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Stability of λ-Harmonic Maps

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Abstract: In this paper, λ-harmonic maps from a Finsler manifold to a Riemannian manifold are studied. Then, some properties of this kind of harmonic maps are presented and some examples are given. Finally, the stability of the λ-harmonic maps from a Finsler manifold to the standard unit sphere $S^n(n > 2)$ is investigated.

Keywords: harmonic maps; stability; variational problem

1. Introduction

Harmonic maps between Riemannian manifolds were first introduced by Eells and Sampson in 1964. They showed that any map $\phi_0 : (M, g) \longrightarrow (N, h)$ from any compact Riemannian manifold (M, g) into a Riemannian manifold (N, h) with non-positive sectional curvature can be deformed into a harmonic maps. This is so-called the fundamental existence theorem for harmonic maps. In view of physics, harmonic maps have been studied in various fields of physics, such as super conductor, ferromagnetic material, liquid crystal, etc. [1–9].

Lichnerowicz first studied f-harmonic maps between Riemannian manifolds as a generalization of harmonic maps in 1970 [10]. Recently, in Cherif et al. [11], the researchers proved that any stable f-harmonic map ψ from sphere $\mathbb{S}^n(n > 2)$ to Riemannian manifold N is constant. Course [12] studied the f-harmonic flow on surfaces. Ou [13] analysed the f-harmonic morphisms as a subclass of harmonic maps which pull back harmonic functions to f-harmonic functions. Many scholars have studied and done research on f-harmonic maps, see for instance, [10,11,13–17].

The concept of harmonic maps from a Finsler manifold to a Riemannian manifold was first introduced by Mo [18]. On the workshop of Finsler Geometry in 2000, Professor S. S. Chern conjectured that the fundamental existence theorem for harmonic maps on Finsler spaces is true. In [19], Mo and Yang, the researchers have proved this conjecture and shown that any smooth map from a compact Finsler manifold to a compact Riemannian manifold of non-positive sectional curvature can be deformed into a harmonic map which has minimum energy in its homotopy class. Shen and Zhang [20] extended Mo's work to Finsler target manifold and obtained the first and second variation formulas.

As an application, He and Shen [21] proved that any harmonic map from an Einstein Riemannian manifold to a Finsler manifold with certain conditions is totally geodesic and there is no stable harmonic map from an Euclidean unit sphere S^n to any Finsler manifolds. Harmonic maps between Finsler manifolds have been studied extensively by various researchers, see for instance, [18–22].

In [23], J. Lie introduced the notion of $\mathcal{F}$-harmonic maps between Finsler manifolds. Let $\mathcal{F} : [0, \infty) \longrightarrow [0, \infty)$ be a C^2 function such that $\mathcal{F}' > 0$ on $(0, \infty)$. The smooth map $\psi : (M, F) \longrightarrow (N, h)$ is said to be $\mathcal{F}$-harmonic if it is an extermal point of the $\mathcal{F}$-energy functional $E_{\mathcal{F}}(\psi) := \frac{1}{c_{m-1}} \int_{SM} \mathcal{F}(e(\psi))dV_{SM}$, where $e(\psi)$ is the energy density of ψ, c_{m-1} denotes the volume of the standard $(m-1)$-dimensional sphere and dV_{SM} is the canonical volume element of SM.

The $\mathcal{F}$-energy functional is the energy , the p-energy, the $\alpha-$energy of Sacks-Uhlenbeck and exponential energy when $\mathcal{F}(t)$ is equal to t, $\frac{(2t)^{\frac{p}{2}}}{p}$ $(p \geq 4)$, $(1+2t)^{\alpha}(\alpha > 1, \dim M = 2)$ and e^t, respectively [20,21,23–25].

In view of physics, when (M,F) is a Riemannian manifold, p-harmonic maps have been extensively applied in image processing for denoising color images [26,27]. Furthermore, exponential harmonic maps have been studied on gravity [28]. The concept of $\mathcal{F}$-harmonic maps, as an extension of harmonic, p-harmonic and exponential harmonic maps, have an important role in physics and physical cosmology. For instance, instead of the scalar field in the Lagrangian, some of the $\mathcal{F}$-harmonic maps, such as the trigonometric functions, are studied in order to reproduce the inflation. Moreover, there are other $\mathcal{F}$-harmonic maps, such as exponential harmonic maps, are investigated in order to depict the phenomenon of the quintessence [5,29,30].

Let $\psi : (M,F) \longrightarrow (N,h)$ be a smooth map from a Finsler manifold (M,F) into a Riemannian manifold (N,h) and $\lambda : SM \times N \longrightarrow (0,\infty)$ be a smooth positive function. A map $\psi : (M,F) \longrightarrow (N,h)$ is said to be λ-harmonic if it is a critical point of the λ-energy functional

$$E_{\lambda}(\psi) := \frac{1}{c_{m-1}} \int_{SM} \lambda_{\psi} Tr_g \psi^* h \, dV_{SM}, \tag{1}$$

where g is the fundamental tensor of (M,F), $\psi^* h$ is the pull-back of the metric h by the map ψ and λ_ψ is a smooth function given by $(x,y) \in SM \longrightarrow \lambda_\psi(x,y) := \lambda(x,y,\psi(x))$. By considering the Euler-Lagrange equation associated to the λ-energy functional, it can be seen that any $\mathcal{F}$-harmonic map $\psi : (M,F) \longrightarrow (N,h)$ from a Finsler manifold (M,F) into a Riemannian manifold (N,h) without critical points (i.e., $\mid d\psi_x \mid \neq 0$ for all $x \in M$), is a λ-harmonic map with $\lambda = \mathcal{F}'(e(\psi))$.

In particular, when $grad_h \lambda = 0$ and (M,F) is a Riemannian manifold, λ-harmonic maps can be considered as the stationary solutions of inhomogeneous Heisenberg spin system, see for instance [13,14]. Furthermore, the intersection of λ-harmonicity with curvature conditions justifies their application for gleaning valuable information on weighted manifolds and gradient Ricci solitons, see [15–17].

The current paper is organized as follows. In Section 2, a few concepts of Finsler geometry are reviewed. In Section 3, the λ-energy functional of a smooth map from a Finsler manifold to a Riemannian manifold is introduced and the corresponding Euler-Lagrange equation is obtained via calculating the first variation formula of the λ-energy functional and an example is given. In Section 4, the second variation formula of the λ-energy functional for a λ-harmonic map is derived. As an application, the stability theorems for λ-harmonic maps are given.

2. Preliminaries

Throughout this paper, let (M,F) be an m-dimensional smooth, oriented, compact Finsler manifold without boundary. In the local coordinates (x^i, y^i) on $TM \setminus \{0\}$, the fundamental tensor of (M,F) is defined as follows:

$$g = g_{ij} dx^i \otimes dx^j, \qquad g_{ij} := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}.$$

In this paper, the following conventions of index ranges are used

$$1 \leqslant i,j,k,... \leqslant m, \qquad 1 \leqslant a,b,c,... \leqslant m-1, \qquad 1 \leqslant A,B,C,... \leqslant 2m-1.$$

Let $\rho : SM \longrightarrow M$ be the natural projection on the projective sphere bundle SM. The Finsler structure F determines two important quantities on the pull-back bundle $\rho^* T^* M$ as follows:

$$\omega = \frac{\partial F}{\partial y^i} dx^i, \qquad A := A_{ijk} dx^i \otimes dx^j \otimes dx^k, \qquad A_{ijk} := \frac{F}{2} \frac{\partial g_{ij}}{\partial y^k},$$

which are called the Hilbert form and Cartan tensor, respectively. The dual of the Hilbert form ω is the distinguished section $\ell := \frac{y^i}{F}\frac{\partial}{\partial x^i}$ of the pull-back bundle ρ^*TM. Note that, all indices related to ρ^*TM are raised and lowered with the metric g.

On the pull-back bundle ρ^*TM, there exists uniquely the Chern connection ∇^c whose connection 1-forms $\{\omega_i^j\}$ are satisfied the following equations

$$d(dx^i) - dx^k \wedge \omega_k^i = 0, \qquad \text{and} \qquad dg_{ij} - g_{ik}\omega_j^k - g_{jk}\omega_i^k = 2A_{ijk}\frac{\delta y^k}{F},$$

where $\delta y^i := dy^i + N_j^i dx^j$ [31]. Here $N_j^i := \gamma_{jk}^i y^k - A_{jk}^i y^p y^q$ and γ_{jk}^i are the formal Christoffel symbols of the second kind for g_{ij}. The curvature 2-forms of the Chern connection, $\Omega_j^i := d\omega_j^i - \omega_j^k \wedge \omega_k^i$, have the following structure

$$\Omega_j^i := \frac{1}{2}R_{j\ kl}^{\ i}dx^k \wedge dx^l + P_{j\ kl}^{\ i}dx^k \wedge \frac{\delta y^l}{F}.$$

See [22] for an expository proof. The Landsberg curvature of (M, F) is defined as follows:

$$L := L_{ijk}dx^i \otimes dx^j \otimes dx^k, \qquad L_{ijk} := g_{il}\frac{y^m}{F}P_{m\ jk}^{\ l}.$$

By [22], we have

$$L_{ijk} = -\dot{A}_{ijk},$$

where "." denotes the covariant derivative along the Hilbert form. Consider a g-orthonormal frame $\{\omega^i = v_j^i dx^j\}$ for any fibre of ρ^*T^*M where ω^m is the Hilbert form ω, and let $\{e_i = u_i^j \frac{\partial}{\partial x^j}\}$ be its dual frame where e_m is the distinguished section ℓ dual to the Hilbert form $\omega(= \omega^m)$. Set

$$\begin{aligned} e_i^H &:= u_i^j \frac{\delta}{\delta x^j}, & \hat{e}_{m+a} &:= u_a^i F\frac{\partial}{\partial y^i}, \\ \omega^i &:= v_j^i dx^j, & \omega_m^a &:= v_j^a \frac{\delta y^j}{F}, \end{aligned} \tag{2}$$

where $\frac{\delta}{\delta x^j} = \frac{\partial}{\partial x^j} - N_j^i \frac{\partial}{\partial y^i}$. It can be seen that $\{\omega^A\}_{A=1}^{2m-1} = \{\omega^i, \omega_m^a\}$ is a local basis for the cotangent bundle T^*SM and $\{e_A\}_{A=1}^{2m-1} = \{e_i^H, \hat{e}_{m+a}\}$ is a local basis for TSM. By (2), it can be seen that

$$\omega^i(e_j^H) = \delta_j^i, \qquad \omega^i(\hat{e}_{m+a}) = 0, \qquad \omega_m^a(e_j^H) = 0, \qquad \omega_m^b(\hat{e}_{m+a}) = \delta_a^b. \tag{3}$$

Tangent vectors on SM which are annihilated by all $\{\omega_m^a\}$'s form the horizontal sub-bundle HSM of TSM. The fibres of HSM are m-dimensional. On the other hand, let $VSM := \cup_{x\in M} TS_xM$ be the vertical sub-bundle of TSM, its fibres are $m-1$ dimensional. The decomposition $TSM = HSM \oplus VSM$ holds because HSM and VSM are direct summands, (see [32], p. 7). The inner product $g = g_{ij}dx^i dx^j$ on ρ^*TM induces a Riemannian metric $\hat{g}$ on SM as follows:

$$\hat{g} = \delta_{ij}\omega^i \otimes \omega^j + \delta_{ab}\omega_m^a \otimes \omega_m^b.$$

Furthermore, the volume element dV_{SM} of SM with respect to $\hat{g}$ is defined as follows

$$dV_{SM} := \omega^1 \wedge \cdots \omega^m \wedge \omega_m^1 \cdots \wedge \omega_m^{m-1}.$$

Lemma 1. *[18] For $\psi = \psi_i\omega^i \in \Gamma(\rho^*T^*M)$, we have*

$$div_{\hat{g}}\psi = \sum_i \psi_{i|i} + \sum_{a,b}\psi_a L_{bba} = \sum_i ({}^c\nabla_{e_i^H}\psi)(e_i) + \sum_{a,b}\psi_a L_{bba}, \tag{4}$$

where "$|$" denotes the horizontal covariant derivative with respect to the Chern connection and e_i^H is defined in (2).

3. The First Variation Formula

Let $\psi : (M^m, F) \longrightarrow (N^n, h)$ be a smooth map from a Finsler manifold (M, F) into a Riemannian manifold (N, h), $\rho : SM \longrightarrow M$ a natural projection on SM and $\tilde{\psi} = \psi \circ \rho$. In the sequel, we denote the Chern connection on ρ^*TM by ${}^c\nabla$, the connection induced by the Chern connection of (M, F) on the pulled-back bundle ψ^*TN over SM by ∇ and the Levi-Civita connection on (N, h) by ∇^N.

Let $\lambda : SM \times N \longrightarrow (0, \infty)$ be a smooth positive function. The *λ-energy density* of ψ is the function $e_\lambda(\psi) : SM \longrightarrow \mathbb{R}$, defined by

$$e_\lambda(\psi)(x, y) := \frac{1}{2}\lambda(x, y, \psi(x))Tr_g\psi^*h, \tag{5}$$

where ψ^*h is the pull-back of h by the map ψ and Tr_g stands for taking the trace with respect to g (the fundamental tensor of F) at $(x, y) \in SM$. By making use of (5), the λ-energy functional is defined as follows:

$$E_\lambda(\psi) := \frac{1}{c_{m-1}} \int_{SM} e_\lambda(\psi) dV_{SM}, \tag{6}$$

where c_{m-1} denotes the volume of the standard $(m-1)$-dimensional sphere and dV_{SM} is the canonical volume element of SM. A map $\psi : (M, F) \longrightarrow (N, h)$ is said to be *λ-harmonic* if it is a critical point of the λ-energy functional.

Let $\{\psi_t\}_{t \in I}$ be a smooth variation of $\psi_0 = \psi$ with the variational vector field

$$V = \frac{\partial \psi_t}{\partial t}\Big|_{t=0} := V^\alpha \frac{\partial}{\partial \tilde{x}^\alpha} \circ \psi.$$

For any $t \in I$, in local coordinate (x^i, U) on M and $(\tilde{x}^\alpha, V)$ on N, the λ-energy density of ψ_t can be written as follows:

$$e_\lambda(\psi_t)(x, y) = \frac{1}{2} f(x, y, \psi_t(x)) g^{ij} \psi^\alpha_{t|i} \psi^\beta_{t|j} h_{\alpha\beta}(\tilde{x}), \tag{7}$$

where $\tilde{x} = \psi_t(x)$ and $d\psi_t(\frac{\partial}{\partial x^i}) = \psi^\alpha_{t|i} \frac{\partial}{\partial \tilde{x}^\alpha} \circ \psi$. Due to the fact that $\{V^\alpha\}$ is independent of y and using (7), we have

$$\begin{aligned} \frac{\partial}{\partial t} e_\lambda(\psi_t)\Big|_{t=0} &= \frac{1}{2}\frac{\partial}{\partial t}\left(\lambda_{\psi_t} g^{ij} \psi^\alpha_{t|i} \psi^\beta_{t|j} h_{\alpha\beta}\right)\Big|_{t=0} \\ &= \frac{1}{2} g^{ij} \psi^\alpha_i \psi^\beta_j h_{\alpha\beta} \frac{\partial \lambda_{\psi_t}}{\partial t}\Big|_{t=0} + \frac{1}{2}\lambda_\psi g^{ij} \psi^\alpha_i \psi^\beta_j \frac{\partial h_{\alpha\beta}}{\partial t}\Big|_{t=0} + \lambda_\psi g^{ij} \psi^\alpha_i \frac{\delta V^\beta}{\delta x^j} h_{\alpha\beta} \\ &= \frac{1}{2} g^{ij} \psi^\alpha_i \psi^\beta_j h_{\alpha\beta} \frac{\partial \lambda_{\psi_t}}{\partial t}\Big|_{t=0} + \sum_i \lambda_\psi h(\nabla_{e_i} V, d\psi(e_i)) \\ &= I + II, \end{aligned} \tag{8}$$

where λ_ψ is a smooth function given by $(x, y) \in SM \longrightarrow \lambda_\psi(x, y) := \lambda(x, y, \psi(x))$. By using the definition of the gradient operator, we get

$$\begin{aligned} I &= e(\psi)(x, y) \frac{\partial}{\partial t} \lambda(x, y, \psi_t(x))\Big|_{t=0} \\ &= e(\psi)(x, y) d\lambda_{(x,y)}(V) \\ &= h(e(\psi)(x, y)(grad_h \lambda_{(x,y)})_{\psi(x)}, V(x)), \end{aligned} \tag{9}$$

where $e(\psi) := \frac{1}{2} Tr_g h(d\psi, d\psi)$ and $\lambda_{(x,y)}$ is the smooth function $z \in N \longrightarrow \lambda_{(x,y)}(z) = \lambda(x, y, z)$. The function $e(\psi)$ is called the *energy density* of ψ. Let $\Psi := \lambda_\psi h(V, d\psi(e_i))\omega^i$, which is a section of p^*T^*M . By Lemma 4, we have

$$
\begin{aligned}
div_{\hat{g}}\Psi &= \sum_i ({}^c\nabla_{e_i^H}\Psi)(e_i) + \Psi_a P^a_{bb} \\
&= \sum_i \lambda_\psi \{h(\nabla_{e_i^H} V, d\psi(e_i)) + h(V, (\nabla_{e_i^H}\lambda d\psi)(e_i))\} - h(V, d\psi(e_a))\dot{A}^a \\
&= h\Big(V, \lambda_\psi Tr_g \nabla d\psi + d\psi \circ p(grad^H \lambda_\psi) - \lambda_\psi d\psi \circ p(K^H)\Big) \\
&+ \sum_i \lambda_\psi h(\nabla_{e_i^H} V, d\psi(e_i)),
\end{aligned}
\tag{10}
$$

where

$$K := \dot{A}^i \frac{\partial}{\partial x^i} \tag{11}$$

is a section of ρ^*TM. By (10), we get

$$II = div_{\hat{g}}\Psi - h\Big(V, \lambda_\psi Tr_g \nabla d\psi + d\tilde{\psi}(grad^H \lambda_\psi) - \lambda_\psi d\tilde{\psi}(K^H)\Big). \tag{12}$$

By substituting (9) and (12) in (8) and considering the Green's theorem, the first variation formula of the λ-energy functional is obtained as follows:

$$\frac{d}{dt} E_\lambda(\psi_t)\big|_{t=0} = -\frac{1}{c_{m-1}} \int_{SM} h(\tau_\lambda(\psi), V) dV_{SM},$$

where

$$\tau_\lambda(\psi) := \lambda_\psi Tr_g \nabla d\psi + d\tilde{\psi}(grad_{\hat{g}}^H \lambda_\psi) - \lambda_\psi d\psi(K) - e(\psi)(grad_h \lambda) \circ \tilde{\psi}. \tag{13}$$

Here $grad^H \lambda$ denotes the horizontal part of $grad_{\hat{g}}\, \lambda \in \Gamma(TSM)$ and K is defined by (11). The field $\tau_\lambda(\psi)$ is said to be the λ-*tension field* of ψ.

Theorem 1. *Let $\psi : (M, F) \longrightarrow (N, h)$ be a smooth map from a Finsler manifold (M, F) to a Riemannian manifold (N, h) and $\lambda \in C^\infty(SM \times N)$. Then, ψ is the λ-harmonic map if and only if $\tau_\lambda(\psi) \equiv 0$.*

Example 1. *Assume that $(\mathbb{R}^2, F)$ is a locally Minkowski manifold and $(\mathbb{R}^3, \langle,\rangle)$ be the three-dimensional Euclidean space. Consider the map $\psi : (\mathbb{R}^2, F) \longrightarrow (\mathbb{R}^3, \langle,\rangle)$ defined by*

$$\psi(x^1, x^2) := (2x^2, x^1 + 2x^2, 3x^1 - x^2).$$

Let $\lambda : S\mathbb{R}^2 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a positive smooth map such that $\lambda(x^1, x^2, y^1, y^2, \psi(x^1, x^2)) := exp(\frac{y^2(y^2-2y^1)}{(y^2)^2+(y^1)^2})$. Due to the fact that the Landsberg curvature of locally Minkowski manifold vanishes and considering Theorem 1 and Equation (13), one can see that ψ is λ-harmonic.

Now, we discuss the relation between λ-harmonic maps and $\mathcal{F}$-harmonic maps from a Finsler manifolds to a Riemannian manifolds. Let $\mathcal{F} : [0, \infty) \longrightarrow [0, \infty)$ be a C^2 function such that $\mathcal{F}' > 0$ on $(0, \infty)$. The smooth map $\psi : (M, F) \longrightarrow (N, h)$ is said to be $\mathcal{F}$-harmonic if it is an external point of the $\mathcal{F}$-energy functional

$$E_{\mathcal{F}}(\psi) := \int_{SM} \mathcal{F}(e(\psi)) dV_{SM}. \tag{14}$$

The concept of $\mathcal{F}$-harmonic maps from a Finsler manifold was first introduced by J. Li [23] in 2010. The $\mathcal{F}$-energy functional is the energy , the p-energy, the α-energy of Sacks-Uhlenbeck and exponential energy when $\mathcal{F}(t)$ is equal to t, $\frac{(2t)^{\frac{p}{2}}}{p}$ $(p \geq 4)$, $(1+2t)^\alpha (\alpha > 1, \dim M = 2)$ and e^t, respectively. The Euler-Lagrange equation associated to $\mathcal{F}$-energy functional is given by

$$\tau_{\mathcal{F}}(\psi) := Tr_g \nabla(\mathcal{F}'(e(\psi)) d\psi) - \mathcal{F}'(e(\psi)) d\psi(K) = 0. \tag{15}$$

For more details, see [23]. The field $\tau_{\mathcal{F}}(\psi)$ is called the $\mathcal{F}$-tension field of ψ.

Proposition 1. *Let $\psi : (M, F) \longrightarrow (N, h)$ be an $\mathcal{F}$-harmonic map from a Finsler manifold to a Riemannian manifold without critical points (i.e., $| \, d\psi_x \, | \neq 0$ for all $x \in M$). Then, ψ is a λ-harmonic map with $\lambda = \mathcal{F}'(e(\psi))$.*

Proof. It is obtained from (13) and (15) immediately. □

4. Stability of λ-Harmonic Maps

In this section, the second variation formula of the λ-energy functional for a λ-harmonic map from a Finsler manifold to a Riemannian manifold is obtained. As an application, it is shown that any stable λ-harmonic map $\psi : (M, F) \longrightarrow \mathbb{S}^n$ from a Finsler manifold (M, F) to the standard sphere $\mathbb{S}^n (n > 2)$ is constant.

Theorem 2. *(The second variation formula). Let (M^m, F) be a Finsler manifold and (N, h) be a Riemannian manifold and let $\psi : (M, F) \longrightarrow (N, h)$ be a λ-harmonic map. Assume that $\{\psi_t\}_{t \in I}$ is a smooth variation of $\psi_0 = \psi$ with the variation vector field $V = \frac{\partial \psi_t}{\partial t}|_{t=0}$. Then, the second variation of λ-energy functional is*

$$\begin{aligned} &\frac{d^2}{dt^2} E_\lambda(\psi_t)|_{t=0} \\ &= \frac{1}{c_{m-1}} \int_{SM} h\Big(V, e(\psi)(\nabla^N_V grad_h \lambda) \circ \psi + 2Tr_g < \nabla V, d\psi > (grad_h \lambda) \circ \psi - \lambda_\psi Tr_g(\nabla^2 V) \\ &- \lambda_\psi Tr_g R^N(V, d\psi) d\psi - \nabla_{grad^H_{\tilde{g}} \lambda_\psi} V + \lambda_\psi \nabla_{K^H} V \Big) dV_{SM}, \end{aligned} \tag{16}$$

where R^N is the curvature tensor on (N, h) and K is defined by (11).

Proof. Let $p : SM \longrightarrow M$ be a natural projection and $\tilde{\psi} = \psi \circ p$, and let ${}^c\nabla$ and ∇ be the Chern connection on p^*TM and the pull-back Chern connection on $\tilde{\psi}^*TN$, respectively. For any $t \in I$, we shall use the same notation of ∇ for the pull-back Chern connection on $\tilde{\psi}^*TN$. By (5), it can be shown that

$$\begin{aligned} &\frac{\partial^2}{\partial t^2} E_\lambda(\psi_t)\Big|_{t=0} \\ &= \frac{1}{2c_{m-1}} \sum_i \int_{SM} \frac{\partial^2}{\partial t^2} \Big\{ \lambda(x, y, \psi_t(x)) h(d\psi_t(e_i), d\psi_t(e_i)) \Big\}\Big|_{t=0} dV_{SM} \\ &= \frac{1}{2c_{m-1}} \sum_i \int_{SM} \Big\{ h(d\psi_t(e_i), d\psi_t(e_i)) \frac{\partial^2}{\partial t^2} \lambda(x, y, \psi_t(x)) \Big\}\Big|_{t=0} dV_{SM} \\ &+ \frac{2}{c_{m-1}} \sum_i \int_{SM} \Big\{ \frac{\partial}{\partial t} (h(\nabla_{\frac{\partial}{\partial t}} d\psi_t(e_i), d\psi_t(e_i))) \frac{\partial}{\partial t} \lambda(x, y, \psi_t(x)) \Big\}\Big|_{t=0} dV_{SM} \\ &+ \frac{1}{2c_{m-1}} \sum_i \int_{SM} \Big\{ \lambda(x, y, \psi_t(x)) \frac{\partial^2}{\partial t^2} h(d\psi_t(e_i), d\psi_t(e_i)) \Big\}\Big|_{t=0} dV_{SM} \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{17}$$

Now, we calculate each term of the right hand side (RHS) of (17). By (9) and considering the definition of gradient operator, we get

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\lambda(x,y,\psi_t(x))\Big|_{t=0} &= \frac{\partial}{\partial t} < (grad_h\lambda_{(x,y)})_{\psi_t(x)}, d\psi_t(\frac{\partial}{\partial t}) > \Big|_{t=0} \\ &= h(\nabla^N_V grad_h\lambda_{(x,y)}, V)\circ\bar{\psi} + h\big((grad_h\lambda)\circ\bar{\psi}, \nabla_{\frac{\partial}{\partial t}} d\psi(\frac{\partial}{\partial t})\Big|_{t=0}\big).\end{aligned} \tag{18}$$

Thus, the first term of the RHS of (17) is obtained as follows:

$$\begin{aligned}I_1 = \frac{1}{c_{m-1}}\int_{SM}\Big\{h\Big(V, e(\psi)(\nabla^N_V grad_h\lambda)\circ\bar{\psi}\Big) \\ + e(\psi)h\Big((grad_h\lambda)\circ\bar{\psi}, \nabla_{\frac{\partial}{\partial t}} d\psi_t(\frac{\partial}{\partial t})\Big|_{t=0}\Big)\Big\}dV_{SM}.\end{aligned} \tag{19}$$

By calculating the second term of the RHS of (17), we get

$$\begin{aligned}I_2 =&\frac{2}{c_{m-1}}\sum_i\int_{SM}\frac{\partial}{\partial t}(\lambda(x,y,\psi_t(x)))h(\nabla_{\frac{\partial}{\partial t}} d\psi_t(e_i), d\psi_t(e_i))\Big|_{t=0} dV_{SM} \\ =& \frac{2}{c_{m-1}}\sum_i\int_{SM} h\big(grad_h\lambda_{(x,y)}, V\big)h\big(\nabla_{e_i^H}V, d\psi(e_i)\big)dV_{SM} \\ =& \frac{2}{c_{m-1}}\int_{SM} Tr_g < \nabla V, d\psi > h\big((grad_h\lambda)\circ\psi, V\big)dV_{SM}.\end{aligned} \tag{20}$$

Now, we calculate the last term of the RHS of (17). By definition of the function λ_ψ, we have

$$\begin{aligned}&\frac{1}{2}\lambda(x,y,\psi_t(x))\frac{\partial^2}{\partial t^2}h(d\psi_t(e_i), d\psi_t(e_i)) \\ &= \lambda_{\psi_t}(x,y)\frac{\partial}{\partial t}h(\nabla_{\frac{\partial}{\partial t}} d\psi_t(e_i), d\psi_t(e_i)) \\ &= \lambda_{\psi_t}(x,y)\frac{\partial}{\partial t}h(\nabla_{e_i^H} d\psi_t(\frac{\partial}{\partial t}), d\psi_t(e_i)) \\ &= \lambda_{\psi_t}(x,y)\Big\{h(\nabla_{\frac{\partial}{\partial t}}\nabla_{e_i^H} d\psi_t(\frac{\partial}{\partial t}), d\psi_t(e_i)) + h(\nabla_{e_i^H} d\psi_t(\frac{\partial}{\partial t}), \nabla_{e_i^H} d\psi_t(\frac{\partial}{\partial t}))\Big\} \\ &= \lambda_{\psi_t}(x,y)\Big\{h(\nabla_{e_i^H}\nabla_{\frac{\partial}{\partial t}} d\psi_t(\frac{\partial}{\partial t}), d\psi_t(e_i)) + h(\nabla_{e_i^H} d\psi_t(\frac{\partial}{\partial t}), \nabla_{e_i^H} d\psi_t(\frac{\partial}{\partial t})) \\ &+ h(R^N(d\psi_t(\frac{\partial}{\partial t}), d\psi_t(e_i))d\psi_t(\frac{\partial}{\partial t}), d\psi_t(e_i))\Big\}.\end{aligned} \tag{21}$$

Therefore,

$$\begin{aligned}I_3 =& \frac{1}{c_{m-1}}\sum_i\int_{SM}\lambda_{\psi_t}\Big\{h(\nabla_{e_i^H} d\psi_t(\frac{\partial}{\partial t}), \nabla_{e_i^H} d\psi_t(\frac{\partial}{\partial t})) \\ &+ h(R^N(d\psi_t(\frac{\partial}{\partial t}), d\psi_t(e_i))d\psi_t(\frac{\partial}{\partial t}), d\psi_t(e_i))\Big\}\Big|_{t=0} dV_{SM} \\ &+ \frac{1}{c_{m-1}}\sum_i\int_{SM}\lambda_{\psi_t}h(\nabla_{e_i^H}\nabla_{\frac{\partial}{\partial t}} d\psi_t(\frac{\partial}{\partial t}), d\psi_t(e_i))\Big|_{t=0} dV_{SM} \\ &= I_5 + I_6.\end{aligned} \tag{22}$$

Let $\Psi := \lambda_{\psi_t} h(\nabla_{e_i^H} d\psi_t(\frac{\partial}{\partial t}), d\psi_t(\frac{\partial}{\partial t}))\omega^i$, which is a section of (p^*T^*M). By Lemma 4, we get

$$\begin{aligned} div_{\hat{g}}\Psi &= \sum_i ({}^c\nabla_{e_i^H}\Psi)(e_i) + \sum_b \Psi_a L^a_{bb} \\ &= \sum_i \Big\{ e_i^H(\lambda_{\psi_t}) h(\nabla_{e_i^H} d\psi_t(\frac{\partial}{\partial t}), d\psi_t(\frac{\partial}{\partial t})) + \lambda_{\psi_t} h(\nabla_{e_i^H}\nabla_{e_i^H} d\psi_t(\frac{\partial}{\partial t}), d\psi_t(\frac{\partial}{\partial t})) \\ &+ \lambda_{\psi_t} h(\nabla_{e_i^H} d\psi_t(\frac{\partial}{\partial t}), \nabla_{e_i^H} d\psi_t(\frac{\partial}{\partial t})) + \lambda_{\psi_t} h(\nabla_{e_j^H} d\psi_t(\frac{\partial}{\partial t}), d\psi_t(\frac{\partial}{\partial t}))({}^c\nabla_{e_i^H}\omega^j)(e_i) \Big\} \\ &- \sum_b \lambda_{\psi_t} h(\nabla_{e_a^H} d\psi_t(\frac{\partial}{\partial t}), d\psi_t(\frac{\partial}{\partial t})) \dot{A}^a_{bb}. \end{aligned} \tag{23}$$

By Green's theorem and Equation (23), the first term of the RHS of (22) is obtained as follows:

$$\begin{aligned} I_5 = -\frac{1}{c_{m-1}} \int_{SM} h\Big(&\lambda_\psi Tr_g(\nabla^2 V) + \lambda_\psi Tr_g R^N(d\psi, V)d\psi \\ &+ \nabla_{grad^H_{\hat{g}}\lambda_\psi} V - \lambda_\psi \nabla_{K^H} V, V \Big) dV_{SM}. \end{aligned} \tag{24}$$

Similarly, let $\bar{\Psi} := h(\nabla_{\frac{\partial}{\partial t}} d\psi_t(\frac{\partial}{\partial t}), \lambda_{\psi_t} d\psi_t(e_i))\omega^i$, which is a section of p^*T^*M. By Lemma (4), we also have

$$\begin{aligned} div_{\hat{g}}\bar{\Psi} = \sum_i \Big\{ &\lambda_{\psi_t} h(\nabla_{e_i^H}\nabla_{\frac{\partial}{\partial t}} d\psi_t(\frac{\partial}{\partial t}), d\psi_t(e_i)) + h\big(\nabla_{\frac{\partial}{\partial t}} d\psi_t(\frac{\partial}{\partial t}), (\nabla_{e_i^H}\lambda_{\psi_t} d\psi_t)(e_i)\big) \Big\} \\ &- \lambda_{\psi_t} \sum_b h(\nabla_{\frac{\partial}{\partial t}} d\psi_t(\frac{\partial}{\partial t}), d\psi(K)). \end{aligned} \tag{25}$$

By (25), we get the second term of the RHS of (22) as follows:

$$I_6 = -\frac{1}{c_{m-1}} \int_{SM} h(\nabla_{\frac{\partial}{\partial t}} d\psi_t(\frac{\partial}{\partial t}) \mid_{t=0}, Tr_g \nabla(\lambda_\psi d\psi) - \lambda_\psi d\psi(K)) dV_{SM}. \tag{26}$$

By making use of (24) and (26), we have

$$\begin{aligned} I_3 = &-\frac{1}{c_{m-1}} \int_{SM} h\Big(\lambda Tr_g(\nabla^2 V) + \lambda_\psi R^N(d\psi, V)d\psi + \nabla_{grad^H_{\hat{g}}\lambda} V - \lambda\nabla_{K^H} V, V\Big) dV_{SM} \\ &- \frac{1}{c_{m-1}} \int_{SM} h(\nabla_{\frac{\partial}{\partial t}} d\psi_t(\frac{\partial}{\partial t}) \mid_{t=0}, Tr_g \nabla\lambda_\psi d\psi - \lambda_\psi d\psi(K)) dV_{SM}. \end{aligned} \tag{27}$$

By substituting (19), (20) and (27) in (17), the Equation (16) is obtained and hence completes the proof. □

Definition 1. *By considering the assumptions of Theorem 2, set*

$$Q^\psi_\lambda(V) := \frac{d^2}{dt^2} E_\lambda(\psi_t)\big|_{t=0}.$$

A λ-harmonic map ψ is said to be stable λ-harmonic map if $Q^\psi_\lambda(V) \geq 0$ for any vector field V along ψ.

Stability of λ-Harmonic Maps to $\mathbb{S}^n$

Let $\mathbb{S}^n$ denote the unit n-sphere as a submanifold of Euclidean space $(\mathbb{R}^{n+1}, \langle, \rangle)$. At any point $x_0 \in \mathbb{S}^n$, every vector field W on the Euclidean space $\mathbb{R}^{n+1}$ can be split into two parts

$$W = W^\top + W^\perp = W^\top + \langle W, x_0 \rangle x_0, \tag{28}$$

where $W^\top$ is the tangential part of W to $\mathbb{S}^n$ and $W^\perp = \langle W, x_0\rangle x_0$ is the normal part to $\mathbb{S}^n$. Denote the second fundamental form of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ by B, the Levi-Civita connection on $\mathbb{R}^{n+1}$ by ∇^R and the Levi-Civita connection on $\mathbb{S}^n$ by ∇^S. Then, we have the following relation

$$\nabla^R_V Z = \nabla^S_V Z + B(V, Z), \tag{29}$$

where V and Z are smooth vector fields on $\mathbb{S}^n$. Let X be a normal vector field on $\mathbb{S}^n$, the *shape operator* corresponding to X, denoted by A^X, is defined as follows:

$$A^X(V) := -(\nabla^R_V X)^\top, \qquad V \in \chi(\mathbb{S}^n). \tag{30}$$

At each point $x_0 \in \mathbb{S}^n$, the second fundamental form and the shape operator are related by

$$\langle A^X(V), Z\rangle = \langle B(V, Z), X\rangle = -\langle V, Z\rangle\langle x_0, X\rangle, \tag{31}$$

where V and Z are tangent vector fields on $\mathbb{S}^n$ and X is a normal vector field on $\mathbb{S}^n$.

Definition 2. *Let (M, F) be a Finsler manifold and $(\mathbb{S}^n, h)$ the n-dimensional Euclidean sphere. A smooth function $\lambda : SM \times \mathbb{S}^n \longrightarrow \mathbb{R}$ is called an adopted function if there exists a parallel orthonormal frame field $\{Y_\alpha\}_{\alpha=1}^{n+1}$ in $\mathbb{R}^{n+1}$ such that*

$$\sum_\alpha \mu_\alpha(x) h(grad_h \lambda, Y_\alpha^\top)(z, x) \geq 0, \tag{32}$$

for any point $(z, x) \in SM \times \mathbb{S}^n$, where $Y_\alpha^\top$ is the tangential part of Y_α to $\mathbb{S}^n$ and the function $\mu_\alpha : \mathbb{S}^n \longrightarrow \mathbb{R}$ is defined as follows

$$\mu_\alpha(x) := \langle (Y_\alpha)_x, x\rangle, \qquad \forall x \in \mathbb{S}^n. \tag{33}$$

Here $(Y_\alpha)_x$ denotes Y_α at the point x. The frame field $\{Y_1, \cdots, Y_{n+1}\}$ which satisfies Equation (32), is called the λ-frame field in $\mathbb{R}^{n+1}$.

Now, we provide an example of a function λ satisfying Definition 2.

Example 2. *Let $(\mathbb{S}^3, h)$ be the unit 3-sphere as a submanifold of Euclidean space $(\mathbb{R}^4, \langle, \rangle)$ and let $\{Y_1 = (1, 0, 0, 0), Y_2 = (0, 1, 0, 0), Y_3 = (0, 0, 1, 0), Y_4 = (0, 0, 0, 1)\}$ be the standard basis for the tangent space of Euclidean space $(\mathbb{R}^4, \langle, \rangle)$. Consider the function $\lambda : SM \times \mathbb{S}^3 \longrightarrow \mathbb{R}$ defined by*

$$\lambda(z, x) = K(z)(\mu_1^2(x) + 2), \tag{34}$$

for any $z \in SM$ and $x = (x^1, x^2, x^3, x^4) \in \mathbb{S}^3$, where $K : SM \longrightarrow (0, \infty)$ is a smooth positive function on SM and μ_1 is defined in (33). By (28), (33) and (34), it follows that

$$(grad_h \lambda)_{(z,x)} = 2K(z)\mu_1(x)(Y_1^\top)_x, \tag{35}$$

where $(Y_1^\top)_x$ *is the tangential part of* Y_1 *to* $\mathbb{S}^n$ *at the point* x. *By* (35), *at any point* $(z, x) \in SM \times \mathbb{S}^3$, *the left hand side of* (32) *for the function* λ *which is defined in* (34), *can be calculated as follows:*

$$\begin{aligned} \sum_{\alpha=1}^{4} \mu_\alpha(x) h(grad_h \lambda, Y_\alpha^\top)(z, x) &= 2K(z) \sum_{\alpha=1}^{4} \mu_\alpha(x)\mu_1(x)\langle Y_1^\top, Y_\alpha^\top\rangle(x) \\ &= 2K(z)(x^1)^2\Big(1 - (x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2\Big) \\ &= 0, \end{aligned} \tag{36}$$

where we use $| \ x \ |^2 = 1$, *in the last equality. Due to the fact that* (z, x) *is an arbitrary point on* $SM \times \mathbb{S}^3$ *and considering Equation* (36), *it can be seen that the function* λ *satisfies Definition* 2. *Thus,* λ *is an an adopted function.*

Remark 1. *Any smooth function* $\lambda \in C^\infty(SM)$ *is an adopted function on* $\mathbb{S}^n$.

Based on the above notations, we prove the following result.

Theorem 3. *Let* (M^m, F) *be a Finsler manifold and* $(\mathbb{S}^n, h)$ *the n-dimensional Euclidean sphere* $(n > 2)$, *and let* $\lambda : SM \times \mathbb{S}^n \longrightarrow \mathbb{R}$ *be an adopted function on* $\mathbb{S}^n$. *Then, any non-constant* λ*-harmonic map* $\psi : (M, F) \longrightarrow \mathbb{S}^n$ *is unstable.*

Proof. Choose an arbitrary point $z_0 \in SM$. Then, set $\tilde{\psi} = \psi \circ \rho$ and $x_0 = \tilde{\psi}(z_0)$, where $\rho : SM \longrightarrow M$ is the natural projection on SM. By (2), one can see that [22]

$$\rho_*(\hat{e}_{m+a}) = 0, \qquad \rho_*(e_i^H) = e_i. \tag{37}$$

Let $\{Y_1, \cdots, Y_{n+1}\}$ be a λ-frame field in $\mathbb{R}^{n+1}$ at x_0. By (2), we have

$$\begin{aligned} \sum_{\alpha=1}^{n+1} Q_\lambda^\psi(Y_\alpha^\top) = \frac{1}{c_{m-1}} \sum_{\alpha=1}^{n+1} \int_{SM} h\Big(&2Tr_g < \nabla Y_\alpha^\top, d\tilde{\psi} > (grad_h \lambda) \circ \tilde{\psi} \\ &- \nabla_{grad_{\hat{g}}^H \lambda_\psi} Y_\alpha^\top + \lambda_\psi \nabla_{K^H} Y_\alpha^\top - \lambda_\psi Tr_g(\nabla^2 Y_\alpha^\top) \\ &- \lambda_\psi Tr_g R^S(Y_\alpha^\top, d\tilde{\psi})d\tilde{\psi}, Y_\alpha^\top\Big) dV_{SM}, \end{aligned} \tag{38}$$

where ∇ and R^S denote the induced connection on the pull-back bundle $(\psi \circ \rho)^{-1}T\mathbb{S}^n$ and the curvature tensor of $\mathbb{S}^n$, respectively. Now we discuss at the point x_0. Since $\{Y_\alpha\}$ is a parallel frame field in $\mathbb{R}^{n+1}$ and considering the definition of shape operator, we obtain

$$\begin{aligned} \nabla_{e_i^H} Y_\alpha^\top &= \nabla^S_{d\tilde{\psi}(e_i^H)} Y_\alpha^\top = (\nabla^R_{d\tilde{\psi}(e_i^H)} Y_\alpha^\top)^\top \\ &= (\nabla^R_{d\tilde{\psi}(e_i^H)}(Y_\alpha - Y_\alpha^\perp))^\top = -(\nabla^R_{d\tilde{\psi}(e_i^H)} Y_\alpha^\perp)^\top \\ &= A^{Y_\alpha^\perp}(d\tilde{\psi}(e_i^H)). \end{aligned} \tag{39}$$

According to the definition of the function μ_α in (33), one can easily check that

$$A^{Y_\alpha^\perp}(V) = -\mu_\alpha V, \qquad V \in \Gamma(T\mathbb{S}^n). \tag{40}$$

By (2), (37), (39) and (40) and considering the definition of the energy density, we have

$$
\begin{aligned}
&2Tr_g < \nabla \Upsilon_\alpha^\top, d\tilde{\psi} > h\Big((grad_h\lambda)\circ\tilde{\psi}, \Upsilon_\alpha^\top\Big) \\
&= 2\sum_{A=1}^{2m-1} < \nabla_{e_A}\Upsilon_\alpha^\top, d\tilde{\psi}(e_A) > h\Big((grad_h\lambda)\circ\tilde{\psi}, \Upsilon_\alpha^\top\Big) \\
&= 2\sum_{i=1}^{m} < \nabla_{e_i^H}\Upsilon_\alpha^\top, d\tilde{\psi}(e_i^H) > h\Big((grad_h\lambda)\circ\tilde{\psi}, \Upsilon_\alpha^\top\Big) \\
&+ 2\sum_{a=1}^{m-1} < \nabla_{\hat{e}_{m+a}}\Upsilon_\alpha^\top, d\tilde{\psi}(\hat{e}_{m+a}) > h\Big((grad_h\lambda)\circ\tilde{\psi}, \Upsilon_\alpha^\top\Big) \\
&= 2\sum_{i=1}^{m} < \nabla_{e_i^H}\Upsilon_\alpha^\top, d\tilde{\psi}(e_i^H) > h\Big((grad_h\lambda)\circ\tilde{\psi}, \Upsilon_\alpha^\top\Big) \\
&= 2\sum_{i=1}^{m} < A^{\Upsilon_\alpha^\perp}(d\tilde{\psi}(e_i^H)), d\tilde{\psi}(e_i^H) > h\Big((grad_h\lambda)\circ\tilde{\psi}, \Upsilon_\alpha^\top\Big) \\
&= 2\sum_{i=1}^{m} < -\mu_\alpha d\tilde{\psi}(e_i^H), d\tilde{\psi}(e_i^H) > h\Big((grad_h\lambda)\circ\tilde{\psi}, \Upsilon_\alpha^\top\Big) \\
&= -2\sum_{i=1}^{m} \mu_\alpha < d\psi\circ\rho(e_i^H), d\psi\circ\rho(e_i^H) > h\Big((grad_h\lambda)\circ\tilde{\psi}, \Upsilon_\alpha^\top\Big) \\
&= -4\mu_\alpha e(\psi) h\Big((grad_h\lambda)\circ\tilde{\psi}, \Upsilon_\alpha^\top\Big),
\end{aligned}
\tag{41}
$$

where we use

$$
\begin{aligned}
< \nabla_{\hat{e}_{m+a}}\Upsilon_\alpha^\top, d\tilde{\psi}(\hat{e}_{m+a}) > &=< \nabla^S_{d\tilde{\psi}(\hat{e}_{m+a})}\Upsilon_\alpha^\top, d\tilde{\psi}(\hat{e}_{m+a}) >=< \nabla^S_{\psi_*(\rho_*(\hat{e}_{m+a})}\Upsilon_\alpha^\top, \psi_*(\rho_*(\hat{e}_{m+a})) > \\
&= 0
\end{aligned}
\tag{42}
$$

in the third equality. Here $\{e_A\} = \{e_i^H, \hat{e}_{m+a}\}$ is a local basis for TSM which is defined in (2). By similar calculation, we get

$$
\big\langle \nabla_{grad^H_{\tilde{g}}\lambda_\psi}\Upsilon_\alpha^\top, \Upsilon_\alpha^\top \big\rangle = -\mu_\alpha\circ\tilde{\psi}\,\big\langle d\tilde{\psi}(grad^H_{\tilde{g}}\lambda_\psi), \Upsilon_\alpha^\top \big\rangle, \tag{43}
$$

$$
\lambda_\psi \big\langle \nabla_{K^H}\Upsilon_\alpha^\top, \Upsilon_\alpha^\top \big\rangle = -\lambda_\psi\mu_\alpha\circ\tilde{\psi}\,\big\langle d\tilde{\psi}(K^H), \Upsilon_\alpha^\top \big\rangle. \tag{44}
$$

By considering (39) and (40), it can be concluded that

$$
\begin{aligned}
\sum_i \nabla_{e_i^H}\nabla_{e_i^H}\Upsilon_\alpha^\top &= \sum_i \nabla_{e_i^H} A^{\Upsilon_\alpha^\perp}(d\tilde{\psi}(e_i^H)) \\
&= -\sum_i \nabla_{e_i^H}(\mu_\alpha\circ\tilde{\psi}\ d\tilde{\psi}(e_i^H)) \\
&= -d\tilde{\psi}\,(grad\ \mu_\alpha\circ\tilde{\psi}) - \mu_\alpha\circ\tilde{\psi}\ \ \sum_i \nabla_{e_i^H} d\psi(e_i).
\end{aligned}
\tag{45}
$$

Since $grad\ \mu_\alpha = \Upsilon_\alpha^\top$ and using the definition of gradient operator, it can be shown that

$$
\begin{aligned}
d\tilde{\psi}(grad\ \mu_\alpha\circ\tilde{\psi}) &= \sum_i \big\langle d\tilde{\psi}(e_i^H), (grad\ \mu_\alpha)\circ\tilde{\psi}\big\rangle\, d\tilde{\psi}(e_i^H) \\
&= \sum_i \big\langle d\tilde{\psi}(e_i^H), \Upsilon_\alpha^\top\circ\tilde{\psi}\big\rangle\, d\tilde{\psi}(e_i^H).
\end{aligned}
\tag{46}
$$

By means of (45) and (46), we have

$$\sum_\alpha \lambda_\psi \langle Tr_g(\nabla^2 Y_\alpha^\top), Y_\alpha^\top \rangle = -\sum_\alpha \mu_\alpha \circ \tilde{\psi} \ \langle \lambda_\psi Tr_g \nabla d\psi, Y_\alpha^\top \rangle - \lambda_\psi \mid d\psi \mid^2 . \tag{47}$$

Since $\mathbb{S}^n$ has a constant curvature and considering Equation (47), it can be seen that

$$\sum_\alpha \lambda_\psi \langle Tr_g \ R^S(Y_\alpha^\top, d\psi) d\psi, Y_\alpha^\top \rangle = (n-1)\lambda_\psi \mid d\psi \mid^2 . \tag{48}$$

On the other hand, due to the fact that λ is an adopted function and $e(\psi)$ is a non-negative function and considering Equations (32) and (33), it can be seen that

$$\begin{aligned} -\frac{3}{c_{m-1}} \sum_\alpha \int_{SM} \mu_\alpha e(\psi) h\Big((grad_h \lambda) \circ \tilde{\psi}, Y_\alpha^\top\Big) &= -\frac{3}{c_{m-1}} \int_{SM} e(\psi) \sum_\alpha \mu_\alpha h\Big((grad_h \lambda) \circ \tilde{\psi}, Y_\alpha^\top\Big) \\ &\leq 0. \end{aligned} \tag{49}$$

By substituting (41)–(44) and (47)–(49) in (38) and considering the assumptions of this theorem, it follows that

$$\begin{aligned} \sum_\alpha Q_\lambda^\psi(Y_\alpha^\top) = & \frac{2-n}{c_{m-1}} \int_{SM} \lambda_\psi \mid d\psi \mid^2 dV_{SM} \\ & + \frac{1}{c_{m-1}} \sum_\alpha \int_{SM} \mu_\alpha \circ \tilde{\psi} \ \langle \tau_\lambda(\psi), Y_\alpha^\top \rangle dV_{SM} \\ & - \frac{3}{c_{m-1}} \sum_\alpha \int_{SM} \mu_\alpha e(\psi) h\Big((grad_h \lambda) \circ \tilde{\psi}, Y_\alpha^\top\Big) \\ & \leq \frac{2-n}{c_{m-1}} \int_{SM} \lambda_\psi \mid d\psi \mid^2 dV_{SM} < 0. \end{aligned} \tag{50}$$

Thus, the map ψ is unstable and hence completes the proof. □

By considering the Theorem 1 and the Remark 3, we obtain the following result.

Corollary 1. *[17] Let $\psi : (M, F) \longrightarrow (\mathbb{S}^n, h)$ be a non-constant λ-harmonic map from a Finsler manifold (M, F) to the standard n-dimensional sphere $\mathbb{S}^n (n > 2)$, and let $grad_h \lambda = 0$. Then, ψ is unstable.*

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