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Near Fixed Point Theorems in Hyperspaces

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Abstract: The hyperspace consists of all the subsets of a vector space. It is well-known that the hyperspace is not a vector space because it lacks the concept of inverse element. This also says that we cannot consider its normed structure, and some kinds of fixed point theorems cannot be established in this space. In this paper, we shall propose the concept of null set that will be used to endow a norm to the hyperspace. This normed hyperspace is clearly not a conventional normed space. Based on this norm, the concept of Cauchy sequence can be similarly defined. In addition, a Banach hyperspace can be defined according to the concept of Cauchy sequence. The main aim of this paper is to study and establish the so-called near fixed point theorems in Banach hyperspace.

Keywords: Cauchy sequence; near fixed point; Banach hyperspace; normed hyperspace; null set

MSC: 47H10; 54H25

1. Introduction

Given a universal set X, the collection of all nonempty subsets of X is called the hyperspace. The topic of set-valued analysis is to study the mathematical properties on hyperspace by referring to the monographs [1–7]. Especially, the set-valued optimization and differential inclusion have been well-studied. The purpose of this paper is to study the near fixed point theorem in hyperspace. The concept of near fixed point is the first attempt at studying the fixed point theory in hyperspace. The conventional fixed point theorem in normed space can be used to study the existence of solutions for some mathematical problems. Especially, the application of fixed point theorems in economics can refer to the articles [8–10]. Therefore, the potential applications of near fixed point theorems established in this paper can be used to study the existence of solutions for the mathematical problems that involve the set-valued mappings. This could be future research.

Let X be a topological vector space, and let $\mathcal{K}(X)$ be the collection of all nonempty compact and convex subsets of X. Given $A, B \in \mathcal{K}(X)$, the set addition is defined by

$$A \oplus B = \{a + b : a \in A \text{ and } b \in B\}$$

and the scalar multiplication in $\mathcal{K}(X)$ is defined by

$$\lambda A = \{\lambda a : a \in A\},$$

where λ is a constant in $\mathbb{R}$. It is clear to see that $\mathcal{K}(X)$ cannot be a vector space under the above set addition and scalar multiplication. The main reason is that there is no additive inverse element for each element in $\mathcal{K}(X)$.

The *substraction* between A and B is denoted and defined by

$$A \ominus B \equiv A \oplus (-B) = \{a - b : a \in A \text{ and } b \in B\}.$$

Let θ_X be the zero element of the vector space X. It is clear to see that the singleton set $\{\theta_X\}$, denoted by $\theta_{\mathcal{K}(X)}$, can be regarded as the zero element of $\mathcal{K}(X)$ since

$$A \oplus \theta_{\mathcal{K}(X)} = A \oplus \{\theta_X\} = A. \tag{1}$$

On the other hand, since $A \ominus A \neq \{\theta_X\}$, it means that $A \ominus A$ is not the zero element of $\mathcal{K}(X)$. In other words, the additive inverse element of A in $\mathcal{K}(X)$ does not exist. This says that $\mathcal{K}(X)$ cannot form a vector space under the above set addition and scalar multiplication. The following set

$$\Omega = \{A \ominus A : A \in \mathcal{K}(X)\}$$

is called the *null set* of $\mathcal{K}(X)$, which can be regarded as a kind of "zero element" of $\mathcal{K}(X)$. We also recall that the true zero element of $\mathcal{K}(X)$ is $\theta_{\mathcal{K}(X)} \equiv \{\theta_X\}$, since (1) is satisfied.

Recall that the (conventional) normed space is based on the vector space by referring to the monographs [11–17]. Since $\mathcal{K}(X)$ is not a vector space, we cannot consider the (conventional) normed space $(\mathcal{K}(X), \|\cdot\|)$. Therefore we cannot study the fixed point theorem in $(\mathcal{K}(X), \|\cdot\|)$ using the conventional way. In this paper, although $\mathcal{K}(X)$ is not a vector space, we still can endow a norm to $\mathcal{K}(X)$ in which the axioms are almost the same as the axioms of conventional norm. The only difference is that the concept of null set is involved in the axioms. Under these settings, we shall study the so-called near fixed point theorem in the normed hyperspace $(\mathcal{K}(X), \|\cdot\|)$.

Let $\mathfrak{T} : \mathcal{K}(X) \to \mathcal{K}(X)$ be a function from $\mathcal{K}(X)$ into itself. We say that $A \in \mathcal{K}(X)$ is a fixed point if and only if $\mathfrak{T}(A) = A$. Since $\mathcal{K}(X)$ lacks the vector structure, we cannot expect to obtain the fixed point of the mapping $\mathfrak{T}$ using conventional methods. In this paper, we shall try to construct a subset A of X satisfying $\mathfrak{T}(A) \oplus \omega_1 = A \oplus \omega_2$ for some $\omega_1, \omega_2 \in \Omega$. Since the null set Ω can play the role of zero element in $\mathcal{K}(X)$, i.e., the elements ω_1 and ω_2 can be ignored in some sense, this kind of subset A is said to be a near fixed point of the mapping $\mathfrak{T}$.

In Sections 2 and 3, the concept of normed hyperspace is proposed, where some interesting properties are derived in order to study the near fixed point theorem. In Section 4, according to the norm, the concept of Cauchy sequence can be similarly defined. In addition, the Banach hyperspace is defined based on the Cauchy sequence. In Section 5, we present many near fixed point theorems that are established using the almost identical concept in normed hyperspace.

2. Hyperspaces

Let X be a vector space with zero element θ_X, and let $\mathcal{K}(X)$ be the collection of all subsets of X. Under the set addition and scalar multiplication in $\mathcal{K}(X)$, it is clear to see that $\mathcal{K}(X)$ cannot form a vector space. One of the reasons is that, given any $A \in \mathcal{K}(X)$, the difference $A \ominus A$ is not a zero element of $\mathcal{K}(X)$. It is clear to see that the singleton set $\{\theta_X\}$ is a zero element, since

$$A \oplus \{\theta_X\} = \{\theta_X\} \oplus A = A$$

for any $A \in \mathcal{K}(X)$. However, when $A \in \mathcal{K}(X)$ is not a singleton set, we cannot have $A \ominus A = \{\theta_X\}$. In this section, we shall present some properties involving the null set Ω, which will be used for establishing the so-called near fixed point theorems in $\mathcal{K}(X)$.

Remark 1. For further discussion, we first recall some well-known properties given below:

- $(A \oplus B) \oplus C = A \oplus (B \oplus C)$;
- $\lambda(A \oplus B) = \lambda A \oplus \lambda B$ for $\lambda \in \mathbb{R}$;
- $\lambda_1(\lambda_2 A) = (\lambda_1\lambda_2)A$ for $\lambda_1, \lambda_2 \in \mathbb{R}$;
- if A is a convex subset of X and λ_1 and λ_2 have the same sign, then $(\lambda_1 + \lambda_2)A = \lambda_1 A \oplus \lambda_2 A$.

We also recall that the following family

$$\Omega = \{A \ominus A : A \in \mathcal{K}(X)\}$$

is called the null set of $\mathcal{K}(X)$. For further discussion, we present some useful properties.

Proposition 1. *The following statements hold true:*

- *The singleton set* $\{\theta_X\} \equiv \theta_{\mathcal{K}(X)}$ *is in the null set* Ω*;*
- $\omega \in \Omega$ *implies* $-\omega = \omega$*;*
- $\lambda\Omega = \Omega$ *for* $\lambda \in \mathbb{R}$ *with* $\lambda \neq 0$*;*
- Ω *is closed under the set addition; that is,* $\omega_1 \oplus \omega_2 \in \Omega$ *for any* $\omega_1, \omega_2 \in \Omega$*.*

Since the null set Ω is treated as a zero element, we can propose the almost identical concept for elements in $\mathcal{K}(X)$.

Definition 1. Given any $A, B \in \mathcal{K}(X)$, we say that A and B are *almost identical* if and only if there exist $\omega_1, \omega_2 \in \Omega$ such that $A \oplus \omega_1 = B \oplus \omega_2$. In this case, we write $A \stackrel{\Omega}{=} B$.

When A, B and C are not singleton sets, if $A \ominus B = C$, then we cannot have $A = B \oplus C$. However, we can have $A \stackrel{\Omega}{=} B \oplus C$. Indeed, since $A \ominus B = C$, by adding B on both sides, we obtain $A \oplus \omega = B \oplus C$, where $\omega = B \ominus B \in \Omega$. This says that $A \stackrel{\Omega}{=} B \oplus C$.

Proposition 2. *The binary relation* $\stackrel{\Omega}{=}$ *is an equivalence relation.*

Proof. For any $A \in \mathcal{K}(X)$, $A = A$ implies $A \stackrel{\Omega}{=} A$, which shows the reflexivity. The symmetry is obvious by the definition of the binary relation $\stackrel{\Omega}{=}$. Regarding the transitivity, for $A \stackrel{\Omega}{=} B$ and $B \stackrel{\Omega}{=} C$, we want to claim $A \stackrel{\Omega}{=} C$. By definition, we have

$$A \oplus \omega_1 = B \oplus \omega_2 \text{ and } B \oplus \omega_3 = C \oplus \omega_4$$

for some $\omega_i \in \Omega$ for $i = 1, \cdots, 4$. Then

$$A \oplus \omega_1 \oplus \omega_3 = B \oplus \omega_3 \oplus \omega_2 = C \oplus \omega_4 \oplus \omega_2,$$

which shows $A \stackrel{\Omega}{=} C$, since Ω is closed under the set addition. This completes the proof. □

According to the equivalence relation $\stackrel{\Omega}{=}$, for any $A \in \mathcal{K}(X)$, we define the equivalence class

$$[A] = \left\{B \in \mathcal{K}(X) : A \stackrel{\Omega}{=} B\right\}.$$

The family of all classes $[A]$ for $A \in \mathcal{K}(X)$ is denoted by $[\mathcal{K}(X)]$. In this case, the family $[\mathcal{K}(X)]$ is called the quotient set of $\mathcal{K}(X)$. We also have that $B \in [A]$ implies $[A] = [B]$. In other words, the family of all equivalence classes form a partition of the whole set $\mathcal{K}(X)$. We also remark that the quotient set $[\mathcal{K}(X)]$ is still not a vector space. The reason is

$$(\alpha + \beta)[A] \neq \alpha[A] + \beta[A]$$

for $\alpha\beta < 0$, since $(\alpha + \beta)A \neq \alpha A + \beta A$ for $A \in \mathcal{K}(X)$ with $\alpha\beta < 0$.

3. Normed Hyperspaces

Notice that $\mathcal{K}(X)$ is not a vector space. Therefore we cannot consider the normed space $(\mathcal{K}(X), \|\cdot\|)$. However, we can propose the so-called normed hyperspace involving the null set Ω as follows.

Definition 2. Given the nonnegative real-valued function $\|\cdot\|$: $\mathcal{K}(X) \to \mathbb{R}_+$, we consider the following conditions:

(i) $\| \alpha A \| = |\alpha| \, \| A \|$ for any $A \in \mathcal{K}(X)$ and $\alpha \in \mathbb{F}$;
(i′) $\| \alpha A \| = |\alpha| \, \| A \|$ for any $A \in \mathcal{K}(X)$ and $\alpha \in \mathbb{F}$ with $\alpha \neq 0$;
(ii) $\| A \oplus B \| \leq \| A \| + \| B \|$ for any $A, B \in \mathcal{K}(X)$;
(iii) $\| A \| = 0$ implies $A \in \Omega$.

We say that $\|\cdot\|$ satisfies the *null condition* when condition (iii) is replaced by $\| A \| = 0$ if and only if $A \in \Omega$. Different kinds of normed hyperspaces are defined below.

- We say that $(\mathcal{K}(X), \|\cdot\|)$ is a *pseudo-seminormed hyperspace* if and only if conditions (i′) and (ii) are satisfied.
- We say that $(\mathcal{K}(X), \|\cdot\|)$ is a *seminormed hyperspace* if and only if conditions (i) and (ii) are satisfied.
- We say that $(\mathcal{K}(X), \|\cdot\|)$ is a *pseudo-normed hyperspace* if and only if conditions (i′), (ii) and (iii) are satisfied.
- We say that $(\mathcal{K}(X), \|\cdot\|)$ is a *normed hyperspace* if and only if conditions (i), (ii) and (iii) are satisfied.

Now we consider the following definitions:

- We say that $\|\cdot\|$ satisfies the *null super-inequality* if and only if $\| A \oplus \omega \| \geq \| A \|$ for any $A \in \mathcal{K}(X)$ and $\omega \in \Omega$.
- We say that $\|\cdot\|$ satisfies the *null sub-inequality* if and only if $\| A \oplus \omega \| \leq \| A \|$ for any $A \in \mathcal{K}(X)$ and $\omega \in \Omega$.
- We say that $\|\cdot\|$ satisfies the *null equality* if and only if $\| A \oplus \omega \| = \| A \|$ for any $A \in \mathcal{K}(X)$ and $\omega \in \Omega$.

For any $A, B \in \mathcal{K}(X)$, since $-(B \ominus A) = A \ominus B$, we have

$$\| A \ominus B \| = \| B \ominus A \| .$$

Example 1. We consider the (conventional) normed space $(X, \|\cdot\|_X)$. For any $A \in \mathcal{K}(X)$, we define

$$\| A \| = \sup_{a \in A} \| a \|_X .$$

Then we have the following properties.

- $\| A \| = 0$ if and only if $A = \{\theta_X\} \in \Omega$. Indeed, if $A = \{\theta_X\}$, then it is obvious that $\| A \| = 0$. For the converse, if $\| A \| = 0$, then we have $\| a \|_X = 0$ for all $a \in A$, i.e., $A = \{\theta_X\}$.
- We have
$$\| \lambda A \| = \sup_{a \in \lambda A} \| a \|_X = \sup_{b \in A} \| \lambda b \|_X = |\lambda| \sup_{b \in A} \| b \|_X = |\lambda| \, \| A \| .$$
- We want to claim $\| A \oplus B \| \leq \| A \| + \| B \|$. We denote by
$$\zeta_1 = \sup_{\{(a,b): a \in A, b \in B\}} \| a \|_X \text{ and } \zeta_2 = \sup_{\{(a,b): a \in A, b \in B\}} \| b \|_X .$$

Then we see that $\| a \|_X + \| b \|_X \leq \zeta_1 + \zeta_2$ for all $a \in A$ and $b \in B$.

Therefore we obtain

$$\sup_{\{(a,b):a\in A,b\in B\}} (\| a \|_X + \| b \|_X) \leq \zeta_1 + \zeta_2 = \sup_{\{(a,b):a\in A,b\in B\}} \| a \|_X + \sup_{\{(a,b):a\in A,b\in B\}} \| b \|_X .$$

Now we have

$$\begin{aligned}
\| A \oplus B \| = \sup_{c\in A\oplus B} \| c \|_X &= \sup_{\{(a,b):a\in A,b\in B\}} \| a+b \|_X \\
&\leq \sup_{\{(a,b):a\in A,b\in B\}} (\| a \|_X + \| b \|_X) \\
&\leq \sup_{\{(a,b):a\in A,b\in B\}} \| a \|_X + \sup_{\{(a,b):a\in A,b\in B\}} \| b \|_X \\
&= \sup_{a\in A} \| a \|_X + \sup_{b\in B} \| b \|_X = \| A \| + \| B \| .
\end{aligned}$$

We conclude that $(\mathcal{K}(X), \| \cdot \|)$ is a normed hyperspace. For $\omega \in \Omega$, it means that $\omega = B \ominus B$ for some $B \in \mathcal{K}(X)$. Then we have

$$\| \omega \| = \| B \ominus B \| = \sup_{\{(b_1,b_2):b_1,b_2\in B\}} \| b_1 - b_2 \|_X .$$

Since $\| \omega \|$ is not equal to zero in general, it means that the null condition is not satisfied.

Proposition 3. *Let $(\mathcal{K}(X), \| \cdot \|)$ be a pseudo-seminormed hyperspace such that $\| \cdot \|$ satisfies the null super-inequality. For any $A, C, B_1, \cdots, B_m \in \mathcal{K}(X)$, we have*

$$\| A \ominus C \| \leq \| A \ominus B_1 \| + \| B_1 \ominus B_2 \| + \cdots + \| B_j \ominus B_{j+1} \| + \cdots + \| B_m \ominus C \| .$$

Proof. We have

$$\begin{aligned}
\| A \ominus C \| &\leq \| A \oplus (-C) \oplus B_1 \oplus \cdots \oplus B_m \oplus (-B_1) \oplus \cdots \oplus (-B_m) \| \\
&\quad \text{(using the null super-inequality for } m \text{ times)} \\
&= \| [A \oplus (-B_1)] \oplus [B_1 \oplus (-B_2)] + \cdots + [B_j \oplus (-B_{j+1})] + \cdots + [B_m \oplus (-C)] \| \\
&\leq \| A \ominus B_1 \| + \| B_1 \ominus B_2 \| + \cdots + \| B_j \ominus B_{j+1} \| + \cdots + \| B_m \ominus C \| \\
&\quad \text{(using the triangle inequality).}
\end{aligned}$$

This completes the proof. □

Proposition 4. *The following statements hold true.*

(i) *Let $(\mathcal{K}(X), \| \cdot \|)$ be a pseudo-seminormed hyperspace such that $\| \cdot \|$ satisfies the null equality. For any $A, B \in \mathcal{K}(X)$, if $A \stackrel{\Omega}{=} B$, then $\| A \| = \| B \|$.*

(ii) *Let $(\mathcal{K}(X), \| \cdot \|)$ be a pseudo-normed hyperspace. For any $A, B \in \mathcal{K}(X)$, $\| A \ominus B \| = 0$ implies $A \stackrel{\Omega}{=} B$.*

(iii) *Let $(\mathcal{K}(X), \| \cdot \|)$ be a pseudo-seminormed hyperspace such that $\| \cdot \|$ satisfies the null super-inequality and null condition. For any $A, B \in \mathcal{K}(X)$, $A \stackrel{\Omega}{=} B$ implies $\| A \ominus B \| = 0$.*

Proof. To prove part (i), we see that $A \stackrel{\Omega}{=} B$ implies $A \oplus \omega_1 = B \oplus \omega_2$ for some $\omega_1, \omega_2 \in \Omega$. Therefore, using the null equality, we have $\| A \| = \| A \oplus \omega_1 \| = \| B \oplus \omega_2 \| = \| B \|$. To prove part (ii), suppose that $\| A \ominus B \| = 0$. Then $A \ominus B \in \Omega$, i.e., $A \ominus B = \omega_1$ for some $\omega_1 \in \Omega$. Then, by adding B on both sides, we have $A \oplus \omega_2 = B \oplus \omega_1$ for some $\omega_2 \in \Omega$, which says that $A \stackrel{\Omega}{=} B$. To prove part (iii), for $A \stackrel{\Omega}{=} B$, we have $A \oplus \omega_1 = B \oplus \omega_2$ for some $\omega_1, \omega_2 \in \Omega$. Since Ω is closed under the vector addition, it follows that

$$A \ominus B \oplus \omega_1 = A \oplus \omega_1 \ominus B = B \oplus \omega_2 \ominus B = \omega_3 \tag{2}$$

for some $\omega_3 \in \Omega$. Using the null super-inequality, null condition and (2), we have

$$\| A \ominus B \| \leq \| A \ominus B \oplus \omega_1 \| = \| \omega_3 \| = 0.$$

This completes the proof. □

4. Cauchy Sequences

Let $(\mathcal{K}(X), \| \cdot \|)$ be a pseudo-seminormed hyperspace. Given a sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{K}(X)$, it is clear that $\| A_n \ominus A \| = \| A \ominus A_n \|$. The concept of limit is defined below.

Definition 3. Let $(\mathcal{K}(X), \| \cdot \|)$ be a pseudo-seminormed hyperspace. A sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{K}(X)$ is said to *converge* to $A \in \mathcal{K}(X)$ if and only if

$$\lim_{n\to\infty} \| A_n \ominus A \| = \lim_{n\to\infty} \| A \ominus A_n \| = 0.$$

We have the following interesting results.

Proposition 5. *Let $(\mathcal{K}(X), \| \cdot \|)$ be a pseudo-normed hyperspace with the null set Ω.*

(i) *If the sequence $\{A_n\}_{n=1}^{\infty}$ in $(\mathcal{K}(X), \| \cdot \|)$ converges to A and B simultaneously, then $[A] = [B]$.*
(ii) *Suppose that $\| \cdot \|$ satisfies the null equality. If the sequence $\{A_n\}_{n=1}^{\infty}$ in $(\mathcal{K}(X), \| \cdot \|)$ converges to $A \in \mathcal{K}(X)$, then, give any $B \in [A]$, the sequence $\{A_n\}_{n=1}^{\infty}$ converges to B.*

Proof. To prove the first case of part (i), we have

$$\lim_{n\to\infty} \| A \ominus A_n \| = \lim_{n\to\infty} \| A_n \ominus B \| = 0.$$

By Proposition 3, we have

$$0 \leq \| A \ominus B \| \leq \| A \ominus A_n \| + \| A_n \ominus B \| \to 0 + 0 = 0, \tag{3}$$

which says that $\| A \ominus B \| = 0$. By Definition 2, we see that $A \ominus B \in \Omega$, i.e. $A \stackrel{\Omega}{=} B$, which also says that B is in the equivalence class $[A]$.

To prove part (ii), for any $B \in [A]$, i.e., $A \oplus \omega_1 = B \oplus \omega_2$ for some $\omega_1, \omega_2 \in \Omega$, using the null equality, we have

$$0 \leq \| A_n \ominus B \| = \| B \ominus A_n \| = \| \omega_2 \oplus B \ominus A_n \| = \| \omega_1 \oplus A \ominus A_n \| = \| A \ominus A_n \| \to 0.$$

This completes the proof. □

Inspired by part (ii) of Proposition 5, we propose the following concept of limit.

Definition 4. Let $(\mathcal{K}(X), \| \cdot \|)$ be a pseudo-seminormed hyperspace. If the sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{K}(X)$ converges to some $A \in \mathcal{K}(X)$, then the equivalence class $[A]$ is called the *class limit* of $\{A_n\}_{n=1}^{\infty}$. We also write

$$\lim_{n\to\infty} A_n = [A] \text{ or } A_n \to [A].$$

We need to remark that if $[A]$ is a class limit and $B \in [A]$ then it is not necessarily that the sequence $\{A_n\}_{n=1}^{\infty}$ converges to y unless $\| \cdot \|$ satisfies the null equality. In other words, for the class limit $[A]$, if $\| \cdot \|$ satisfies the null equality, then part (ii) of Proposition 5 says that sequence $\{A_n\}_{n=1}^{\infty}$ converges to B for any $B \in [A]$.

Proposition 6. *Let $(\mathcal{K}(X), \| \cdot \|)$ be a pseudo-normed hyperspace such that $\| \cdot \|$ satisfies the null super-inequality. Then the class limit is unique.*

Proof. Suppose that the sequence $\{A_n\}_{n=1}^{\infty}$ is convergent with the class limits $[A]$ and $[B]$. Then, by definition, we have

$$\lim_{n\to\infty} \| A \ominus A_n \| = \lim_{n\to\infty} \| A_n \ominus A \| = \lim_{n\to\infty} \| B \ominus A_n \| = \lim_{n\to\infty} \| A_n \ominus B \| = 0,$$

which says that $\| A \ominus B \| = 0$ by referring to (3). By part (ii) of Proposition 4, we have $A \stackrel{\Omega}{=} B$, i.e., $[A] = [B]$. This shows the uniqueness in the sense of class limit. □

Definition 5. Let $(\mathcal{K}(X), \| \cdot \|)$ be a pseudo-seminormed hyperspace. A sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{K}(X)$ is called a *Cauchy sequence* if and only if, given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\| A_n \ominus A_m \| = \| A_m \ominus A_n \| < \epsilon$$

for $m, n > N$ with $m \neq n$. If every Cauchy sequence in $\mathcal{K}(X)$ is convergent, then we say that $\mathcal{K}(X)$ is *complete*.

Proposition 7. *Let $(\mathcal{K}(X), \| \cdot \|)$ be a pseudo-seminormed hyperspace such that $\| \cdot \|$ satisfies the null super-inequality. Then every convergent sequence is a Cauchy sequence.*

Proof. If $\{A_n\}_{n=1}^{\infty}$ is a convergent sequence, then, given any $\epsilon > 0$, $\| A_n \ominus A \| = \| A \ominus A_n \| < \epsilon/2$ for sufficiently large n. Therefore, by Proposition 3, we have

$$\| A_n \ominus A_m \| = \| A_m \ominus A_n \| \leq \| A_m \ominus A \| + \| A \ominus A_n \| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for sufficiently large n and m, which says that $\{A_n\}_{n=1}^{\infty}$ is a Cauchy sequence. This completes the proof. □

Definition 6. Different kinds of Banach hyperspaces are defined below.

- Let $(\mathcal{K}(X), \| \cdot \|)$ be a pseudo-seminormed hyperspace. If $\mathcal{K}(X)$ is complete, then it is called a *pseudo-semi-Banach hyperspace*.
- Let $(\mathcal{K}(X), \| \cdot \|)$ be a seminormed hyperspace. If $\mathcal{K}(X)$ is complete, then it is called a *semi-Banach hyperspace*.
- Let $(\mathcal{K}(X), \| \cdot \|)$ be a pseudo-normed hyperspace. If $\mathcal{K}(X)$ is complete, then it is called a *pseudo-Banach hyperspace*.
- Let $(\mathcal{K}(X), \| \cdot \|)$ be a normed hyperspace. If $\mathcal{K}(X)$ is complete, then it is called a *Banach hyperspace*.

Example 2. Continued from Example 1, we further assume that $(X, \| \cdot \|_X)$ is a (conventional) Banach space. Then we want to show that the normed hyperspace $(\mathcal{K}(X), \| \cdot \|)$ is complete. Suppose that $\{A_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathcal{K}(X), \| \cdot \|)$. Let $\mathcal{A}$ be the collection of all sequences induced from the sequence $\{A_n\}_{n=1}^{\infty}$. More precisely, each element in $\mathcal{A}$ is a sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n \in A_n$ for all n. Firstly, we need to claim that each sequence in $\mathcal{A}$ is convergent. Since $\{A_n\}_{n=1}^{\infty}$ is a Cauchy sequence, i.e., $\| A_n \ominus A_m \| < \epsilon$ for $m, n > N$ with $m \neq n$, we have

$$\epsilon > \| A_n \ominus A_m \| = \sup_{x \in A_n - A_m} \| x \|_X = \sup_{\{(a_n, a_m) : a_n \in A_n, a_m \in A_m\}} \| a_n - a_m \|_X, \tag{4}$$

which says that $\| a_n - a_m \|_X < \epsilon$ for any sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n \in A_n$ for all n, where ϵ is independent of a_n and a_m. By the completeness of $(X, \| \cdot \|_X)$, it follows that each sequence $\{a_n\}_{n=1}^{\infty}$ is convergent to some $a \in X$, i.e., $\| a_n - a \|_X \to 0$ as $n \to \infty$ in the uniform sense, which means $\| a_n - a \|_X < \epsilon$ such that ϵ is independent of a_n and a. Indeed, if ϵ is dependent of a_n and a, then

$$\| a_n - a_m \|_X \leq \| a_n - a \|_X + \| a - a_m \|_X$$

says that ϵ is dependent of a_n and a_m, which is a contradiction. We can define a subset A of X that collects all of the limit points of each sequence in $\mathcal{A}$. Then, finally, we want to claim $\| A_n \ominus A \| \to 0$ as $n \to \infty$. For any $x \in A_n - A$, we have $x = a_n - a$ for some $a_n \in A_n$ and $a \in A$. Since a is a limit point of some sequence $\{a_n^{\circ}\}_{n=1}^{\infty}$, for $m > n > N$, using (4), we have

$$\| a_n - a \|_X \leq \| a_n - a_m^{\circ} \|_X + \| a_m^{\circ} - a \|_X < \epsilon + \| a_m^{\circ} - a \|_X,$$

where ϵ is independent of a_n and a_m° by referring to (4) again. Since $\| a_m^{\circ} - a \|_X \to 0$ as $m \to \infty$ in the uniform sense, it follows that $\| a_n - a \|_X \to 0$ as $n \to \infty$ in the uniform sense. Therefore we obtain

$$\| A_n \ominus A \| = \sup_{x \in A_n - A} \| x \|_X = \sup_{\{(a_n, a): a_n \in A_n, a \in A\}} \| a_n - a \|_X \to 0 \text{ as } n \to \infty.$$

This shows that the sequence $\{A_n\}_{n=1}^{\infty}$ is convergent, i.e., $(\mathcal{K}(X), \| \cdot \|)$ is a Banach hyperspace.

5. Near Fixed Point Theorems

Let $\mathfrak{T} : \mathcal{K}(X) \to \mathcal{K}(X)$ be a function from $\mathcal{K}(X)$ into itself. We say that $A \in \mathcal{K}(X)$ is a fixed point if and only if $\mathfrak{T}(A) = A$. This concept is completely different from the concept of fixed point in set-valued functions. Some conventional fixed point theorems are based on the normed space that is also a vector space. Since $(\mathcal{K}(X), \| \cdot \|)$ is not a vector space, we cannot study the corresponding fixed point theorems based on $(\mathcal{K}(X), \| \cdot \|)$. However, we can study the so-called near fixed point that is defined below.

Definition 7. Let $\mathfrak{T} : \mathcal{K}(X) \to \mathcal{K}(X)$ be a function defined on $\mathcal{K}(X)$ into itself. A point $A \in \mathcal{K}(X)$ is called a *near fixed point* of $\mathfrak{T}$ if and only if $\mathfrak{T}(A) \stackrel{\Omega}{=} A$.

By definition, we see that $\mathfrak{T}(A) \stackrel{\Omega}{=} A$ if and only if there exist $\omega_1, \omega_2 \in \Omega$ such that $\mathfrak{T}(A) = A$, $\mathfrak{T}(A) \oplus \omega_1 = A$, or $\mathfrak{T}(A) = A \oplus \omega_1$ or $\mathfrak{T}(A) \oplus \omega_1 = A \oplus \omega_2$.

Definition 8. Let $(\mathcal{K}(X), \| \cdot \|)$ be a pseudo-seminormed hyperspace. A function $\mathfrak{T} : (\mathcal{K}(X), \| \cdot \|) \to (\mathcal{K}(X), \| \cdot \|)$ is called a *contraction on* $\mathcal{K}(X)$ if and only if there is a real number $0 < \alpha < 1$ such that

$$\| \mathfrak{T}(A) \ominus \mathfrak{T}(B) \| \leq \alpha \| A \ominus B \|$$

for any $A, B \in \mathcal{K}(X)$.

Given any initial element $A_0 \in \mathcal{K}(X)$, we define the iterative sequence $\{A_n\}_{n=1}^{\infty}$ using the function $\mathfrak{T}$ as follows:

$$A_1 = \mathfrak{T}(A_0), \quad A_2 = \mathfrak{T}(A_1) = \mathfrak{T}^2(A_0), \cdots, A_n = \mathfrak{T}^n(A_0). \tag{5}$$

Under some suitable conditions, we are going to show that the sequence $\{A_n\}_{n=1}^{\infty}$ can converge to near a fixed point.

Theorem 1. *Let $(\mathcal{K}(X), \| \cdot \|)$ be a Banach hyperspace with the null set Ω such that $\| \cdot \|$ satisfies the null equality. Suppose that the function $\mathfrak{T} : (\mathcal{K}(X), \| \cdot \|) \to (\mathcal{K}(X), \| \cdot \|)$ is a contraction on $\mathcal{K}(X)$. Then $\mathfrak{T}$ has a near fixed point $A \in \mathcal{K}(X)$ satisfying $\mathfrak{T}(A) \stackrel{\Omega}{=} A$. Moreover, the near fixed point A is obtained by the limit*

$$\| A \ominus A_n \| = \| A_n \ominus A \| \to 0 \text{ as } n \to \infty$$

in which the sequence $\{A_n\}_{n=1}^{\infty}$ is generated according to (5). We also have the following properties.

- *The uniqueness is in the sense that there is a unique equivalence class $[A]$ such that any $\bar{A} \notin [A]$ cannot be near a fixed point.*
- *Each point $\bar{A} \in [A]$ is also a near fixed point of $\mathfrak{T}$ satisfying $\mathfrak{T}(\bar{A}) \stackrel{\Omega}{=} \bar{A}$ and $[\bar{A}] = [A]$.*
- *If $\bar{A}$ is a near fixed point of $\mathfrak{T}$, then $\bar{A} \in [A]$, i.e., $[\bar{A}] = [A]$. Equivalently, if A and $\bar{A}$ are the near fixed points of $\mathfrak{T}$, then $A \stackrel{\Omega}{=} \bar{A}$.*

Proof. Given any initial element $A_0 \in \mathcal{K}(X)$, we are going to show that $\{A_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since $\mathfrak{T}$ is a contraction on $\mathcal{K}(X)$, we have

$$\begin{aligned} \| A_{m+1} \ominus A_m \| &= \| \mathfrak{T}(A_m) \ominus \mathfrak{T}(A_{m-1}) \| \leq \alpha \| A_m \ominus A_{m-1} \| \\ &= \alpha \| \mathfrak{T}(A_{m-1}) \ominus \mathfrak{T}(A_{m-2}) \| \leq \alpha^2 \| A_{m-1} \ominus A_{m-2} \| \\ &\leq \cdots \leq \alpha^m \| A_1 \ominus A_0 \| . \end{aligned}$$

For $n < m$, using Proposition 3, we obtain

$$\begin{aligned} \| A_m \ominus A_n \| &\leq \| A_m \ominus A_{m-1} \| + \| A_{m-1} \ominus A_{m-2} \| + \cdots + \| A_{n+1} \ominus A_n \| \\ &\leq \left(\alpha^m + \alpha^{m+1} + \cdots + \alpha^n \right) \cdot \| A_1 \ominus A_0 \| \\ &= \alpha^n \cdot \frac{1 - \alpha^{m-n}}{1 - \alpha} \cdot \| A_1 \ominus A_0 \| . \end{aligned}$$

Since $0 < \alpha < 1$, we have $1 - \alpha^{m-n} < 1$ in the numerator, which says that

$$\| A_m \ominus A_n \| \leq \frac{\alpha^n}{1 - \alpha} \cdot \| A_1 \ominus A_0 \| \to 0 \text{ as } n \to \infty.$$

This proves that $\{A_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since $\mathcal{K}(X)$ is complete, there exists $A \in \mathcal{K}(X)$ such that

$$\| A \ominus A_n \| = \| A_n \ominus A \| \to 0 \text{ as } n \to \infty.$$

We are going to show that any point $\bar{A} \in [A]$ is a near fixed point. Now we have $\bar{A} \oplus \omega_1 = A \oplus \omega_2$ for some $\omega_1, \omega_2 \in \Omega$. Using the triangle inequality and the fact of contraction on $\mathcal{K}(X)$, we have

$$\begin{aligned} \| \bar{A} \ominus \mathfrak{T}(\bar{A}) \| &= \| (\bar{A} \oplus \omega_1) \ominus \mathfrak{T}(\bar{A}) \| \quad \text{(since } \| \cdot \| \text{ satisfies the null equality)} \\ &\leq \| (\bar{A} \oplus \omega_1) \ominus A_m \| + \| A_m \ominus \mathfrak{T}(\bar{A}) \| \quad \text{(using Proposition 3)} \\ &= \| (\bar{A} \oplus \omega_1) \ominus A_m \| + \| \mathfrak{T}(A_{m-1}) \ominus \mathfrak{T}(\bar{A}) \| \\ &\leq \| (\bar{A} \oplus \omega_1) \ominus A_m \| + \alpha \| A_{m-1} \ominus \bar{A} \| \\ &= \| (\bar{A} \oplus \omega_1) \ominus A_m \| + \alpha \| A_{m-1} \ominus \bar{A} \oplus (-\omega_1) \| \\ &\quad \text{(since } -\omega_1 \in \Omega \text{ and } \| \cdot \| \text{ satisfies the null equality)} \\ &= \| (\bar{A} \oplus \omega_1) \ominus A_m \| + \alpha \| A_{m-1} \ominus (\bar{A} \oplus \omega_1) \| \quad \text{(using Remark 1)} \\ &= \| (A \oplus \omega_2) \ominus A_m \| + \alpha \| A_{m-1} \ominus (A \oplus \omega_2) \| \\ &= \| A \ominus A_m \| + \alpha \| A_{m-1} \ominus A \| \\ &\quad \text{(using } -\omega_2 \in \Omega \text{, the null equality and Remark 1),} \end{aligned}$$

which implies $\| \bar{A} \ominus \mathfrak{T}(\bar{A}) \| = 0$ as $m \to \infty$. We conclude that $\mathfrak{T}(\bar{A}) \stackrel{\Omega}{=} \bar{A}$ for any point $\bar{A} \in [A]$ by part (ii) of Proposition 4.

Now assume that there is another near fixed point $\bar{A}$ of $\mathfrak{T}$ with $\bar{A} \notin [A]$, i.e., $\bar{A} \stackrel{\Omega}{=} \mathfrak{T}(\bar{A})$. Then

$$\bar{A} \oplus \omega_1 = \mathfrak{T}(\bar{A}) \oplus \omega_2 \text{ and } A \oplus \omega_3 = \mathfrak{T}(A) \oplus \omega_4$$

for some $\omega_i \in \Omega$, $i = 1, \cdots, 4$. Since $\mathfrak{T}$ is a contraction on $\mathcal{K}(X)$ and $\| \cdot \|$ satisfies the null equality, we obtain

$$\begin{aligned}
\| \bar{A} \ominus A \| &= \| (\bar{A} \oplus \omega_1) \ominus (A \oplus \omega_3) \| \\
&\quad \text{(using } -\omega_3 \in \Omega \text{, the null equality and Remark 1)} \\
&= \| (\mathfrak{T}(\bar{A}) \oplus \omega_2) \ominus (\mathfrak{T}(A) \oplus \omega_4) \| = \| \mathfrak{T}(\bar{A}) \ominus \mathfrak{T}(A) \| \\
&\quad \text{(using } -\omega_4 \in \Omega \text{, the null equality and Remark 1)} \\
&\leq \alpha \| \bar{A} \ominus A \| .
\end{aligned}$$

Since $0 < \alpha < 1$, we conclude that $\| \bar{A} \ominus A \| = 0$, i.e., $\bar{A} \stackrel{\Omega}{=} A$, which contradicts $\bar{A} \notin [A]$. Therefore, any $\bar{A} \notin [A]$ cannot be the near fixed point. Equivalently, if $\bar{A}$ is a near fixed point of $\mathfrak{T}$, then $\bar{A} \in [A]$. This completes the proof. □

Definition 9. Let $(\mathcal{K}(X), \| \cdot \|)$ be a pseudo-normed hyperspace. A function $\mathfrak{T} : (\mathcal{K}(X), \| \cdot \|) \to (\mathcal{K}(X), \| \cdot \|)$ is called a *weakly strict contraction on* $\mathcal{K}(X)$ if and only if the following conditions are satisfied:

- $A \stackrel{\Omega}{=} B$, i.e., $[A] = [B]$ implies $\| \mathfrak{T}(A) \ominus \mathfrak{T}(B) \| = 0$.
- $A \stackrel{\Omega}{\neq} B$, i.e., $[A] \neq [B]$ implies $\| \mathfrak{T}(A) \ominus \mathfrak{T}(B) \| < \| A \ominus B \|$.

By part (ii) of Proposition 4, we see that if $A \stackrel{\Omega}{\neq} B$, then $\| A \ominus B \| \neq 0$, which says that the weakly strict contraction is well-defined. In other words, $(\mathcal{K}(X), \| \cdot \|)$ should be assumed to be a pseudo-normed hyperspace rather than pseudo-seminormed hyperspace. We further assume that $\| \cdot \|$ satisfies the null super-inequality and null condition. Part (iii) of Proposition 4 says that if T is a contraction on $\mathcal{K}(X)$, then it is also a weakly strict contraction on $\mathcal{K}(X)$.

Theorem 2. *Let $(\mathcal{K}(X), \| \cdot \|)$ be a Banach hyperspace with the null set Ω Suppose that $\| \cdot \|$ satisfies the null super-inequality and null condition, and that the function $\mathfrak{T} : (\mathcal{K}(X), \| \cdot \|) \to (\mathcal{K}(X), \| \cdot \|)$ is a weakly strict contraction on $\mathcal{K}(X)$. If $\{\mathfrak{T}^n(A_0)\}_{n=1}^{\infty}$ forms a Cauchy sequence for some $A_0 \in \mathcal{K}(X)$, then $\mathfrak{T}$ has a near fixed point $A \in \mathcal{K}(X)$ satisfying $\mathfrak{T}(A) \stackrel{\Omega}{=} A$. Moreover, the near fixed point A is obtained by the limit*

$$\| \mathfrak{T}^n(A_0) \ominus A \| = \| A \ominus \mathfrak{T}^n(A_0) \| \to 0 \text{ as } n \to \infty.$$

Assume further that $\| \cdot \|$ satisfies the null equality. Then we also have the following properties.

- *The uniqueness is in the sense that there is a unique equivalence class $[A]$ such that any $\bar{A} \notin [A]$ cannot be a near fixed point.*
- *Each point $\bar{A} \in [A]$ is also a near fixed point of $\mathfrak{T}$ satisfying $\mathfrak{T}(\bar{A}) \stackrel{\Omega}{=} \bar{A}$ and $[\bar{A}] = [A]$.*
- *If $\bar{A}$ is a near fixed point of $\mathfrak{T}$, then $\bar{A} \in [A]$, i.e., $[\bar{A}] = [A]$. Equivalently, if A and $\bar{A}$ are the near fixed points of $\mathfrak{T}$, then $A \stackrel{\Omega}{=} \bar{A}$.*

Proof. Since $\{\mathfrak{T}^n(A_0)\}_{n=1}^{\infty}$ is a Cauchy sequence, the completeness says that there exists $A \in \mathcal{K}(X)$ such that

$$\| \mathfrak{T}^n(A_0) \ominus A \| = \| A \ominus \mathfrak{T}^n(A_0) \| \to 0.$$

Therefore, given any $\epsilon > 0$, there exists an integer N such that $\| \mathfrak{T}^n(A_0) \ominus A \| < \epsilon$ for $n \geq N$. We consider the following two cases.

- Suppose that $\mathfrak{T}^n(A_0) \stackrel{\Omega}{=} A$. Since $\mathfrak{T}$ is a weakly strict contraction on $\mathcal{K}(X)$, it follows that

$$\| \mathfrak{T}^{n+1}(A_0) \ominus \mathfrak{T}(A) \| = 0 < \epsilon$$

  by part (iii) of Proposition 4.
- Suppose that $\mathfrak{T}^n(A_0) \stackrel{\Omega}{\neq} A$. Since $\mathfrak{T}$ is a weakly strict contraction on $\mathcal{K}(X)$, we have

$$\| \mathfrak{T}^{n+1}(A_0) \ominus \mathfrak{T}(A) \| < \| \mathfrak{T}^n(A_0) \ominus A \| < \epsilon \text{ for } n \geq N.$$

The above two cases say that $\| \mathfrak{T}^{n+1}(A_0) \ominus \mathfrak{T}(A) \| \to 0$. Using Proposition 3, we obtain

$$\| A \ominus \mathfrak{T}(A) \| \leq \| A \ominus \mathfrak{T}^{n+1}(A_0) \| + \| \mathfrak{T}^{n+1}(A_0) \ominus \mathfrak{T}(A) \| \to 0 \text{ as } n \to \infty,$$

which says that $\| A \ominus \mathfrak{T}(A) \| = 0$, i.e., $\mathfrak{T}(A) \stackrel{\Omega}{=} A$ by part (ii) of Proposition 4. This shows that A is a near fixed point.

Assume that $\| \cdot \|$ satisfies the null equality. We are going to claim that each point $\bar{A} \in [A]$ is also a near fixed point of $\mathfrak{T}$. Since $\bar{A} \stackrel{\Omega}{=} A$, we have $\bar{A} \oplus \omega_1 = A \oplus \omega_2$ for some $\omega_1, \omega_2 \in \Omega$. Then, using the null equality for $\| \cdot \|$, we obtain

$$\begin{aligned} \| \mathfrak{T}^n(A_0) \ominus \bar{A} \| &= \| \bar{A} \ominus \mathfrak{T}^n(A_0) \| = \| (\bar{A} \oplus \omega_1) \ominus \mathfrak{T}^n(A_0) \| = \| (A \oplus \omega_2) \ominus \mathfrak{T}^n(A_0) \| \\ &= \| A \ominus \mathfrak{T}^n(A_0) \| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Using the above argument, we can also obtain $\| \mathfrak{T}^{n+1}(A_0) \ominus \mathfrak{T}(\bar{A}) \| \to 0$ as $n \to \infty$. Using Proposition 3, we have

$$\| \bar{A} \ominus \mathfrak{T}(\bar{A}) \| \leq \| \bar{A} \ominus \mathfrak{T}^{n+1}(A_0) \| + \| \mathfrak{T}^{n+1}(A_0) \ominus \mathfrak{T}(\bar{A}) \| \to 0 \text{ as } n \to \infty,$$

which says that $\| \bar{A} \ominus \mathfrak{T}(\bar{A}) \| = 0$. Therefore we conclude that $\mathfrak{T}(\bar{A}) \stackrel{\Omega}{=} \bar{A}$ for any point $\bar{A} \in [A]$ by part (ii) of Proposition 4.

Suppose that $\bar{A} \notin [A]$ is another near fixed point of $\mathfrak{T}$. Then $\mathfrak{T}(\bar{A}) \stackrel{\Omega}{=} \bar{A}$ and $[\bar{A}] \neq [A]$, i.e., $A \stackrel{\Omega}{\neq} \bar{A}$. Then $\mathfrak{T}(A) \oplus \omega_1 = A \oplus \omega_2$ and $\mathfrak{T}(\bar{A}) \oplus \omega_3 = \bar{A} \oplus \omega_4$, where $\omega_i \in \Omega$ for $i = 1, 2, 3, 4$. Therefore we obtain

$$\begin{aligned} \| A \ominus \bar{A} \| &= \| (A \oplus \omega_2) \ominus (\bar{A} \oplus \omega_4) \| \\ &\quad \text{(using } -\omega_4 \in \Omega \text{, the null equality and Remark 1)} \\ &= \| (\mathfrak{T}(A) \oplus \omega_1) \ominus (\mathfrak{T}(\bar{A}) \oplus \omega_3) \| = \| \mathfrak{T}(A) \ominus \mathfrak{T}(\bar{A}) \| \\ &\quad \text{(using } -\omega_3 \in \Omega \text{, the null equality and Remark 1)} \\ &< \| A \ominus \bar{A} \| \quad \text{(since } A \stackrel{\Omega}{\neq} \bar{A} \text{ and } \mathfrak{T} \text{ is a weakly strict contraction).} \end{aligned}$$

This contradiction says that $\bar{A}$ cannot be a near fixed point of $\mathfrak{T}$. Equivalently, if $\bar{A}$ is a near fixed point of $\mathfrak{T}$, then $\bar{A} \in [A]$. This completes the proof. $\square$

Now we consider another fixed point theorem based on the concept of weakly uniformly strict contraction which was proposed by Meir and Keeler [18].

Definition 10. Let $(\mathcal{K}(X), \|\cdot\|)$ be a pseudo-normed hyperspace with the null set Ω. A function $\mathfrak{T} : (\mathcal{K}(X), \|\cdot\|) \to (\mathcal{K}(X), \|\cdot\|)$ is called a *weakly uniformly strict contraction on* $\mathcal{K}(X)$ if and only if the following conditions are satisfied:

- for $A \stackrel{\Omega}{=} B$, i.e., $[A] = [B]$, $\| \mathfrak{T}(A) \ominus \mathfrak{T}(B) \| = 0$;
- given any $\epsilon > 0$, there exists $\delta > 0$ such that $\epsilon \leq \| A \ominus B \| < \epsilon + \delta$ implies $\| \mathfrak{T}(A) \ominus \mathfrak{T}(B) \| < \epsilon$ for any $A \stackrel{\Omega}{\neq} B$, i.e., $[A] \neq [B]$.

By part (ii) of Proposition 4, we see that if $A \stackrel{\Omega}{\neq} B$, then $\| A \ominus B \| \neq 0$, which says that the weakly uniformly strict contraction is well-defined. In other words, $(\mathcal{K}(X), \|\cdot\|)$ should be assumed to be a pseudo-normed hyperspace rather than pseudo-seminormed hyperspace.

Remark 2. We observe that if $\mathfrak{T}$ is a weakly uniformly strict contraction on $\mathcal{K}(X)$, then $\mathfrak{T}$ is also a weakly strict contraction on $\mathcal{K}(X)$.

Lemma 1. *Let $(\mathcal{K}(X), \|\cdot\|)$ be a pseudo-normed hyperspace with the null set Ω, and let $\mathfrak{T} : (\mathcal{K}(X), \|\cdot\|) \to (\mathcal{K}(X), \|\cdot\|)$ be a weakly uniformly strict contraction on $\mathcal{K}(X)$. Then the sequence $\{\| \mathfrak{T}^n(A) \ominus \mathfrak{T}^{n+1}(A) \|\}_{n=1}^{\infty}$ is decreasing to zero for any $A \in \mathcal{K}(X)$.*

Proof. For convenience, we write $\mathfrak{T}^n(A) = A_n$ for all n. Let $c_n = \| A_n \ominus A_{n+1} \|$.

- Suppose that $[A_{n-1}] \neq [A_n]$. By Remark 2, we have

$$c_n = \| A_n \ominus A_{n+1} \| = \| \mathfrak{T}^n(A) \ominus \mathfrak{T}^{n+1}(A) \| < \| \mathfrak{T}^{n-1}(A) \ominus \mathfrak{T}^n(A) \| = \| A_{n-1} \ominus A_n \| = c_{n-1}.$$

- Suppose that $[A_{n-1}] = [A_n]$. Then, by the first condition of Definition 10,

$$c_n = \| \mathfrak{T}^n(A) \ominus \mathfrak{T}^{n+1}(A) \| = \| \mathfrak{T}(A_{n-1}) \ominus \mathfrak{T}(A_n) \| = 0 < c_{n-1}.$$

The above two cases say that the sequence $\{c_n\}_{n=1}^{\infty}$ is decreasing. We consider the following cases.

- Let m be the first index in the sequence $\{A_n\}_{n=1}^{\infty}$ such that $[A_{m-1}] = [A_m]$. Then we want to claim $c_{m-1} = c_m = c_{m+1} = \cdots = 0$. Since $A_{m-1} \stackrel{\Omega}{=} A_m$, we have

$$c_{m-1} = \| A_{m-1} \ominus A_m \| = 0.$$

Using the first condition of Definition 10, we also have

$$0 = \| \mathfrak{T}(A_{m-1}) \ominus \mathfrak{T}(A_m) \| = \| \mathfrak{T}^m(A) \ominus \mathfrak{T}^{m+1}(A) \| = \| A_m \ominus A_{m+1} \| = c_m,$$

which says that $A_m \stackrel{\Omega}{=} A_{m+1}$, i.e., $[A_m] = [A_{m+1}]$. Using the similar arguments, we can obtain $c_{m+1} = 0$ and $[A_{m+1}] = [A_{m+2}]$. Therefore the sequence $\{c_n\}_{n=1}^{\infty}$ is decreasing to zero.
- Suppose that $[A_{m+1}] \neq [A_m]$ for all $m \geq 1$. Since the sequence $\{c_n\}_{n=1}^{\infty}$ is decreasing, we assume that $c_n \downarrow \epsilon > 0$, i.e., $c_n \geq \epsilon > 0$ for all n. There exists $\delta > 0$ such that $\epsilon \leq c_m < \epsilon + \delta$ for some m, i.e.,

$$\epsilon \leq \| A_m \ominus A_{m+1} \| < \epsilon + \delta.$$

By the second condition of Definition 10, we have

$$c_{m+1} = \| A_{m+1} \ominus A_{m+2} \| = \| \mathfrak{T}^{m+1}(A) \ominus \mathfrak{T}^{m+2}(A) \| = \| \mathfrak{T}(A_m) \ominus \mathfrak{T}(A_{m+1}) \| < \epsilon,$$

which contradicts $c_{m+1} \geq \epsilon$.

This completes the proof. □

Theorem 3. *Let $(\mathcal{K}(X), \| \cdot \|)$ be a Banach hyperspace with the null set Ω. Suppose that $\| \cdot \|$ satisfies the null super-inequality, and that the function $\mathfrak{T} : (\mathcal{K}(X), \| \cdot \|) \to (\mathcal{K}(X), \| \cdot \|)$ is a weakly uniformly strict contraction on $\mathcal{K}(X)$. Then $\mathfrak{T}$ has a near fixed point satisfying $\mathfrak{T}(A) \stackrel{\Omega}{=} A$. Moreover, the near fixed point A is obtained by the limit*

$$\| \mathfrak{T}^n(A_0) \ominus A \| = \| A \ominus \mathfrak{T}^n(A_0) \| \to 0 \text{ as } n \to \infty.$$

Assume further that $\| \cdot \|$ satisfies the null equality. Then we also have the following properties.

- *The uniqueness is in the sense that there is a unique equivalence class $[A]$ such that any $\bar{A} \notin [A]$ cannot be a near fixed point.*
- *Each point $\bar{A} \in [A]$ is also a near fixed point of $\mathfrak{T}$ satisfying $\mathfrak{T}(\bar{A}) \stackrel{\Omega}{=} \bar{A}$ and $[\bar{A}] = [A]$.*
- *If $\bar{A}$ is a near fixed point of $\mathfrak{T}$, then $\bar{A} \in [A]$, i.e., $[\bar{A}] = [A]$. Equivalently, if A and $\bar{A}$ are the near fixed points of $\mathfrak{T}$, then $A \stackrel{\Omega}{=} \bar{A}$.*

Proof. According to Theorem 2 and Remark 2, we just need to claim that if $\mathfrak{T}$ is a weakly uniformly strict contraction, then $\{\mathfrak{T}^n(A_0)\}_{n=1}^{\infty} = \{A_n\}_{n=1}^{\infty}$ forms a Cauchy sequence. Suppose that $\{A_n\}_{n=1}^{\infty}$ is not a Cauchy sequence. Then there exists $2\epsilon > 0$ such that, given any N, there exist $n > m \geq N$ satisfying $\| A_m \ominus A_n \| > 2\epsilon$. Since $\mathfrak{T}$ is a weakly uniformly strict contraction on $\mathcal{K}(X)$, there exists $\delta > 0$ such that

$$\epsilon \leq \| A \ominus B \| < \epsilon + \delta \text{ implies } \| \mathfrak{T}(A) \ominus \mathfrak{T}(B) \| < \epsilon \text{ for any } A \stackrel{\Omega}{\neq} B.$$

Let $\delta' = \min\{\delta, \epsilon\}$. We are going to claim

$$\epsilon \leq \| A \ominus B \| < \epsilon + \delta' \text{ implies } \| \mathfrak{T}(A) \ominus \mathfrak{T}(B) \| < \epsilon \text{ for any } A \stackrel{\Omega}{\neq} B. \tag{6}$$

Indeed, if $\delta' = \epsilon$, i.e., $\epsilon < \delta$, then $\epsilon + \delta' = \epsilon + \epsilon < \epsilon + \delta$.

Let $c_n = \| A_n \ominus A_{n+1} \|$. Since the sequence $\{c_n\}_{n=1}^{\infty}$ is decreasing to zero by Lemma 1, we can find N such that $c_N < \delta'/3$. For $n > m \geq N$, we have

$$\| A_m \ominus A_n \| > 2\epsilon \geq \epsilon + \delta', \tag{7}$$

which implicitly says that $A_m \stackrel{\Omega}{\neq} A_n$. Since the sequence $\{c_n\}_{n=1}^{\infty}$ is decreasing by Lemma 1 again, we obtain

$$\| A_m \ominus A_{m+1} \| = c_m \leq c_N < \frac{\delta'}{3} \leq \frac{\epsilon}{3} < \epsilon. \tag{8}$$

For j with $m < j \leq n$, using Proposition 3, we have

$$\| A_m \ominus A_{j+1} \| \leq \| A_m \ominus A_j \| + \| A_j \ominus A_{j+1} \| . \tag{9}$$

We want to show that there exists j with $m < j \leq n$ such that $A_m \stackrel{\Omega}{\neq} A_j$ and

$$\epsilon + \frac{2\delta'}{3} < \| A_m \ominus A_j \| < \epsilon + \delta'. \tag{10}$$

Let $\gamma_j = \| A_m \ominus A_j \|$ for $j = m+1, \cdots, n$. Then (7) and (8) says that $\gamma_{m+1} < \epsilon$ and $\gamma_n > \epsilon + \delta'$. Let j_0 be an index such that

$$j_0 = \max\left\{ j \in [m+1, n] : \gamma_j \leq \epsilon + \frac{2\delta'}{3} \right\}.$$

Then we see that $j_0 < n$, since $\gamma_n > \epsilon + \delta'$. By the definition of j_0, we also see that $j_0 + 1 \leq n$ and $\gamma_{j_0+1} > \epsilon + \frac{2\delta'}{3}$, which also says that $A_m \overset{\Omega}{\neq} A_{j_0+1}$. Therefore expression (10) will be obtained if we can show that

$$\epsilon + \frac{2\delta'}{3} < \gamma_{j_0+1} < \epsilon + \delta'.$$

Suppose that this is not true, i.e., $\gamma_{j_0+1} \geq \epsilon + \delta'$. From (9), we have

$$\frac{\delta'}{3} > c_N \geq c_{j_0} = \| A_{j_0} \ominus A_{j_0+1} \| \geq \gamma_{j_0+1} - \gamma_{j_0} \geq \epsilon + \delta' - \epsilon - \frac{2\delta'}{3} = \frac{\delta'}{3}.$$

This contradiction says that (10) is sound. Since $A_m \overset{\Omega}{\neq} A_j$, using (6), we see that (10) implies

$$\| A_{m+1} \ominus A_{j+1} \| = \| \mathfrak{T}(A_m) \ominus \mathfrak{T}(A_j) \| < \epsilon. \tag{11}$$

Therefore we obtain

$$\begin{aligned} \| A_m \ominus A_j \| &\leq \| A_m \ominus A_{m+1} \| + \| A_{m+1} \ominus A_{j+1} \| + \| A_{j+1} \ominus A_j \| \ \text{(by Proposition 3)} \\ &< c_m + \epsilon + c_j \ \text{(by (11))} \\ &< \frac{\delta'}{3} + \epsilon + \frac{\delta'}{3} = \epsilon + \frac{2\delta'}{3}, \end{aligned}$$

which contradicts (10). This contradiction says that the sequence $\{\mathfrak{T}^n(A)\}_{n=1}^{\infty} = \{A_n\}_{n=1}^{\infty}$ is a Cauchy sequence, and the proof is complete. □

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References

1. Agarwal, R.P.; O'Regan, D. *Set Valued Mappings with Applications in Nonlinear Analysis*; Taylor and Francis: London, UK, 2002.
2. Aubin, J.-P.; Frankowska, H. *Set-Valued Analysis*; Springer: New York, NY, USA, 2009.
3. Burachik, R.S.; Iusem, A.N. *Set-Valued Mappings and Enlargements of Monotone Operators*; Springer: New York, NY, USA, 2008.
4. Chen, G.-Y.; Huang, X.; Yang, X. *Vector Optimization: Set-Valued and Variational Analysis*; Springer: New York, NY, USA, 2005.
5. Hu, S.; Papageorgiou, N.S. *Handbook of Multivalued Analysis I: Theory*; Kluwer Academic Publishers: Boston, MA, USA, 1997.
6. Hu, S.; Papageorgiou, N.S. *Handbook of Multivalued Analysis II: Applications*; Kluwer Academic Publishers: Boston, MA, USA, 2000.
7. Tarafdar, E.U.; Chowdhury, M.S.R. *Topological Methods for Set-Valued Nonlinear Analysis*; World Scientific Publishing Company: Singapore, 2008.
8. Heikkilä, S.; Reffett, K. Fixed Point Theorems and Their Applications to Theory of Nash Equilibria. *Nonlin. Anal.* **2006**, *64*, 1415–1436. [CrossRef]
9. Horvath, C.D.; Ciscar, J.V.L. Maximal Elements and Fixed Points for Binary Relations on Topological Ordered Spaces. *J. Math. Econ.* **1996**, *25*, 291–306. [CrossRef]
10. Tarafdar, E. A Fixed Point Theorem and Equilibrium Point of an Abstract Economy. *J. Math. Econ.* **1991**, *20*, 211–218. [CrossRef]
11. Aubin, J.-P. *Applied Functional Analysis*, 2nd ed.; John Wiley & Sons: New York, NY, USA, 2000.
12. Adasch, N.; Ernst, B.; Keim, D. *Topological Vector Spaces: The Theory without Convexity Conditions*; Springer: New York, NY, USA, 1978.

13. Conway, J.B. *A Course in Functional Analysis*, 2nd ed.; Springer: New York, NY, USA, 1990.
14. Khaleelulla, S.M. *Counterexamples in Topological Vector Spaces*; Springer: New York, NY, USA, 1982.
15. Schaefer, H.H. *Topological Vector Spaces*; Springer: New York, NY, USA, 1966.
16. Peressini, A.L. *Ordered Topological Vector Spaces*; Harper and Row: New York, NY, USA, 1967.
17. Wong, Y.-C.; Ng, K.-F. *Partially Ordered Topological Vector Spaces*; Oxford University Press: Londn, UK, 1973.
18. Meir, A.; Keeler, E. A Theorem on Contraction Mappings. *J. Math. Anal. Appl.* **1969**, *28*, 326–329. [CrossRef]