y, z) + \frac{1}{t}d(x, y)} \\ &\leq \frac{1}{1 + \frac{1}{t+s}d(x, z)} = \frac{t+s}{t+s+d(x, z)} = M(x, z, t+s). \end{aligned}$$

This shows that  $(X, M, *)$  defined above is indeed a fuzzy semi-metric space satisfying the  $\bowtie$ -triangle inequality.

Given a fuzzy semi-metric space  $(X, M)$ , when we say that the mapping  $M$  satisfies some kinds of (strong) triangle inequalities, it implicitly means that the  $t$ -norm is considered in  $(X, M)$ .

- Suppose that  $M$  satisfies the (strong)  $\triangleright$ -triangle inequality. Then,

$$M(x, y, t) * M(z, y, s) \leq M(x, z, t+s) \text{ and } M(z, y, t) * M(x, y, s) \leq M(z, x, t+s).$$

Since the  $t$ -norm is commutative, it follows that

$$M(x, y, t) * M(z, y, s) = M(z, y, t) * M(x, y, s) \leq \min \{M(x, z, t+s), M(z, x, t+s)\}.$$

- Suppose that  $M$  satisfies the (strong)  $\triangleleft$ -triangle inequality. Then, we similarly have

$$M(y, x, t) * M(y, z, s) = M(y, z, t) * M(y, x, s) \leq \min \{M(x, z, t + s), M(z, x, t + s)\}.$$

**Definition 3.** Let  $(X, M)$  be a fuzzy semi-metric space.

- We say that  $M$  is nondecreasing if and only if, given any fixed  $x, y \in X$ ,  $M(x, y, t_1) \geq M(x, y, t_2)$  for  $t_1 > t_2 > 0$ . We say that  $M$  is strongly nondecreasing if and only if, given any fixed  $x, y \in X$ ,  $M(x, y, t_1) \geq M(x, y, t_2)$  for  $t_1 > t_2 \geq 0$ .
- We say that  $M$  is symmetrically nondecreasing if and only if, given any fixed  $x, y \in X$ ,  $M(x, y, t_1) \geq M(y, x, t_2)$  for  $t_1 > t_2 > 0$ . We say that  $M$  is symmetrically strongly nondecreasing if and only if, given any fixed  $x, y \in X$ ,  $M(x, y, t_1) \geq M(y, x, t_2)$  for  $t_1 > t_2 \geq 0$ .

**Proposition 4.** Let  $(X, M)$  be a fuzzy semi-metric space. Then, we have the following properties:

- If  $M$  satisfies the  $\bowtie$ -triangle inequality, then  $M$  is nondecreasing.
- If  $M$  satisfies the  $\triangleright$ -triangle inequality or the  $\triangleleft$ -triangle inequality, then  $M$  is both nondecreasing and symmetrically nondecreasing.
- If  $M$  satisfies the  $\diamond$ -triangle inequality, then  $M$  is symmetrically nondecreasing.

**Proof.** Given any fixed  $x, y \in X$ , for  $t_1 > t_2 > 0$ , we have the following inequalities.

- Suppose that  $M$  satisfies the  $\bowtie$ -triangle inequality. Then,

$$M(x, y, t_1) \geq M(x, y, t_2) * M(y, y, t_1 - t_2) = M(x, y, t_2) * 1 = M(x, y, t_2).$$

- Suppose that  $M$  satisfies the  $\triangleright$ -triangle inequality. Then,

$$M(x, y, t_1) \geq M(x, y, t_2) * M(y, y, t_1 - t_2) = M(x, y, t_2) * 1 = M(x, y, t_2)$$

and

$$M(x, y, t_1) \geq M(x, x, t_1 - t_2) * M(y, x, t_2) = 1 * M(y, x, t_2) = M(y, x, t_2).$$

- Suppose that  $M$  satisfies the  $\triangleleft$ -triangle inequality. Then,

$$M(x, y, t_1) \geq M(x, x, t_1 - t_2) * M(x, y, t_2) = 1 * M(x, y, t_2) = M(x, y, t_2)$$

and

$$M(x, y, t_1) \geq M(y, x, t_2) * M(y, y, t_1 - t_2) = M(y, x, t_2) * 1 = M(y, x, t_2).$$

- Suppose that  $M$  satisfies the  $\diamond$ -triangle inequality. Then,

$$M(x, y, t_1) \geq M(x, x, t_1 - t_2) * M(y, x, t_2) = 1 * M(y, x, t_2) = M(y, x, t_2).$$

This completes the proof.  $\square$

**Definition 4.** Let  $(X, M)$  be a fuzzy semi-metric space.

- We say that  $M$  is left-continuous with respect to the distance at  $t_0 > 0$  if and only if, for any fixed  $x, y \in X$ , given any  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $0 < t_0 - t < \delta$  implies  $|M(x, y, t) - M(x, y, t_0)| < \epsilon$ ; that is, the mapping  $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$  is left-continuous at  $t_0$ . We say that  $M$  is left-continuous with respect to the distance on  $(0, \infty)$  if and only if the mapping  $M(x, y, \cdot)$  is left-continuous on  $(0, \infty)$  for any fixed  $x, y \in X$ .

- We say that  $M$  is right-continuous with respect to the distance at  $t_0 \geq 0$  if and only if, for any fixed  $x, y \in X$ , given any  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $0 < t - t_0 < \delta$  implies  $|M(x, y, t) - M(x, y, t_0)| < \epsilon$ ; that is, the mapping  $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$  is right-continuous at  $t_0$ . We say that  $M$  is right-continuous with respect to the distance on  $[0, \infty)$  if and only if the mapping  $M(x, y, \cdot)$  is left-continuous on  $[0, \infty)$  for any fixed  $x, y \in X$ .
- We say that  $M$  is continuous with respect to the distance at  $t_0 \geq 0$  if and only if, for any fixed  $x, y \in X$ , given any  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $|t - t_0| < \delta$  implies  $|M(x, y, t) - M(x, y, t_0)| < \epsilon$ ; that is, the mapping  $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$  is continuous at  $t_0$ . We say that  $M$  is continuous with respect to the distance on  $[0, \infty)$  if and only if the mapping  $M(x, y, \cdot)$  is continuous on  $[0, \infty)$  for any fixed  $x, y \in X$ .
- We say that  $M$  is symmetrically left-continuous with respect to the distance at  $t_0 > 0$  if and only if, for any fixed  $x, y \in X$ , given any  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $0 < t_0 - t < \delta$  implies  $|M(x, y, t) - M(y, x, t_0)| < \epsilon$ . We say that  $M$  is symmetrically left-continuous with respect to the distance on  $(0, \infty)$  if and only if it is symmetrically left-continuous with respect to the distance at each  $t > 0$ .
- We say that  $M$  is symmetrically right-continuous with respect to the distance at  $t_0 \geq 0$  if and only if, for any fixed  $x, y \in X$ , given any  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $0 < t - t_0 < \delta$  implies  $|M(x, y, t) - M(y, x, t_0)| < \epsilon$ . We say that  $M$  is symmetrically right-continuous with respect to the distance on  $[0, \infty)$  if and only if it is symmetrically right-continuous with respect to the distance at each  $t \geq 0$ .
- We say that  $M$  is symmetrically continuous with respect to the distance at  $t_0 \geq 0$  if and only if, for any fixed  $x, y \in X$ , given any  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $|t - t_0| < \delta$  implies  $|M(x, y, t) - M(y, x, t_0)| < \epsilon$ . We say that  $M$  is symmetrically continuous with respect to the distance on  $[0, \infty)$  if and only if it is symmetrically continuous with respect to the distance at each  $t \geq 0$ .

**Proposition 5.** Let  $(X, M)$  be a fuzzy semi-metric space such that the  $\circ$ -triangle inequality is satisfied for  $\circ \in \{\triangleright, \triangleleft, \diamond\}$ . Then, we have the following properties:

- Suppose that  $M$  is left-continuous or symmetrically left-continuous with respect to the distance at  $t > 0$ . Then  $M(x, y, t) = M(y, x, t)$ . In other words, if  $M$  is left-continuous or symmetrically left-continuous with respect to the distance on  $(0, \infty)$ . Then  $M$  satisfies the symmetric condition.
- Suppose that  $M$  is right-continuous or symmetrically right-continuous with respect to the distance at  $t \geq 0$ . Then  $M(x, y, t) = M(y, x, t)$ . In other words, if  $M$  is right-continuous or symmetrically right-continuous with respect to the distance on  $[0, \infty)$ . Then  $M$  satisfies the strongly symmetric condition.

**Proof.** To prove part (i), given any  $t > 0$ , there exists  $n_t \in \mathbb{N}$  satisfying  $t - \frac{1}{n_t} > 0$ . We consider the following cases:

- Suppose that the  $\triangleright$ -triangle inequality is satisfied. Then,

$$M\left(y, x, t - \frac{1}{n_t}\right) = 1 * M\left(y, x, t - \frac{1}{n_t}\right) = M\left(x, x, \frac{1}{n_t}\right) * M\left(y, x, t - \frac{1}{n_t}\right) \leq M(x, y, t)$$

and

$$M\left(x, y, t - \frac{1}{n_t}\right) = 1 * M\left(x, y, t - \frac{1}{n_t}\right) = M\left(y, y, \frac{1}{n_t}\right) * M\left(x, y, t - \frac{1}{n_t}\right) \leq M(y, x, t).$$

Using the left-continuity of  $M$ , it follows that  $M(y, x, t) \leq M(x, y, t)$  and  $M(x, y, t) \leq M(y, x, t)$  by taking  $n_t \rightarrow \infty$ . This shows that  $M(x, y, t) = M(y, x, t)$  for all  $t > 0$ . On the other hand, we also have

$$M\left(x, y, t - \frac{1}{n_t}\right) = M\left(x, y, t - \frac{1}{n_t}\right) * 1 = M\left(x, y, t - \frac{1}{n_t}\right) * M\left(y, y, \frac{1}{n_t}\right) \leq M(x, y, t)$$

and

$$M\left(y, x, t - \frac{1}{n_t}\right) = M\left(y, x, t - \frac{1}{n_t}\right) * 1 = M\left(y, x, t - \frac{1}{n_t}\right) * M\left(x, x, \frac{1}{n_t}\right) \leq M(y, x, t).$$

Using the symmetric left-continuity of  $M$ , it follows that  $M(y, x, t) \leq M(x, y, t)$  and  $M(x, y, t) \leq M(y, x, t)$  by taking  $n_t \rightarrow \infty$ . This shows that  $M(x, y, t) = M(y, x, t)$  for all  $t > 0$ .

- Suppose that the  $\triangleleft$ -triangle inequality is satisfied. Then,

$$M\left(y, x, t - \frac{1}{n_t}\right) = M\left(y, x, t - \frac{1}{n_t}\right) * 1 = M\left(y, x, t - \frac{1}{n_t}\right) * M\left(y, y, \frac{1}{n_t}\right) \leq M(x, y, t)$$

and

$$M\left(x, y, t - \frac{1}{n_t}\right) = M\left(x, y, t - \frac{1}{n_t}\right) * 1 = M\left(x, y, t - \frac{1}{n_t}\right) * M\left(x, x, \frac{1}{n_t}\right) \leq M(y, x, t).$$

The left-continuity of  $M$  shows that  $M(x, y, t) = M(y, x, t)$  for all  $t > 0$ . We can similarly obtain the desired result using the symmetric left-continuity of  $M$ .

- Suppose that the  $\diamond$ -triangle inequality is satisfied. Then, this is the same situation as the  $\triangleright$ -triangle inequality.

To prove part (ii), given any  $t \geq 0$  and  $n \in \mathbb{N}$ , we consider the following cases.

- Suppose that the  $\triangleright$ -triangle inequality is satisfied. Then,

$$M\left(y, x, t + \frac{1}{n}\right) = 1 * M\left(y, x, t + \frac{1}{n}\right) = M\left(x, x, \frac{1}{n}\right) * M\left(y, x, t + \frac{1}{n}\right) \leq M\left(x, y, t + \frac{2}{n}\right)$$

and

$$M\left(x, y, t + \frac{1}{n}\right) = 1 * M\left(x, y, t + \frac{1}{n}\right) = M\left(y, y, \frac{1}{n}\right) * M\left(x, y, t + \frac{1}{n}\right) \leq M\left(y, x, t + \frac{2}{n}\right).$$

The right-continuity of  $M$  shows that  $M(x, y, t) = M(y, x, t)$  for all  $t \geq 0$ . We can similarly obtain the desired result using the symmetric right-continuity of  $M$ .

- Suppose that the  $\triangleleft$ -triangle inequality is satisfied. Then,

$$M\left(y, x, t + \frac{1}{n}\right) = M\left(y, x, t + \frac{1}{n}\right) * 1 = M\left(y, x, t + \frac{1}{n}\right) * M\left(y, y, \frac{1}{n}\right) \leq M\left(x, y, t + \frac{2}{n}\right)$$

and

$$M\left(x, y, t + \frac{1}{n}\right) = M\left(x, y, t + \frac{1}{n}\right) * 1 = M\left(x, y, t + \frac{1}{n}\right) * M\left(x, x, \frac{1}{n}\right) \leq M\left(y, x, t + \frac{2}{n}\right).$$

The right-continuity of  $M$  shows that  $M(x, y, t) = M(y, x, t)$  for all  $t \geq 0$ . We can similarly obtain the desired result using the symmetric right-continuity of  $M$ .

- Suppose that the  $\diamond$ -triangle inequality is satisfied. Then, this is the same situation as the  $\triangleright$ -triangle inequality.

This completes the proof.  $\square$

From Proposition 5, if  $M$  is left-continuous or symmetrically left-continuous with respect to the distance on  $(0, \infty)$ , or right-continuous and or symmetrically right-continuous with respect to the distance on  $(0, \infty]$ , then we can just consider the  $\bowtie$ -triangle inequality.

**Proposition 6.** Let  $(X, M)$  be a fuzzy semi-metric space such that  $M$  is left-continuous or symmetrically left-continuous with respect to the distance on  $(0, \infty)$ , or right-continuous and or symmetrically right-continuous with respect to the distance on  $(0, \infty]$ . Suppose that  $M(x, x, 0) = 1$  for any  $x \in X$ . Then,  $M$  satisfies the  $\circ$ -triangle inequality if and only if  $M$  satisfies the strong  $\circ$ -triangle inequality for  $\circ \in \{\triangleright, \triangleleft, \diamond\}$ .

**Proof.** We first note that the converse is obvious. Now, we assume that  $M$  satisfies the  $\triangleleft$ -triangle inequality.

- Suppose that  $s = t = 0$ . If  $x \neq y$  or  $y \neq z$ , then  $M(y, x, 0) = 0$  or  $M(y, z, 0) = 0$ , which implies

$$M(y, x, 0) * M(y, z, 0) = 0 \leq M(x, z, 0).$$

If  $x = y = z$ , then  $M(y, x, 0) = 1 = M(y, z, 0) = M(x, z, 0)$ , which implies

$$M(y, x, 0) * M(y, z, 0) = 1 * 1 = 1 = M(x, z, 0).$$

- Suppose that  $s > 0$  and  $t = 0$ . If  $x \neq y$ , then  $M(y, x, 0) = 0$ , which implies

$$M(y, x, t) * M(y, z, s) = M(y, x, 0) * M(y, z, s) = 0 \leq M(x, z, t + s).$$

If  $x = y$ , then  $M(y, x, t) = M(x, x, 0) = 1$ , which implies

$$M(y, x, t) * M(y, z, s) = 1 * M(y, z, s) = M(y, z, s) = M(x, z, t + s).$$

- Suppose that  $s = 0$  and  $t > 0$ . If  $y \neq z$ , then  $M(y, z, 0) = 0$ , which implies

$$M(y, x, t) * M(y, z, s) = M(y, x, t) * M(y, z, 0) = 0 \leq M(x, z, t + s).$$

If  $y = z$ , then  $M(y, z, s) = M(y, y, 0) = 1$ . Using Proposition 5, we have

$$M(y, x, t) * M(y, z, s) = M(y, x, t) * 1 = M(y, x, t) = M(x, y, t) = M(x, z, t + s).$$

We can similarly obtain the desired results for  $\circ \in \{\triangleleft, \diamond\}$ . This completes the proof.  $\square$

**Proposition 7.** Let  $(X, M)$  be a fuzzy semi-metric space. Suppose that  $M$  satisfies the  $\bowtie$ -triangle inequality, and that  $M$  is left-continuous with respect to the distance at  $t > 0$ . Given any fixed  $x, y \in X$ , if  $M(x, y, t) > 1 - r$ , then there exists  $t_0$  with  $0 < t_0 < t$  such that  $M(x, y, t_0) > 1 - r$ .

**Proof.** Let  $\epsilon = M(x, y, t) - (1 - r) > 0$ . Using the left-continuity of  $M$ , there exists  $t_0$  with  $0 < t_0 < t$  such that  $|M(x, y, t) - M(x, y, t_0)| < \epsilon$ . From part (i) of Proposition 4, we also have  $0 \leq M(x, y, t) - M(x, y, t_0) < \epsilon$ , which implies  $M(x, y, t_0) > 1 - r$ . This completes the proof.  $\square$

#### 4. $T_1$ -Spaces

Let  $(X, M, *)$  be a fuzzy metric space, i.e., the symmetric condition is satisfied. Given  $t > 0$  and  $0 < r < 1$ , the  $(r, t)$ -ball of  $x$  is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) = M(y, x, t) > 1 - r\}$$

by referring to Wu [13]. In this paper, since the symmetric condition is not satisfied, two different concepts of open ball will be proposed below. Therefore, the  $T_1$ -spaces generated from these two different open balls will generalize the results obtained in Wu [13].

**Definition 5.** Let  $(X, M)$  be a fuzzy semi-metric space. Given  $t > 0$  and  $0 < r < 1$ , the  $(r, t)$ -balls centered at  $x$  are denoted and defined by

$$B^{\triangleleft}(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$$

and

$$B^{\triangleright}(x, r, t) = \{y \in X : M(y, x, t) > 1 - r\}.$$

Let  $\mathcal{B}^{\triangleleft}$  denote the family of all  $(r, t)$ -balls  $B^{\triangleleft}(x, r, t)$ , and let  $\mathcal{B}^{\triangleright}$  denote the family of all  $(r, t)$ -balls  $B^{\triangleright}(x, r, t)$ .

It is clearly that if the symmetric condition for  $M$  is satisfied, then

$$B^{\triangleleft}(x, r, t) = B^{\triangleright}(x, r, t).$$

In this case, we simply write  $B(x, r, t)$  to denote the  $(r, t)$ -balls centered at  $x$ , and write  $\mathcal{B}$  to denote the family of all  $(r, t)$ -balls  $B(x, r, t)$ .

We also see that  $B^{\triangleleft}(x, r, t) \neq \emptyset$  and  $B^{\triangleright}(x, r, t) \neq \emptyset$ , since  $x \in B^{\triangleleft}(x, r, t)$  and  $x \in B^{\triangleright}(x, r, t)$  by the fact of  $M(x, x, t) = 1$  for all  $t > 0$ . Since  $0 < r < 1$ , it is obvious that if  $M(x, y, t) = 0$ , then  $y \notin B^{\triangleleft}(x, r, t)$ . In other words, if  $y \in B^{\triangleleft}(x, r, t)$ , then  $M(x, y, t) > 0$ . Similarly, if  $y \in B^{\triangleright}(x, r, t)$ , then  $M(y, x, t) > 0$ .

**Proposition 8.** Let  $(X, M)$  be a fuzzy semi-metric space.

- (i) For each  $x \in X$ , we have  $x \in B^{\triangleleft}(x, r, t) \in \mathcal{B}^{\triangleleft}$  and  $x \in B^{\triangleright}(x, r, t) \in \mathcal{B}^{\triangleright}$ .
- (ii) If  $x \neq y$ , then there exist  $B^{\triangleleft}(x, r, t)$  and  $B^{\triangleright}(x, r, t)$  such that  $y \notin B^{\triangleleft}(x, r, t)$  and  $y \notin B^{\triangleright}(x, r, t)$ .

**Proof.** Part (i) is obvious. To prove part (ii), since  $x \neq y$ , there exists  $t_0 > 0$  such that  $M(x, y, t_0) < 1$ . There also exists  $r_0$  such that  $M(x, y, t_0) < r_0 < 1$ . Suppose that  $y \in B^{\triangleleft}(x, 1 - r_0, t_0)$ . Then, we have

$$M(x, y, t_0) > r_0 > M(x, y, t_0).$$

This contradiction says that  $y \notin B^{\triangleleft}(x, 1 - r_0, t_0)$ , and the proof is complete.  $\square$

**Proposition 9.** Let  $(X, M)$  be a fuzzy semi-metric space.

- (i) Suppose that  $M$  satisfies the  $\circ$ -triangle for  $\circ \in \{\bowtie, \triangleright, \triangleleft\}$ . Then, the following statements hold true:
  - Given any  $B^{\triangleleft}(x, r, t) \in \mathcal{B}^{\triangleleft}$ , there exists  $n \in \mathbb{N}$  such that  $B^{\triangleleft}(x, 1/n, 1/n) \subseteq B^{\triangleleft}(x, r, t)$ .
  - Given any  $B^{\triangleright}(x, r, t) \in \mathcal{B}^{\triangleright}$ , there exists  $n \in \mathbb{N}$  such that  $B^{\triangleright}(x, 1/n, 1/n) \subseteq B^{\triangleright}(x, r, t)$ .
- (ii) Suppose that  $M$  satisfies the  $\circ$ -triangle for  $\circ \in \{\triangleright, \triangleleft, \diamond\}$ . Then, the following statements hold true:
  - Given any  $B^{\triangleleft}(x, r, t) \in \mathcal{B}^{\triangleleft}$ , there exists  $n \in \mathbb{N}$  such that  $B^{\triangleright}(x, 1/n, 1/n) \subseteq B^{\triangleleft}(x, r, t)$ .
  - Given any  $B^{\triangleright}(x, r, t) \in \mathcal{B}^{\triangleright}$ , there exists  $n \in \mathbb{N}$  such that  $B^{\triangleleft}(x, 1/n, 1/n) \subseteq B^{\triangleright}(x, r, t)$ .

**Proof.** To prove part (i), it suffices to prove the first case. We take  $n \in \mathbb{N}$  such that  $1/n \leq \min\{r, t\}$ . Then, for  $y \in B^{\triangleleft}(x, 1/n, 1/n)$ , using parts (i) and (ii) of Proposition 4, we have

$$M(x, y, t) \geq M\left(x, y, \frac{1}{n}\right) > 1 - \frac{1}{n} \geq 1 - r,$$

which says that  $y \in B^{\triangleleft}(x, r, t)$ . Part (ii) can be similarly obtained by using parts (ii) and (iii) of Proposition 4, and the following inequalities:

$$M(x, y, t) \geq M\left(y, x, \frac{1}{n}\right) > 1 - \frac{1}{n} \geq 1 - r.$$

This completes the proof.  $\square$

**Proposition 10.** (Left-Continuity for  $M$ ) Let  $(X, M)$  be a fuzzy semi-metric space along with a  $t$ -norm  $*$  such that the following conditions are satisfied:

- $M$  is left-continuous with respect to the distance on  $(0, \infty)$ ;
- the  $t$ -norm  $*$  is left-continuous at 1 with respect to the first or second component.

Suppose that  $M$  satisfies the  $\bowtie$ -triangle inequality. Then, we have the following inclusions:

- Given any  $y \in B^{\triangleleft}(x, r, t)$ , there exists  $B^{\triangleleft}(y, \bar{r}, \bar{t})$  such that  $B^{\triangleleft}(y, \bar{r}, \bar{t}) \subseteq B^{\triangleleft}(x, r, t)$ .
- Given any  $y \in B^{\triangleright}(x, r, t)$ , there exists  $B^{\triangleright}(y, \bar{r}, \bar{t})$  such that  $B^{\triangleright}(y, \bar{r}, \bar{t}) \subseteq B^{\triangleright}(x, r, t)$ .

**Proof.** For  $y \in B^{\triangleleft}(x, r, t)$ , we have  $M(x, y, t) > 1 - r$ . By part (i) of Proposition 7, there exists  $t_0$  with  $0 < t_0 < t$  such that  $M(x, y, t_0) > 1 - r$ . Let  $r_0 = M(x, y, t_0)$ . Then, we have  $r_0 > 1 - r$ . There exists  $s$  with  $0 < s < 1$  such that  $r_0 > 1 - s > 1 - r$ . By part (i) of Proposition 3, there exists  $r_1$  with  $0 < r_1 < 1$  such that  $r_0 * r_1 \geq 1 - s$ . Let  $\bar{r} = 1 - r_1$  and  $\bar{t} = t - t_0$ . Similarly, for  $y \in B^{\triangleright}(x, r, t)$ , we have  $M(y, x, t) > 1 - r$ . In this case, let  $r_0 = M(y, x, t_0)$ .

To prove part (i), for  $y \in B^{\triangleleft}(x, r, t)$  and  $z \in B^{\triangleleft}(y, \bar{r}, \bar{t})$ , we have

$$M(y, z, t - t_0) = M(y, z, \bar{t}) > 1 - \bar{r} = r_1.$$

By the  $\bowtie$ -triangle inequality, we also have

$$M(x, z, t) \geq M(x, y, t_0) * M(y, z, t - t_0) = r_0 * M(y, z, t - t_0) \geq r_0 * r_1 \geq 1 - s > 1 - r.$$

This shows that  $z \in B^{\triangleleft}(x, r, t)$ . Therefore, we obtain the inclusion  $B^{\triangleleft}(y, \bar{r}, \bar{t}) \subseteq B^{\triangleleft}(x, r, t)$ .

To prove part (ii), for  $y \in B^{\triangleright}(x, r, t)$  and  $z \in B^{\triangleright}(y, \bar{r}, \bar{t})$ , we have

$$M(z, y, t - t_0) = M(z, y, \bar{t}) > 1 - \bar{r} = r_1.$$

By the  $\bowtie$ -triangle inequality, we also have

$$M(z, x, t) \geq M(z, y, t - t_0) * M(y, x, t_0) = M(z, y, t - t_0) * r_0 \geq r_0 * r_1 \geq 1 - s > 1 - r.$$

This shows that  $z \in B^{\triangleright}(x, r, t)$ . Therefore, we obtain the inclusion  $B^{\triangleright}(y, \bar{r}, \bar{t}) \subseteq B^{\triangleright}(x, r, t)$ . This completes the proof.  $\square$

According to Proposition 5, since  $M$  is assumed to be left-continuous with respect to the distance on  $(0, \infty)$ , it is not necessarily to consider the  $\circ$ -triangle inequality for  $\circ \in \{\triangleright, \triangleleft, \diamond\}$  in Proposition 10.

**Proposition 11.** (Symmetric Left-Continuity for  $M$ ) Let  $(X, M)$  be a fuzzy semi-metric space along with a  $t$ -norm  $*$  such that the following conditions are satisfied:

- $M$  is symmetrically left-continuous with respect to the distance on  $(0, \infty)$ ;
- the  $t$ -norm  $*$  is left-continuous at 1 with respect to the first or second component.

Suppose that  $M$  satisfies the  $\bowtie$ -triangle inequality. Then, we have the following inclusions:

- Given any  $y \in B^{\triangleleft}(x, r, t)$ , there exists  $B^{\triangleright}(y, \bar{r}, \bar{t})$  such that  $B^{\triangleright}(y, \bar{r}, \bar{t}) \subseteq B^{\triangleright}(x, r, t)$ .
- Given any  $y \in B^{\triangleright}(x, r, t)$ , there exists  $B^{\triangleleft}(y, \bar{r}, \bar{t})$  such that  $B^{\triangleleft}(y, \bar{r}, \bar{t}) \subseteq B^{\triangleleft}(x, r, t)$ .

**Proof.** For  $y \in B^{\triangleleft}(x, r, t)$ , we have  $M(x, y, t) > 1 - r$ . By part (ii) of Proposition 7, there exists  $t_0$  with  $0 < t_0 < t$  such that  $M(y, x, t_0) > 1 - r$ . Let  $r_0 = M(y, x, t_0)$ . Then, we have  $r_0 > 1 - r$ . There exists  $s$  with  $0 < s < 1$  such that  $r_0 > 1 - s > 1 - r$ . By part (i) of Proposition 3, there exists  $r_1$  with

$0 < r_1 < 1$  such that  $r_0 * r_1 \geq 1 - s$ . Let  $\bar{r} = 1 - r_1$  and  $\bar{t} = t - t_0$ . Similarly, for  $y \in B^\triangleright(x, r, t)$ , we have  $M(y, x, t) > 1 - r$ . In this case, let  $r_0 = M(x, y, t_0)$ .

To prove part (i), for  $y \in B^\triangleleft(x, r, t)$  and  $z \in B^\triangleright(y, \bar{r}, \bar{t})$ , we have

$$M(z, y, t - t_0) = M(z, y, \bar{t}) > 1 - \bar{r} = r_1.$$

By the  $\bowtie$ -triangle inequality, we have

$$M(z, x, t) \geq M(z, y, t - t_0) * M(y, x, t_0) = M(z, y, t - t_0) * r_0 \geq r_1 * r_0 \geq 1 - s > 1 - r.$$

This shows that  $z \in B^\triangleright(x, r, t)$ . Therefore, we obtain the inclusion  $B^\triangleright(y, \bar{r}, \bar{t}) \subseteq B^\triangleright(x, r, t)$ .

To prove part (ii), for  $y \in B^\triangleright(x, r, t)$  and  $z \in B^\triangleleft(y, \bar{r}, \bar{t})$ , we have

$$M(y, z, t - t_0) = M(y, z, \bar{t}) > 1 - \bar{r} = r_1.$$

By the  $\bowtie$ -triangle inequality, we have

$$M(x, z, t) \geq M(x, y, t_0) * M(y, z, t - t_0) = r_0 * M(y, z, t - t_0) \geq r_1 * r_0 \geq 1 - s > 1 - r.$$

This shows that  $z \in B^\triangleleft(x, r, t)$ . Therefore, we obtain the inclusion  $B^\triangleleft(y, \bar{r}, \bar{t}) \subseteq B^\triangleleft(x, r, t)$ .

This completes the proof.  $\square$

According to Proposition 5, since  $M$  is assumed to be symmetrically left-continuous with respect to the distance on  $(0, \infty)$ , it is not necessarily to consider the  $\circ$ -triangle inequality for  $\circ \in \{\triangleright, \triangleleft, \diamond\}$  in Proposition 11.

**Proposition 12.** (Left-Continuity for  $M$ ) Let  $(X, M)$  be a fuzzy semi-metric space along with a  $t$ -norm  $*$  such that the following conditions are satisfied:

- $M$  is left-continuous with respect to the distance on  $(0, \infty)$ ;
- the  $t$ -norm  $*$  is left-continuous at 1 with respect to the first or second component.

Suppose that  $M$  satisfies the  $\bowtie$ -triangle inequality. We have the following inclusions:

- (i) If  $x \in B^\triangleleft(x_1, r_1, t_1) \cap B^\triangleleft(x_2, r_2, t_2)$ , then there exists  $B^\triangleleft(x, r_3, t_3)$  such that

$$B^\triangleleft(x, r_3, t_3) \subseteq B^\triangleleft(x_1, r_1, t_1) \cap B^\triangleleft(x_2, r_2, t_2). \quad (7)$$

- (ii) If  $x \in B^\triangleright(x_1, r_1, t_1) \cap B^\triangleright(x_2, r_2, t_2)$ , then there exists  $B^\triangleright(x, r_3, t_3)$  such that

$$B^\triangleright(x, r_3, t_3) \subseteq B^\triangleright(x_1, r_1, t_1) \cap B^\triangleright(x_2, r_2, t_2). \quad (8)$$

**Proof.** Using part (i) of Proposition 10, there exist  $\bar{t}_1, \bar{t}_2, \bar{r}_1, \bar{r}_2$  such that

$$B^\triangleleft(x, \bar{r}_1, \bar{t}_1) \subseteq B^\triangleleft(x_1, r_1, t_1) \text{ and } B^\triangleleft(x, \bar{r}_2, \bar{t}_2) \subseteq B^\triangleleft(x_2, r_2, t_2).$$

We take  $t_3 = \min\{\bar{t}_1, \bar{t}_2\}$  and  $r_3 = \min\{\bar{r}_1, \bar{r}_2\}$ . Then, for  $y \in B^\triangleleft(x, r_3, t_3)$ , using part (i) of Proposition 4, we have

$$M(x, y, \bar{t}_1) \geq M(x, y, t_3) > 1 - r_3 \geq 1 - \bar{r}_1$$

and

$$M(x, y, \bar{t}_2) \geq M(x, y, t_3) > 1 - r_3 \geq 1 - \bar{r}_2,$$

which say that

$$y \in B^\triangleleft(x, \bar{r}_1, \bar{t}_1) \cap B^\triangleleft(x, \bar{r}_2, \bar{t}_2) \subseteq B^\triangleleft(x_1, r_1, t_1) \cap B^\triangleleft(x_2, r_2, t_2).$$

Therefore, we obtain the inclusion of Equation (7). The second inclusion of Equation (8) can be similarly obtained. This completes the proof.  $\square$

**Proposition 13.** (Symmetric Left-Continuity for  $M$ ) Let  $(X, M)$  be a fuzzy semi-metric space along with a  $t$ -norm  $*$  such that the following conditions are satisfied:

- $M$  is symmetrically left-continuous with respect to the distance on  $(0, \infty)$ ;
- the  $t$ -norm  $*$  is left-continuous at 1 with respect to the first or second component.

Suppose that  $M$  satisfies the  $\bowtie$ -triangle inequality. Then, we have the following inclusions:

- (i) If  $x \in B^{\triangleleft}(x_1, r_1, t_1) \cap B^{\triangleleft}(x_2, r_2, t_2)$ , then there exists  $B^{\triangleleft}(x, r_3, t_3)$  such that

$$B^{\triangleleft}(x, r_3, t_3) \subseteq B^{\triangleleft}(x_1, r_1, t_1) \cap B^{\triangleleft}(x_2, r_2, t_2). \quad (9)$$

- (ii) If  $x \in B^{\triangleright}(x_1, r_1, t_1) \cap B^{\triangleright}(x_2, r_2, t_2)$ , then there exists  $B^{\triangleleft}(x, r_3, t_3)$  such that

$$B^{\triangleright}(x, r_3, t_3) \subseteq B^{\triangleright}(x_1, r_1, t_1) \cap B^{\triangleright}(x_2, r_2, t_2). \quad (10)$$

**Proof.** Using part (iv) of Proposition 11, there exist  $\bar{t}_1, \bar{t}_2, \bar{r}_1, \bar{r}_2$  such that

$$B^{\triangleright}(x, \bar{r}_1, \bar{t}_1) \subseteq B^{\triangleright}(x_1, r_1, t_1) \text{ and } B^{\triangleright}(x, \bar{r}_2, \bar{t}_2) \subseteq B^{\triangleright}(x_2, r_2, t_2).$$

We take  $t_3 = \min\{\bar{t}_1, \bar{t}_2\}$  and  $r_3 = \min\{\bar{r}_1, \bar{r}_2\}$ . Then, for  $y \in B^{\triangleleft}(x, r_3, t_3)$ , using part (i) of Proposition 4, we have

$$M(y, x, \bar{t}_1) \geq M(y, x, t_3) > 1 - r_3 \geq 1 - \bar{r}_1$$

and

$$M(y, x, \bar{t}_2) \geq M(y, x, t_3) > 1 - r_3 \geq 1 - \bar{r}_2,$$

which say that

$$y \in B^{\triangleright}(x, \bar{r}_1, \bar{t}_1) \cap B^{\triangleright}(x, \bar{r}_2, \bar{t}_2) \subseteq B^{\triangleright}(x_1, r_1, t_1) \cap B^{\triangleright}(x_2, r_2, t_2).$$

Therefore, we obtain the inclusion of Equation (9). The second inclusion of Equation (10) can be similarly obtained. This completes the proof.  $\square$

The following proposition does not assume the left-continuity or symmetric left-continuity for  $M$ . Therefore, we can consider the different  $\circ$ -triangle inequality for  $\circ \in \{\bowtie, \triangleright, \triangleleft, \diamond\}$ .

**Proposition 14.** Let  $(X, M)$  be a fuzzy semi-metric space along with a  $t$ -norm  $*$  that is left-continuous at 1 with respect to the first or second component. Suppose that  $x \neq y$ . We have the following properties.

- (i) Suppose that  $M$  satisfies the  $\bowtie$ -triangle inequality or the  $\diamond$ -triangle inequality. Then,

$$B^{\triangleleft}(x, r, t) \cap B^{\triangleright}(y, r, t) = \emptyset \text{ and } B^{\triangleright}(x, r, t) \cap B^{\triangleleft}(y, r, t) = \emptyset$$

for some  $r \in (0, 1)$  and  $t > 0$ .

- (ii) Suppose that  $M$  satisfies the  $\triangleright$ -triangle inequality. Then,

$$B^{\triangleleft}(x, r, t) \cap B^{\triangleleft}(y, r, t) = \emptyset$$

for some  $r \in (0, 1)$  and  $t > 0$ .

- (iii) Suppose that  $M$  satisfies the  $\triangleleft$ -triangle inequality. Then,

$$B^{\triangleright}(x, r, t) \cap B^{\triangleright}(y, r, t) = \emptyset$$

for some  $r \in (0, 1)$  and  $t > 0$ .

**Proof.** Since  $x \neq y$ , there exists  $t_0 > 0$  such that  $M(x, y, t_0) < 1$ . There also exists  $r_0$  such that  $M(x, y, t_0) < r_0 < 1$ . By part (ii) of Proposition 3, there exists  $\hat{r}$  with  $0 < \hat{r} < 1$  such that  $\hat{r} * \hat{r} > r_0$ .

- Suppose that  $M$  satisfies the  $\triangleright$ -triangle inequality. We are going to prove that

$$B^{\triangleleft} \left( x, 1 - \hat{r}, \frac{t_0}{2} \right) \cap B^{\triangleleft} \left( y, 1 - \hat{r}, \frac{t_0}{2} \right) = \emptyset$$

by contradiction. Suppose that

$$z \in B^{\triangleleft} \left( x, 1 - \hat{r}, \frac{t_0}{2} \right) \cap B^{\triangleleft} \left( y, 1 - \hat{r}, \frac{t_0}{2} \right).$$

Since  $M$  satisfies the  $\triangleright$ -triangle inequality, it follows that

$$M(x, y, t_0) \geq M \left( x, z, \frac{t_0}{2} \right) * M \left( y, z, \frac{t_0}{2} \right) \geq \hat{r} * \hat{r} > r_0 > M(x, y, t_0),$$

which is a contradiction.

- Suppose that  $M$  satisfies the  $\triangleleft$ -triangle inequality for

$$z \in B^{\triangleright} \left( x, 1 - \hat{r}, \frac{t_0}{2} \right) \cap B^{\triangleright} \left( y, 1 - \hat{r}, \frac{t_0}{2} \right).$$

Since  $M$  satisfies the  $\triangleleft$ -triangle inequality, it follows that

$$M(x, y, t_0) \geq M \left( z, x, \frac{t_0}{2} \right) * M \left( z, y, \frac{t_0}{2} \right) \geq \hat{r} * \hat{r} > r_0 > M(x, y, t_0),$$

which is a contradiction.

- Suppose that  $M$  satisfies the  $\bowtie$ -triangle inequality for

$$z \in B^{\triangleleft} \left( x, 1 - \hat{r}, \frac{t_0}{2} \right) \cap B^{\triangleright} \left( y, 1 - \hat{r}, \frac{t_0}{2} \right).$$

Since  $M$  satisfies the  $\bowtie$ -triangle inequality, it follows that

$$M(x, y, t_0) \geq M \left( x, z, \frac{t_0}{2} \right) * M \left( z, y, \frac{t_0}{2} \right) \geq \hat{r} * \hat{r} > r_0 > M(x, y, t_0),$$

which is a contradiction. On the other hand, for

$$z \in B^{\triangleright} \left( x, 1 - \hat{r}, \frac{t_0}{2} \right) \cap B^{\triangleleft} \left( y, 1 - \hat{r}, \frac{t_0}{2} \right).$$

Since  $M$  satisfies the  $\bowtie$ -triangle inequality, it follows that

$$M(y, x, t_0) \geq M \left( y, z, \frac{t_0}{2} \right) * M \left( z, x, \frac{t_0}{2} \right) \geq \hat{r} * \hat{r} > r_0 > M(x, y, t_0),$$

which is a contradiction.

- Suppose that  $M$  satisfies the  $\diamond$ -triangle inequality. For

$$z \in B^{\triangleright} \left( x, 1 - \hat{r}, \frac{t_0}{2} \right) \cap B^{\triangleleft} \left( y, 1 - \hat{r}, \frac{t_0}{2} \right).$$

Since  $M$  satisfies the  $\diamond$ -triangle inequality, it follows that

$$M(x, y, t_0) \geq M\left(z, x, \frac{t_0}{2}\right) * M\left(y, z, \frac{t_0}{2}\right) \geq \hat{r} * \hat{r} > r_0 > M(x, y, t_0),$$

which is a contradiction. On the other hand, for

$$z \in B^{\triangleleft}\left(x, 1 - \hat{r}, \frac{t_0}{2}\right) \cap B^{\triangleright}\left(y, 1 - \hat{r}, \frac{t_0}{2}\right).$$

Since  $M$  satisfies the  $\diamond$ -triangle inequality, it follows that

$$M(y, x, t_0) \geq M\left(z, y, \frac{t_0}{2}\right) * M\left(x, z, \frac{t_0}{2}\right) \geq \hat{r} * \hat{r} > r_0 > M(x, y, t_0),$$

which is a contradiction.

This completes the proof.  $\square$

**Theorem 1.** Let  $(X, M)$  be a fuzzy semi-metric space along with a  $t$ -norm  $*$  that is left-continuous at 1 with respect to the first or second component. Suppose that  $M$  is left-continuous or symmetrically left-continuous with respect to the distance on  $(0, \infty)$ , and that  $M$  satisfies the  $\bowtie$ -triangle inequality.

(i) We define

$$\tau^{\triangleleft} = \{O^{\triangleleft} \subseteq X : x \in O^{\triangleleft} \text{ if and only if there exist } t > 0 \text{ and } r \in (0, 1) \text{ such that } B^{\triangleleft}(x, r, t) \subseteq O^{\triangleleft}\}.$$

Then, the family  $\mathcal{B}^{\triangleleft}$  induces a  $T_1$ -space  $(X, \tau^{\triangleleft})$  such that  $\mathcal{B}^{\triangleleft}$  is a base for the topology  $\tau^{\triangleleft}$ , in which  $O^{\triangleleft} \in \tau^{\triangleleft}$  if and only if, for each  $x \in O^{\triangleleft}$ , there exist  $t > 0$  and  $r \in (0, 1)$  such that  $B^{\triangleleft}(x, r, t) \subseteq O^{\triangleleft}$ .

(ii) We define

$$\tau^{\triangleright} = \{O^{\triangleright} \subseteq X : x \in O^{\triangleright} \text{ if and only if there exist } t > 0 \text{ and } r \in (0, 1) \text{ such that } B^{\triangleright}(x, r, t) \subseteq O^{\triangleright}\}.$$

Then, the family  $\mathcal{B}^{\triangleright}$  induces a  $T_1$ -space  $(X, \tau^{\triangleright})$  such that  $\mathcal{B}^{\triangleright}$  is a base for the topology  $\tau^{\triangleright}$ , in which  $O^{\triangleright} \in \tau^{\triangleright}$  if and only if, for each  $x \in O^{\triangleright}$ , there exist  $t > 0$  and  $r \in (0, 1)$  such that  $B^{\triangleright}(x, r, t) \subseteq O^{\triangleright}$ .

Moreover, the  $T_1$ -spaces  $(X, \tau^{\triangleleft})$  and  $(X, \tau^{\triangleright})$  satisfy the first axiom of countability.

**Proof.** Using part (i) of Proposition 8, part (i) of Proposition 12 and part (i) of Proposition 13, we see that  $\tau^{\triangleleft}$  is a topology such that  $\mathcal{B}^{\triangleleft}$  is a base for  $\tau^{\triangleleft}$ . Part (ii) of Proposition 8 says that  $(X, \tau^{\triangleleft})$  is a  $T_1$ -space. Part (i) of Proposition 9 says that there exist countable local bases at each  $x \in X$  for  $\tau^{\triangleright}$  and  $\tau^{\triangleleft}$ , respectively, which also says that  $\tau^{\triangleright}$  and  $\tau^{\triangleleft}$  satisfy the first axiom of countability. We can similarly obtain the desired results regarding the topology  $\tau^{\triangleright}$ . This completes the proof.  $\square$

According to Proposition 5, since  $M$  is assumed to be left-continuous or symmetrically left-continuous with respect to the distance on  $(0, \infty)$ , it follows that the topologies obtained in Wu [13] are still valid when we consider the  $\circ$ -triangle inequality for  $\circ \in \{\triangleright, \triangleleft, \diamond\}$ .

**Proposition 15.** Let  $(X, M)$  be a fuzzy semi-metric space along with a  $t$ -norm  $*$  that is left-continuous at 1 with respect to the first or second component. Suppose that  $M$  is left-continuous with respect to the distance on  $(0, \infty)$ , and that  $M$  satisfies the  $\bowtie$ -triangle inequality. Then, regarding the  $T_1$ -spaces  $(X, \tau^{\triangleleft})$  and  $(X, \tau^{\triangleright})$ ,  $B^{\triangleleft}(x, r, t)$  is a  $\tau^{\triangleleft}$ -open set and  $B^{\triangleright}(x, r, t)$  is a  $\tau^{\triangleright}$ -open set.

**Proof.** Using part (i) of Proposition 10, we see that  $B^{\triangleleft}(x, r, t)$  is a  $\tau^{\triangleleft}$ -open set and  $B^{\triangleright}(x, r, t)$  is a  $\tau^{\triangleright}$ -open set. This completes the proof.  $\square$

## 5. Limits in Fuzzy Semi-Metric Space

Since the symmetric condition is not satisfied in the fuzzy semi-metric space, three kinds of limit concepts will also be considered in this paper by referring to Wu [14]. In this section, we shall study the consistency of limit concepts in the induced topologies, which was not addressed in Wu [13].

Let  $(X, d)$  be a metric space. If the sequence  $\{x_n\}_{n=1}^{\infty}$  in  $(X, d)$  converges to  $x$ , i.e.,  $d(x_n, x) \rightarrow 0$  as  $n \rightarrow \infty$ , then it is denoted by  $x_n \xrightarrow{d} x$  as  $n \rightarrow \infty$ . In this case, we also say that  $x$  is a  $d$ -limit of the sequence  $\{x_n\}_{n=1}^{\infty}$ .

**Definition 6.** Let  $(X, M)$  be a fuzzy semi-metric space, and let  $\{x_n\}_{n=1}^{\infty}$  be a sequence in  $X$ .

- We write  $x_n \xrightarrow{M^{\triangleright}} x$  as  $n \rightarrow \infty$  if and only if

$$\lim_{n \rightarrow \infty} M(x_n, x, t) = 1 \text{ for all } t > 0.$$

In this case, we call  $x$  a  $M^{\triangleright}$ -limit of the sequence  $\{x_n\}_{n=1}^{\infty}$ .

- We write  $x_n \xrightarrow{M^{\triangleleft}} x$  as  $n \rightarrow \infty$  if and only if

$$\lim_{n \rightarrow \infty} M(x, x_n, t) = 1 \text{ for all } t > 0.$$

In this case, we call  $x$  a  $M^{\triangleleft}$ -limit of the sequence  $\{x_n\}_{n=1}^{\infty}$ .

- We write  $x_n \xrightarrow{M} x$  as  $n \rightarrow \infty$  if and only if

$$\lim_{n \rightarrow \infty} M(x_n, x, t) = \lim_{n \rightarrow \infty} M(x, x_n, t) = 1 \text{ for all } t > 0.$$

In this case, we call  $x$  a  $M$ -limit of the sequence  $\{x_n\}_{n=1}^{\infty}$ .

**Proposition 16.** Let  $(X, M)$  be a fuzzy semi-metric space along with a  $t$ -norm  $*$  that is left-continuous at 1 with respect to the first or second component, and let  $\{x_n\}_{n=1}^{\infty}$  be a sequence in  $X$ .

- Suppose that  $M$  satisfies the  $\bowtie$ -triangle inequality or the  $\diamond$ -triangle inequality. Then, we have the following properties:

- If  $x_n \xrightarrow{M^{\triangleleft}} x$  and  $x_n \xrightarrow{M^{\triangleright}} y$ , then  $x = y$ .
- If  $x_n \xrightarrow{M^{\triangleright}} x$  and  $x_n \xrightarrow{M^{\triangleleft}} y$ , then  $x = y$ .

- Suppose that  $M$  satisfies the  $\triangleleft$ -triangle inequality. If  $x_n \xrightarrow{M^{\triangleright}} x$  and  $x_n \xrightarrow{M^{\triangleright}} y$ , then  $x = y$ . In other words, the  $M^{\triangleright}$ -limit is unique.

- Suppose that  $M$  satisfies the  $\triangleright$ -triangle inequality. If  $x_n \xrightarrow{M^{\triangleleft}} x$  and  $x_n \xrightarrow{M^{\triangleleft}} y$ , then  $x = y$ . In other words, the  $M^{\triangleleft}$ -limit is unique.

**Proof.** To prove the first case of part (i), we first assume that  $M$  satisfies the  $\bowtie$ -triangle inequality. For any  $t > 0$ , using the left-continuity of  $t$ -norm at 1, we have

$$M(x, y, t) \geq M\left(x, x_n, \frac{t}{2}\right) * M\left(x_n, y, \frac{t}{2}\right) \rightarrow 1 * 1 = 1,$$

which says that  $x = y$ . To prove the second case of part (i), we have

$$M(y, x, t) \geq M\left(y, x_n, \frac{t}{2}\right) * M\left(x_n, x, \frac{t}{2}\right) \rightarrow 1 * 1 = 1,$$

which says that  $x = y$ . Now suppose that  $M$  satisfies the  $\diamond$ -triangle inequality. Then, we have

$$M(y, x, t) \geq M\left(x_n, y, \frac{t}{2}\right) * M\left(x, x_n, \frac{t}{2}\right) \rightarrow 1 * 1 = 1,$$

and

$$M(x, y, t) \geq M\left(x_n, x, \frac{t}{2}\right) * M\left(y, x_n, \frac{t}{2}\right) \rightarrow 1 * 1 = 1.$$

Therefore, we can similarly obtain the desired result.

To prove part (ii), we have

$$M(x, y, t) \geq M\left(x_n, x, \frac{t}{2}\right) * M\left(x_n, y, \frac{t}{2}\right) \rightarrow 1 * 1 = 1,$$

which says that  $x = y$ . To prove part (iii), we have

$$M(x, y, t) \geq M\left(x, x_n, \frac{t}{2}\right) * M\left(y, x_n, \frac{t}{2}\right) \rightarrow 1 * 1 = 1,$$

which says that  $x = y$ . This completes the proof.  $\square$

Let  $(X, \tau)$  be a topological space. The sequence  $\{x_n\}_{n=1}^{\infty}$  in  $X$  converges to  $x \in X$  with respect to the topology  $\tau$  is denoted by  $x_n \xrightarrow{\tau} x$  as  $n \rightarrow \infty$ , where the limit is unique when  $\tau$  is a Hausdorff topology.

**Remark 2.** Let  $(X, M)$  be a fuzzy semi-metric space along with a  $t$ -norm  $*$  and be endowed with a topology  $\tau^{\triangleright}$  given in Theorem 1. Let  $\{x_n\}_{n=1}^{\infty}$  be a sequence in  $X$ . Since  $\mathcal{B}^{\triangleright}$  is a base for  $\tau^{\triangleright}$ , it follows that  $x_n \xrightarrow{\tau^{\triangleright}} x$  as  $n \rightarrow \infty$ , if and only if, given any  $t > 0$  and  $0 < r < 1$ , there exists  $n_{r,t}$  such that  $x_n \in B^{\triangleright}(x, r, t)$  for all  $n \geq n_{r,t}$ . Since  $x_n \in B^{\triangleright}(x, r, t)$  means  $M(x_n, x, t) > 1 - r$ , it says that  $x_n \xrightarrow{\tau^{\triangleright}} x$  as  $n \rightarrow \infty$ , if and only if, given any  $t > 0$  and  $0 < r < 1$ , there exists  $n_{r,t}$  such that  $M(x_n, x, t) > 1 - r$  for all  $n \geq n_{r,t}$ .

**Proposition 17.** Let  $(X, M)$  be a fuzzy semi-metric space along with a  $t$ -norm  $*$ . Suppose that  $M$  is left-continuous or symmetrically left-continuous with respect to the distance on  $(0, \infty)$ , and that  $M$  satisfies the strong  $\bowtie$ -triangle inequality. Then, the following statements hold true:

- (i) Let  $\tau^{\triangleright}$  be the topology induced by  $(X, M, *)$ , and let  $\{x_n\}_{n=1}^{\infty}$  be a sequence in  $X$ . Then,  $x_n \xrightarrow{\tau^{\triangleright}} x$  as  $n \rightarrow \infty$  if and only if  $x_n \xrightarrow{M^{\triangleright}} x$  as  $n \rightarrow \infty$ .
- (ii) Let  $\tau^{\triangleleft}$  be the topology induced by  $(X, M, *)$ , and let  $\{x_n\}_{n=1}^{\infty}$  be a sequence in  $X$ . Then,  $x_n \xrightarrow{\tau^{\triangleleft}} x$  as  $n \rightarrow \infty$  if and only if  $x_n \xrightarrow{M^{\triangleleft}} x$  as  $n \rightarrow \infty$ .

**Proof.** Under the assumptions, Theorem 1 says that we can induce two topologies  $\tau^{\triangleright}$  and  $\tau^{\triangleleft}$ . It suffices to prove part (i). Suppose that  $x_n \xrightarrow{\tau^{\triangleright}} x$  as  $n \rightarrow \infty$ . Fixed  $t > 0$ , given any  $\epsilon \in (0, 1)$ , there exists  $n_{\epsilon,t} \in \mathbb{N}$  such that  $x_n \in B^{\triangleright}(x, \epsilon, t)$  for all  $n \geq n_{\epsilon,t}$ , which says that  $M(x_n, x, t) > 1 - \epsilon$ , i.e.,  $0 \leq 1 - M(x_n, x, t) < \epsilon$  for all  $n \geq n_{\epsilon,t}$ . Therefore, we obtain  $M(x_n, x, t) \rightarrow 1$  as  $n \rightarrow \infty$ . Conversely, given any  $t > 0$ , if  $M(x_n, x, t) \rightarrow 1$  as  $n \rightarrow \infty$ , then, given any  $\epsilon \in (0, 1)$ , there exists  $n_{\epsilon,t} \in \mathbb{N}$  such that  $1 - M(x_n, x, t) < \epsilon$ , i.e.,  $M(x_n, x, t) > 1 - \epsilon$  for all  $n \geq n_{\epsilon,t}$ , which says that  $x_n \in B^{\triangleright}(x, \epsilon, t)$  for all  $n \geq n_{\epsilon,t}$ . This shows that  $x_n \xrightarrow{\tau^{\triangleright}} x$  as  $n \rightarrow \infty$ , and the proof is complete.  $\square$

Let  $(X, M)$  be a fuzzy semi-metric space. We consider the following sets

$$\bar{B}^{\triangleleft}(x, r, t) = \{y \in X : M(x, y, t) \geq 1 - r\}$$

and

$$\bar{B}^{\triangleright}(x, r, t) = \{y \in X : M(y, x, t) \geq 1 - r\}.$$

If the symmetric condition is satisfied, then we simply write  $\bar{B}(x, r, t)$ . We are going to consider the closeness of  $\bar{B}^{\triangleleft}(x, r, t)$  and  $\bar{B}^{\triangleright}(x, r, t)$ .

**Proposition 18.** *Let  $(X, M)$  be a fuzzy semi-metric space along with a  $t$ -norm  $*$  that is left-continuous at 1 with respect to the first or second component. Suppose that  $M$  is continuous or symmetrically continuous with respect to the distance on  $(0, \infty)$ . If  $M$  satisfies the  $\bowtie$ -triangle inequality, then  $\bar{B}^{\triangleleft}(x, r, t)$  and  $\bar{B}^{\triangleright}(x, r, t)$  are  $\tau^{\triangleright}$ -closed and  $\tau^{\triangleleft}$ -closed, respectively. In other words, we have*

$$\tau^{\triangleright}\text{-cl}(\bar{B}^{\triangleleft}(x, r, t)) = \bar{B}^{\triangleleft}(x, r, t) \text{ and } \tau^{\triangleleft}\text{-cl}(\bar{B}^{\triangleright}(x, r, t)) = \bar{B}^{\triangleright}(x, r, t).$$

**Proof.** Under the assumptions, Theorem 1 says that we can induce two topologies  $\tau^{\triangleright}$  and  $\tau^{\triangleleft}$  satisfying the first axiom of countability. To prove the first case, for  $y \in \tau^{\triangleright}\text{-cl}(\bar{B}^{\triangleleft}(x, r, t))$ , since  $(X, \tau^{\triangleright})$  satisfies the first axiom of countability, there exists a sequence  $\{y_n\}_{n=1}^{\infty}$  in  $\bar{B}^{\triangleleft}(x, r, t)$  such that  $y_n \xrightarrow{\tau^{\triangleright}} y$  as  $n \rightarrow \infty$ . We also have  $M(x, y_n, t) \geq 1 - r$  for all  $n$ . By Proposition 17, we have  $M(y_n, y, t) \rightarrow 1$  as  $n \rightarrow \infty$  for all  $t > 0$ . Given any  $\epsilon > 0$ , the  $\bowtie$ -triangle inequality says that

$$M(x, y, t + \epsilon) \geq M(x, y_n, t) * M(y_n, y, \epsilon) \geq (1 - r) * M(y_n, y, \epsilon).$$

Since the  $t$ -norm  $*$  is left-continuous at 1 with respect to each component by Remark 1, we obtain

$$M(x, y, t + \epsilon) \geq (1 - r) * \left( \lim_{n \rightarrow \infty} M(y_n, y, \epsilon) \right) = (1 - r) * 1 = 1 - r.$$

By the right-continuity of  $M$ , we also have

$$M(x, y, t) = \lim_{\epsilon \rightarrow 0+} M(x, y, t + \epsilon) \geq 1 - r,$$

which says that  $y \in \bar{B}^{\triangleleft}(x, r, t)$ .

To prove the second case, for  $y \in \tau^{\triangleleft}\text{-cl}(\bar{B}^{\triangleright}(x, r, t))$ , since  $(X, \tau^{\triangleleft})$  satisfies the first axiom of countability, there exists a sequence  $\{y_n\}_{n=1}^{\infty}$  in  $\bar{B}^{\triangleright}(x, r, t)$  such that  $y_n \xrightarrow{\tau^{\triangleleft}} y$  as  $n \rightarrow \infty$ . We also have  $M(y_n, x, t) \geq 1 - r$  for all  $n$ . By Proposition 17, we have  $M(y, y_n, t) \rightarrow 1$  as  $n \rightarrow \infty$  for all  $t > 0$ . Given any  $\epsilon > 0$ , the  $\bowtie$ -triangle inequality says that

$$M(y, x, t + \epsilon) \geq M(y, y_n, \epsilon) * M(y_n, x, t) \geq M(y, y_n, \epsilon) * (1 - r).$$

Since the  $t$ -norm  $*$  is left-continuous at 1 with respect to each component by Remark 1, we obtain

$$M(y, x, t + \epsilon) \geq \left( \lim_{n \rightarrow \infty} M(y, y_n, \epsilon) \right) * (1 - r) = 1 * (1 - r) = 1 - r.$$

By the right-continuity of  $M$ , we also have

$$M(y, x, t) = \lim_{\epsilon \rightarrow 0+} M(y, x, t + \epsilon) \geq 1 - r,$$

which says that  $y \in \bar{B}^{\triangleright}(x, r, t)$ . This completes the proof.  $\square$

## 6. Conclusions

In fuzzy metric space, the triangle inequality plays an important role. In general, since the symmetric condition is not necessarily to be satisfied, the so-called fuzzy semi-metric space is proposed in this paper. In this situation, four different types of triangle inequalities are proposed and studied. The main purpose of this paper is to establish the  $T_1$ -spaces that are induced by the fuzzy semi-metric spaces along with the special kind of triangle inequality.

On the other hand, the limit concepts in fuzzy semi-metric space are also proposed and studied in this paper. Since the symmetric condition is not satisfied, three kinds of limits in fuzzy semi-metric space are considered. The concepts of uniqueness for the limits are also studied. Finally, we present the consistency of limit concepts in the induced  $T_1$ -spaces.

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