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t-Norm Fuzzy Incidence Graphs

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Abstract: It is the case that, in certain applications of fuzzy graphs, a t-norm, instead of a minimum, is more suitable. This requires the development of a new theory of fuzzy graphs involving an arbitrary t-norm in the basic definition of a fuzzy graph. There is very little known about this type of fuzzy graph. The purpose of this paper is to further develop this type of fuzzy graph. We concentrate on the relatively new concept of fuzzy incidence graphs.

Keywords: fuzzy incidence graphs; t-norms; connectedness; paths; cycles; trees; bridges; cutpairs

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1. Introduction

It is well known that Zadeh is the originator of fuzzy logic [1] and that Rosenfeld [2] and Yeh and Bang [3] are the founders of fuzzy graph theory.

It is the case that in certain applications of fuzzy graphs, a t-norm, instead of a minimum, is more suitable. This requires the development of a new theory of fuzzy graphs involving an arbitrary t-norm in the basic definition of a fuzzy graph. There is very little known about this type of fuzzy graph [4]. It was shown in [4] that many basic results that hold for fuzzy graphs defined using a minimum do not hold when the minimum is replaced by an arbitrary t-norm. The purpose of this paper is to further develop this type of fuzzy graph. We concentrate on the relatively new concept of fuzzy incidence graphs [5–7]. It is the goal of this paper to develop results that allow for the use of this new theory by the new concept of fuzzy incidence graphs. The only application of fuzzy incidence graphs up to now has been to problems of human trafficking and illegal immigration [8–11]. The potential for this application is immense as can be seen by the many applications of fuzzy graphs. In fact, it may be useful to describe diffusion closures as in [12].

In relation to illegal immigration, it was shown in [9,10] that the combining of government response measures and vulnerability measures was more suitably accomplished using certain t-norms and t-conorms rather than using minimum and maximum, respectively. This was also determined to be the case in the combining measures of human trafficking flows from one country to another [8,13]. We provide a brief example. Let $G = (V, E)$, where V is a set of countries and E is a set of edges. Say $V = \{u, v, w\}$ and $E = \{uw, vw\}$. Define the fuzzy subset σ of V by $\sigma(u) = 0.3, \sigma(v) = 0.8$, and $\sigma(w) = 0.2$. Define the fuzzy subset μ of E by $\mu(uw) = 0.3 \wedge 0.2 = 0.2$ and $\mu(vw) = 0.8 \wedge 0.2 = 0.2.$, where $\wedge$ denotes minimum. Let σ denote the vulnerability of a country to illegal immigration and let μ denote the risk for illegal immigrants to move from country u to country w and from country v to w. The vulnerability of u and v plays no role in the risk instead of being larger than 0.2. A t-norm, instead of a minimum, would be better used to determine the risk from the vulnerability. For example, if the t-norm product $\otimes$ were used, then $\mu(uw) = 0.3 \otimes 0.2 = 0.06$ and $\mu(vw) = 0.8 \otimes 0.2 = 0.16$. Foundation material and data can be found in [14,15].

In this paper, we show that, if a minimum is replaced by an arbitrary t-norm in the definition of a fuzzy incidence graph, then certain basic results hold while others do not. These results deal with connectedness, paths, cycles, trees, bridges, and cutpairs. We show that an incidence pair (x, xy) can be an incidence cutpair but may be the weakest incidence pair of an incidence cycle. We also show that the following statement is not true: if there is at most one incidence path with the most incidence strength between any vertex and edge of the fuzzy incidence graph G, then G is a fuzzy incidence forest.

The theory of fuzzy graphs is rich in application. In [16], for example, applications to environmental science, social science, geography, and linguistics are shown. Applications to cluster analysis, pattern classification, and database theory have also been noted [11,17]. Other application areas include traffic light control [18], traffic control [19], job allocation [20], analysis of a multispecies trawl fishery [21], military applications [22], information networks [23], database theory [11,17], chemical structure [24], group structure [25], and human cardiac function [26]. The reader may also find the modern treatment of graph theory and its theoretical developments and algorithmic applications in [13] interesting.

We let $\wedge$ denote a minimum and $\vee$ denote a maximum.

2. Incidence

We introduce the notion of the degree of incidence of a vertex and an edge in a fuzzy graph in fuzzy graph theory. We concentrate on incidence, where the edge is adjacent to the vertex. We determine results concerning bridges, cutvertices, cutpairs, fuzzy incidence paths, and a fuzzy incidence tree for fuzzy incidence graphs. In [5,6], Dinesh introduced the notion of the degree of incidence of a vertex and an edge in fuzzy graph theory. Basic results concerning fuzzy graphs can be found in [2,11].

Let V be a finite set and let E denote a subset of $P(V)$, the power set of V, such that every set in E contains exactly two elements. For $\{x, y\}$ in E, we write $xy = \{x, y\}$. Then $xy = yx$. The elements of V are called **vertices**, and the elements of E are **edges.**The pair (V, E) is called a **graph**.

Definition 1. *Let (V, E) be a graph. Then $G^* = (V, E, I)$ is called an* ***incidence graph****, where $I \subseteq V \times E$.*

We note that, if $V = \{u, v\}, E = \{uv\}$ and $I = \{(v, uv)\}$, then (V, E, I) is an incidence graph even though $(u, uv) \notin I$.

Definition 2. *Let $G^* = (V, E, I)$ be an incidence graph. If $(u, vw) \in I$, then (u, vw) is called an* ***incidence pair*** *or simply a* ***pair****. If $(u, uv), (v, uv), (v, vw), (w, vw) \in I$, then uv and vw are called* ***adjacent edges.***

Definition 3. *An* ***incidence subgraph*** *H^* of an incidence graph G^* is an incidence graph having its vertices, edges, and pairs in G^*.*

It is important to note that, if (V, E, I) is an incidence graph, then $(V \cup E, I)$ is a bipartite graph since $I \subseteq V \times E$, where we interpret (u, uv) as an undirected edge between u and uv and so between uv and u also. This leads us to the following definition.

Definition 4. *(σ, μ, Ψ) is a* ***fuzzy incidence subgraph*** *of (V, E, I) if $(\sigma \cup \mu, \Psi)$ is a fuzzy subgraph of $(V \cup E, I)$, where $(\sigma \cup \mu)(v) = \sigma(v)$ for all $v \in V$, $(\sigma \cup \mu)(uv) = \mu(uv)$ for all $uv \in E$, and $\Psi(u, uv) \leq \sigma(u) \wedge \mu(uv)$ for all $(u, uv) \in I$.*

Note that, in the previous definition, we have not required (σ, μ) to be a fuzzy subgraph of (V, E). If we do require it, then we may, at times, want to consider near fuzzy incidence subgraphs (τ, ν, Ω) of (σ, μ, Ψ), where $\tau(u) = 0$ and $\tau(v) = 0$ and $\nu(uv) > 0$ is allowed. See the following definition.

Definition 5. *Let $G = (\sigma, \mu, \Psi)$ be a fuzzy incidence graph. Then (τ, ν, Ω) is a* ***near fuzzy incidence subgraph*** *of G if $\tau \subseteq \sigma, \nu \subseteq \mu, \Omega \subseteq \Psi$, and $\Omega(u, uv) \leq \sigma(u) \wedge \mu(uv)$ for all $(u, uv) \in I$.*

Note that, in the previous definition, $\nu(uv) \leq \tau(u) \wedge \tau(v)$ for $u, v \in V$ is not required.

Definition 6. *Let $G^* = (V, E, I)$ be an incidence graph. The sequences*

$$S_1 : v_0, (v_0, v_0v_1), v_0v_1, (v_1, v_0v_1), v_1, ..., v_{n-1}, (v_{n-1}, v_{n-1}v_n), v_{n-1}v_n, (v_n, v_{n-1}v_n), v_n$$

$$S_2 : v_0, (v_0, v_0v_1), v_0v_1, (v_1, v_0v_1), v_1, ..., v_{n-1}, (v_{n-1}, v_{n-1}v_n), v_{n-1}v_n, (v_n, v_{n-1}v_n), v_n, (v_n, v_nv_{n+1}), v_nv_{n+1}$$

$$S_3 : uv_0, (v_0, uv_0), v_0, (v_0, v_0v_1), v_0v_1, (v_1, v_0v_1), v_1, ..., v_{n-1}, (v_{n-1}, v_{n-1}v_n), v_{n-1}v_n, (v_n, v_{n-1}v_n), v_n$$

$$S_4 : uv_0, (v_0, uv_0), v_0, (v_0, v_0v_1), v_0v_1, (v_1, v_0v_1), v_1, ..., v_{n-1}, (v_{n-1}, v_{n-1}v_n), v_{n-1}v_n, (v_n, v_{n-1}v_n), v_n, (v_n, v_nv_{n+1}), v_nv_{n+1}$$

are called ***incidence walks****. Then S_1 and S_4 are called* ***closed*** *if $v_0 = v_n$ and $uv_0 = v_nv_{n+1}$, respectively. If the vertices and edges are distinct, then they are called* ***incidence paths****. If S_1 and S_4 are closed incidence paths, then they are called* ***incidence cycles.*** *The shortest incidence cycles have three vertices, three edges, and six pairs.*

By the definition of a cycle, all pairs of vertices and edges are distinct. Thus, from the definition of an incidence path, if uv is on the path, so are (u, uv), (v, uv), but not an incidence pair of the form (u, vw) with $v \neq u \neq w$.

Let (V, E) be a graph and (V, E, I) an incidence graph. Then $I \subseteq V \times E$. We will assume in the following that $I \subseteq \{(u, uv)|uv \in E\}$. Let $E^{(i)} = \{(u, uv)|uv \in E\}$. (Note that, since $uv = vu, (v, uv) \in E^{(i)}$.) Incidence pairs of the form (u, vw), where $v \neq u \neq w$, are not allowed here.

Let (V, E) be a graph and (σ, μ) a fuzzy subgraph of (V, E). Let $G^* = (V, E, I)$ be an incidence graph. Let (σ, μ, Ψ) be a fuzzy incidence graph. Define $\sigma \cup \mu : V \cup E \to [0, 1]$ as follows: if $u \in V, (\sigma \cup \mu)(u) = \sigma(u)$ and if $uv \in E, (\sigma \cup \mu)(uv) = \mu(uv)$. Since $\Psi(u, uv) \leq \sigma(u) \wedge \mu(uv) = (\sigma \cup \mu)(u) \wedge (\sigma \cup \mu)(uv)$, we can consider $(\sigma \cup \mu, \Psi)$ as a fuzzy subgraph of $(V \cup E, E^{(i)})$. That is, the elements of $V \cup E$ are the vertices and the elements of $E^{(i)}$ are the edges. This interpretation will aid in the understanding of the proofs to follow.

Definition 7. *An incidence graph in which all pairs of vertices and all pairs of edges are joined by an incidence path is said to be* ***connected****.*

Definition 8. *An incidence graph having no cycles is called an* ***incidence forest****. If it is connected, then it is called an* ***incidence tree****.*

Since a tree is connected, all pairs of vertices are connected by an incidence path. By the definition of an incidence path, if uv is on the path so are (u, uv), (v, uv) but no incidence pair of the form (u, vw) with $v \neq u \neq w$ is on the path.

A **component** in an incidence graph is a maximally connected incidence subgraph. Recall that the definition of connectedness uses a path which, for incidence graphs, involves (u, uv) and (v, uv) for every uv in the path. Thus the removal of a pair (u, uv) can increase the number of components in an incidence graph. For example, consider the incidence graph $G^* = (\{u, v\}, \{uv\}, \{(u, uv), (v, uv)\})$. G^* is connected, but $H = (\{u, v\}, \{uv\}, \{(v, uv)\})$ is not. H has two components, namely $\{u\}$ and $\{v, uv, (v, uv)\}$.

Definition 9. *If the removal of an edge in an incidence graph increases the number of connected components, then the edge is called an* ***incidence bridge****.*

Definition 10. *If the removal of a vertex in an incidence graph increases the number of connected components, then the vertex is called an* ***incidence cutvertex****.*

Definition 11. *If the removal of an incidence pair in an incidence graph increases the number of connected components, then the incidence pair is called an* ***incidence cutpair.***

Consider the incidence graph $G^* = (\{u, v, w\}, \{uv, uw, vw\}, \{(u, uv), (v, uv), (u, uw), (w, uw), (v, vw), (w, vw)\})$. (u, uv) is not a cutpair since G^* remains connected since there is a path from u to v going through w.

3. Fuzzy Incidence

Definition 12. *Let $\otimes$ be a function of the closed interval $[0, 1]$ into itself. If $\otimes$ satisfies the following properties, it is called a t-norm. For all, $a, b, c \in [0, 1]$,*

(1) $\otimes(a, 1) = a$ *(boundary condition);*
(2) $b \leq c$ *implies* $\otimes(a, b) \leq \otimes(a, c)$ *(monotonicity);*
(3) $\otimes(a, b) = \otimes(b, a)$ *(commutativity);*
(4) $\otimes(a, \otimes(b, c)) = \otimes(\otimes(a, b), c)$ *(associativity).*

If $\otimes$ is a t-norm, we write $a \otimes b$ for $\otimes(a, b), a, b \in [0, 1]$. If $a \otimes a < a$ for all $a \in [0, 1] \backslash \{0, 1\}$, then $\otimes$ is called ***subidempotent****.*

In the following $\otimes$ denotes a t-norm.

Definition 13. *Let $G^* = (V, E)$ be a graph and σ be a fuzzy subset of V and μ a fuzzy subset of E. Let Ψ be a fuzzy subset of $V \times E$. If $\Psi(u, uv) \leq \sigma(u) \otimes \mu(uv)$ for all $u \in V$ and $uv \in E$, then Ψ called a* ***fuzzy incidence*** *of G^* with respect to $\otimes$.*

Definition 14. *Let $G^* = (V, E)$ be a graph and (σ, μ) be a fuzzy subgraph of G^* with respect to $\otimes$. If Ψ is a fuzzy incidence of G^*, then $G = (\sigma, \mu, \Psi)$ is called a* ***fuzzy incidence subgraph graph*** *of G with respect to $\otimes$.*

Definition 15. *Two vertices v_i and v_j joined by an incidence path in a fuzzy incidence graph are said to be* ***connected****.*

Definition 16. *The* ***incidence strength*** *of a fuzzy incidence graph $G = (\sigma, \mu, \Psi)$ is defined to be $\otimes\{\Psi(u, uv) | (u, uv) \in Supp(\Psi)\}$.*

Example 1. *Let $G^* = (V, E, I)$ be an incidence graph and $G = (\sigma, \mu, \Psi)$ be a fuzzy incidence graph associated with G^*, where $V = \{v_1, v_2, v_3, v_4\}$ and*

σ	v_1	v_2	v_3	v_4
	0.5	0.7	1.0	0.8

μ	v_1v_2	v_1v_3	v_2v_3	v_2v_4	v_3v_4	v_1v_4
	0.5	0.3	0.7	0.5	0.8	0.5

Ψ	(v_1, v_1v_2)	(v_2, v_1v_2)	(v_1, v_1v_3)	(v_3, v_1v_3)	(v_2, v_2v_3)	(v_3, v_2v_3)
	0.5	0.4	0.3	0.3	0.6	0.7
Ψ	(v_2, v_2v_4)	(v_4, v_2v_4)	(v_3, v_3v_4)	(v_4, v_3v_4)	(v_1, v_1v_4)	(v_4, v_1v_4)
	0.5	0.4	0.8	0.7	0.4	0.5

Then

$$\begin{aligned} &v_1, (v_1, v_1v_2), v_1v_2, (v_2, v_1v_2), v_2, (v_2, v_2v_3), v_2v_3, (v_3, v_2v_3), \\ &v_3, (v_3, v_3v_4), v_3v_4, (v_4, v_3v_4), v_4, (v_4, (v_2v_4), v_2v_4, (v_2, (v_2v_4), v_2 \end{aligned}$$

is an incidence walk, but not an incidence path since v_2 is repeated. The sequence

$$\begin{aligned} &v_1, (v_1, v_1v_2), v_1v_2, (v_2, v_1v_2), v_2, (v_2, v_2v_3), v_2v_3, (v_3, v_2v_3), \\ &v_3, (v_3, v_3v_4), v_3v_4, (v_4, v_3v_4), v_4 \end{aligned}$$

is an incidence path. The vertices v_1 and v_4 are connected. The incidence strength of this sequence is 0.4 when the t-norm minimum is used.

Definition 17. *The fuzzy incidence graph $G = (\sigma, \mu, \Psi)$ is an* ***incidence cycle*** *if $(Supp(\sigma),\ Supp(\mu),\ Supp(\Psi))$ is a cycle.*

Definition 18. *The fuzzy incidence graph $G = (\sigma, \mu, \Psi)$ is a* ***fuzzy incidence cycle*** *if it is an incidence cycle and no unique $(x, xy) \in Supp(\Psi)$ exists such that $\Psi(x, xy) = \wedge\{\Psi(u, uv) | (u, uv) \in Supp(\Psi)\}$.*

Example 2. *Let $G = (\sigma, \mu, \Psi)$ be the fuzzy incidence cycle with $V = \{v_1, v_2, v_3, v_4\}$ and σ, μ, Ψ defined as follows:*

σ	v_1	v_2	v_3	v_4
	0.2	0.3	0.3	0.4

μ	v_1v_2	v_2v_3	v_3v_4	v_4v_1
	0.2	0.2	0.3	0.2

Ψ	(v_1, v_1v_2)	(v_2, v_1v_2)	(v_2, v_2v_3)	(v_3, v_2v_3)
	0.2	0.2	0.2	0.2

Ψ	(v_3, v_3v_4)	(v_4, v_3v_4)	(v_4, v_4v_1)	(v_1, v_4v_1)
	0.3	0.3	0.2	0.2

Consider the incidence walk

$$v_1, (v_1, v_1v_2), v_1v_2, (v_2, v_1v_2), v_2, (v_2, v_2v_3), v_2v_3, (v_3, v_2v_3),$$
$$v_3, (v_3, v_3v_4), v_3v_4, (v_4, v_3v_4), v_4, (v_4, v_4v_1), v_4v_1, (v_1, v_4v_1), v_1.$$

Then it is a fuzzy incidence cycle since no unique (u, vw) exists such that $\Psi(u, vw) = 0.2$, the incidence strength of G. Note also that there is no unique uv such that $\mu(uv) = 0.2$. There is no unique v such that $\sigma(v) = 0.2$ instead of v_1.

Definition 19. *The fuzzy incidence graph $G = (\sigma, \mu, \Psi)$ is an* ***incidence tree*** *if $(Supp(\sigma),\ Supp(\mu),\ Supp(\Psi))$ is an incidence tree and is an* ***incidence forest*** *if $(Supp(\sigma),\ Supp(\mu), Supp(\Psi))$ is an incidence forest.*

Definition 20. *Let $G = (\sigma, \mu, \Psi)$ be a fuzzy incidence graph. Then $H = (\tau, \nu, \Omega)$ is a* ***fuzzy incidence subgraph*** *of G if $\tau \subseteq \sigma, \nu \subseteq \mu$, and $\Omega \subseteq \Psi$. A fuzzy incidence subgraph H of G is a* ***fuzzy incidence spanning subgraph*** *of G if $\tau = \sigma$ and $\nu = \mu$.*

Definition 21. *Let $G = (\sigma, \mu, \Psi)$ be a fuzzy incidence graph. Define $\Psi^{\infty}(u, vw)$ to be the* ***incidence strength*** *of the incidence path from u to vw of greatest incidence strength.*

Let $u_0 = u, u_{n-1} = v$, and $u_n = w$. Then a path from u_0 to u_n would be

$$u_0, (u_0, u_0u_1), u_0u_1, (u_1, u_0u_1), u_1, (u_1, u_1u_2), u_1u_2, u_2, ..., u_{n-1}, (u_{n-1}, u_{n-1}u_n), u_{n-1}u_n, (u_n, u_{n-1}u_n), u_n.$$

Since $\Psi(u_{i-1}, u_{i-1}u_i) \leq \sigma(u_{i-1}) \otimes \mu(u_{i-1}u_i)$, the strength of the path is $\Psi(u_0, u_0u_1) \otimes ... \otimes \Psi(u_{n-1}, u_{n-1}u_n)$. The strength of the strongest such path is what is meant in the previous definition.

Let $(u, uv) \in V \times E$ and $\Psi' = \Psi|_{V\times E\setminus\{(u,uv)\}}$. By Ψ'^{∞}, we mean the strength of the strongest path from u to uv not including (u, uv). Let $(u, vw) \in V \times E$ with $v \neq u \neq w$ and let $\Psi' = \Psi|_{V\times E\setminus\{(u,vw)\}}$. By Ψ'^{∞}, we mean the strength of the strongest path from u to vw. (Note that (u, vw) cannot be included by the definition of incidence path, where u, v, and w are distinct.)

Definition 22. *Let $G = (\sigma, \mu, \Psi)$ be a fuzzy incidence graph with respect to $\otimes$. Then $(3) \Rightarrow (1) \Leftrightarrow (2)$, where*
$(1)(x, xy)$ *is an incidence cutpair;*
$(2)\Psi'^{\infty}_{\otimes}(x, xy) < \Psi(x, xy)$;
$(3)(x, xy)$ *is not a weakest pair of any incidence cycle.*

Proof. $(2) \Rightarrow (1)$: Suppose (x, xy) is not a cutpair. Then $\Psi'^{\infty}_{\otimes}(x, xy) = \Psi^{\infty}_{\otimes}(x, xy) \geq \Psi(x, xy)$.

$(3) \Rightarrow (2)$: Suppose $\Psi'^{\infty}_{\otimes}(x, xy) \geq \Psi(x, xy)$. Then there is an incidence path from x to xy not involving (x, xy) that has incidence strength $\geq \Psi(x, xy)$. This incidence path together (x, xy) forms an incidence cycle of which (x, xy) is the weakest pair.

$(1) \Rightarrow (2)$: Suppose that (x, xy) is a cutpair with respect to $\otimes$. Then there is $u, v \in \sigma^*$ such that $\Psi'^{\infty}_{\otimes}(u, uv) < \Psi^{\infty}_{\otimes}(u, uv)$. If Q is the strongest incidence path in G from x to xy, then Q must contain (x, xy). Hence $\Psi(x, xy) \geq \Psi^{\infty}_{\otimes}(u, uv) > \Psi'^{\infty}_{\otimes}(u, uv)$. Let P be the strongest incidence path from x to xy. Then $\Psi^{\infty}_{\otimes}(x, xy) \geq \Psi(x, xy)$, and the incidence strength of $(Q - (x, xy)) \cup P$ is thus $\geq$ the incidence strength of Q. Hence, the incidence path $(Q - (x, xy)) \cup P$ must contain (x, xy) and thus P must contain (x, xy). Hence, $\Psi(x, xy) = \Psi^{\infty}_{\otimes}(x, xy)$. Thus, (x, xy) is the strongest incidence path from x to xy. Hence, $\Psi(x, xy) > \Psi'^{\infty}_{\otimes}(x, xy)$, where strict inequality holds since $\Psi'^{\infty}_{\otimes}(x, xy)$ is the strength of a strongest incidence path in $G - (x, xy)$. $\square$

Example 3. *Let $V = \{x, y, z\}$. Let $\sigma(w) = 1$ for all $w \in V$ and $\mu(xy) = \mu(xz) = \mu(yz) = 1$. Let $\Psi(z, zx) = \Psi(z, zy) = 0.4, \Psi(x, xy) = 0.2$ and $\Psi(x, xy) = \Psi(y, yx) = \Psi(y, yx) = 1$. Then $\Psi'^{\infty}_{\otimes}(x, xy) = 0.14 < 0.2 = \Psi(x, xy)$, where $\Psi'(x, xy) = 0$ and, elsewhere, $\Psi' = \Psi$. Thus, (2) holds, but (3) doesn't.*

Proposition 1. *Let (V, E, I) be an incidence graph and let $\otimes$ and $\circledast$ be t-norms. Suppose that, for all $a, b \in [0, 1], a \otimes b \leq a \circledast b$. If (σ, μ, Ψ) is a fuzzy incidence with respect to $\otimes$, then (σ, μ, Ψ) is a fuzzy incidence graph with respect to $\otimes$.*

Proposition 2. *Let (σ, μ, Ψ) be a fuzzy incidence graph with respect to t-norms $\otimes$ and $\circledast$, where $\otimes \subseteq \circledast$. Let $(x, xy) \in I$. If (x, xy) is an incidence cutpair with respect to $\circledast$, then (x, xy) is an incidence cutpair with respect to $\otimes$.*

Proposition 3. *Let (σ, μ, Ψ) be a fuzzy incidence graph with respect to $\otimes$. Then a pair (x, xy) is said to be **effective** if $\Psi(x, xy) = \mu(xy) \otimes \sigma(x)$.*

Proposition 4. *Let (σ, μ, Ψ) be a fuzzy incidence graph with respect to $\otimes$. If the pair (x, xy) is effective, then $\Psi(x, xy) = \Psi^{\infty}_{\otimes}(x, xy)$.*

Proposition 5. *Let (σ, μ, Ψ) be a fuzzy incidence graph with respect to $\otimes$. Then (σ, μ, Ψ) is called **incidence complete** with respect to $\otimes$ if for all $x, y \in V, \Psi(x, xy) = \mu(xy) \otimes \sigma(x)$.*

A fuzzy incidence graph $H = (\tau, \nu, \Omega)$ is called a **partial incidence fuzzy subgraph** of $G = (\sigma, \mu, \Psi)$ if $\tau \subseteq \sigma, \nu \subseteq \mu$, and $\Omega \subseteq \Psi$. The fuzzy incidence subgraph $H = (P, \tau, \nu, \Omega)$ is called a **fuzzy incidence subgraph** of $G = (\sigma, \mu, \Psi)$ if $P \subseteq V, \tau(x) = \sigma(x)$ for all $x \in V, \nu(xy) = \mu(xy)$ for all $xy \in E$, and $\Omega(x, xy) = \Psi(x, xy)$ for all $x, y \in V$.

We say that the partial fuzzy incidence subgraph $H = (\tau, \nu, \Omega)$ **spans** $G = (\sigma, \mu, \Psi)$ if $\sigma = \tau$ and $\nu = \mu$. In this case, we call H a **spanning incidence fuzzy subgraph** of G. For any fuzzy subset τ of V and ν of E such that $\tau \subseteq \sigma$ and $\nu \subseteq \mu$, the partial fuzzy incidence subgraph induced by τ and ν is the **maximal partial fuzzy incidence subgraph** (σ, μ, Ψ) that has vertex τ^* and edge set ν^*. This is the partial fuzzy incidence graph (τ, ν, Ω), where $\Omega(x, xy) = \tau(x) \otimes \tau(y) \otimes \nu(xy)$ for all $x, y \in V$.

A (crisp) incidence graph that has no incidence cycles is called **acyclic** or an **incidence forest**. A connected forest is called an **incidence tree**. A fuzzy incidence graph is called an **incidence forest** if the graph consisting of its nonzero pairs is a forest and an **incidence tree** if this graph is connected. We call the fuzzy incidence graph $G = (\sigma, \mu, \Psi)$ a **fuzzy incidence forest** if it has a partial fuzzy spanning incidence subgraph that is an incidence forest, where for all pairs (x, xy) not in F $(\Omega(x, xy) = 0)$, we have $\Psi(x, xy) < \Omega^{\infty}_{\otimes}(x, xy)$. In other words, if (x, xy) is in G, but not in F, there is an incidence path in F between x and xy whose incidence strength is greater than $\Psi(x, xy)$. It is clear that an incidence forest is a fuzzy incidence forest.

Definition 23. *Let $\otimes$ be a t-norm. A fuzzy incidence graph (σ, μ, Ψ) is a **fuzzy incidence tree** with respect to $\otimes$ if (σ, μ, Ψ) has a partial fuzzy incidence subgraph $F = (\tau, \nu, \Omega)$ that is an incidence tree and $\forall(x, xy)$ not in F, $\Psi(x, xy) < \Omega^{\infty}_{\otimes}(x, xy)$.*

Example 4. *Consider the fuzzy incidence graph in Example 3. We note that (σ, μ, Ψ) is an incidence cycle, but not a fuzzy incidence cycle since there is a unique (x, xy) such that $\Psi(x, xy) = \wedge\{\Psi(u, uv)|(u, v) \in I\}$. It is also the case that (σ, μ, Ψ) is not a fuzzy incidence tree since if it had a fuzzy incidence spanning tree $F = (\tau, \nu, \Omega)$, then $\Omega^{\infty}_{\otimes}(x, xy) = 0.14 > \Psi(x, xy)$, but this is not the case since $\Psi(x, xy) = 0.2$.*

Theorem 1. *A fuzzy incidence graph with respect to $\otimes$ is a fuzzy incidence tree if and only if it has a unique maximum fuzzy incidence spanning tree.*

Proof. Suppose $G = (\sigma, \mu, \Psi)$ is a fuzzy incidence fuzzy tree with respect to $\otimes$. Then, G is a fuzzy incidence tree with respect to $\wedge$. Hence, G has a unique maximum fuzzy spanning subgraph with respect to $\wedge$. Let $F = (\tau, \nu, \Omega)$ be a fuzzy incidence spanning subgraph of G with respect to $\otimes$. Then $\tau = \sigma$ and $\nu = \mu$ and $\forall(x, xy)$ not in $F, \Psi(x, xy) < \Omega^{\infty}_{\otimes}(x, xy) \leq \Omega^{\infty}(x, xy)$. That is, F is a partial fuzzy incidence spanning subgraph of G with respect $\wedge$. We see that if F is a maximum with respect to $\otimes$, i.e., $\Omega(x, xy) = \Psi(x, xy)$, then F is a maximum with respect to $\wedge$ and so is unique for $\otimes$ since it is unique for $\wedge$. □

The following known theorem motivates our next definition.

Theorem 2. *[2] Let (σ, μ) be a cycle. (σ, μ) is a fuzzy cycle with respect to $\wedge$ if and only if (σ, μ) is not a fuzzy tree with respect to $\wedge$.*

Definition 24. *Let $\otimes$ be a t-norm. A fuzzy incidence (σ, μ, Ψ) is a fuzzy incidence cycle with respect to $\otimes$ if $(\sigma^*, \mu^*, \Psi^*)$ is an incidence cycle and there is no unique $(x, xy) \in I$ such that $\Psi(x, xy) = \wedge\{\Psi(u, uv)|(u, uv) \in I\}$ and (σ, μ, Ψ) is not a fuzzy incidence tree with respect to $\otimes$.*

We have that (σ, μ, Ψ) of Example 3 is not a fuzzy incidence cycle with respect to $\otimes$.

We see that a fuzzy incidence cycle with respect to $\otimes$ is a fuzzy incidence cycle with respect to $\wedge$.

Example 5. *Let $V = \{x, y, z\}$, $\sigma(w) = 1$ for all $w \in V$, and $\mu(xy) = \mu(xz) = \mu(yz) = 1$. Let*

$$\begin{aligned}\Psi(x, xy) &= \Psi(y, yx) = 0.9,\\ \Psi(y, yz) &= \Psi(z, zy) = \Psi(z, zx) = \Psi(x, xz) = 1.\end{aligned}$$

(σ, μ, Ψ) is a fuzzy incidence cycle with respect to $\otimes$ since (σ, μ, Ψ) is not a fuzzy incidence tree with respect to $\otimes$.

Example 6. *Let $V = \{x, y, z\}$, $\sigma(w) = 1$ for all $w \in V$, and $\mu(xy) = \mu(xz) = \mu(yz) = 1$. Let*

$$\begin{aligned}\Psi(x, xy) &= \Psi(y, yx) = \Psi(y, yz) = 0.9,\\ \Psi(z, zy) &= \Psi(z, zx) = \Psi(x, xz) = 1.\end{aligned}$$

Then (σ, μ, Ψ) is a fuzzy incidence cycle with respect to $\wedge$ and with respect to $\otimes$, multiplication, but (σ, μ, Ψ) is not a fuzzy incidence tree with respect to $\otimes$ since $0.9 > 0.81$.

Theorem 3. *Suppose (σ, μ, Ψ) is a fuzzy incidence graph with respect to $\otimes$. Suppose that $\otimes$ is subidempotent. Suppose (σ, μ, Ψ) is a fuzzy incidence cycle with respect to $\wedge$. If (σ, μ, Ψ) is not a fuzzy incidence cycle with respect to $\otimes$, then there are two pairs $(x, xy), (u, uv) \in I$ such that $\Psi(x, xy) = \Psi(u, uv) = \wedge\{\Psi(z, zw) | (z, zw) \in I\}$.*

Proof. Suppose there are three or more pairs $(x, xy), (u, uv), (r, rs)$ such that $\Psi(x, xy) = \Psi)u, uv) = \Psi(r, rs) = \wedge\{\Psi(z, zw) | (z, zw) \in I\} = a$, say. Then $\Psi(x, xy) = a > a \otimes a \geq \Psi^{\infty}_{\otimes}(x, xy)$. Thus, (σ, μ, Ψ) is not a fuzzy incidence tree with respect to $\otimes$ and hence is a fuzzy incidence cycle with respect to $\otimes$, a contradiction. Since (σ, μ, Ψ) is a fuzzy incidence cycle with respect to $\wedge$, there is no unique pair (x, xy) such that $\Psi(x, xy) = \wedge\{\Psi(z, zw) | (z, zw) \in I\}$. □

Theorem 4. *Suppose that (σ, μ, Ψ) is a fuzzy incidence cycle with respect to $\wedge$. Then (σ, μ, Ψ) is a fuzzy incidence cycle with respect to $\otimes$ if and only if (σ, μ, Ψ) is not a fuzzy incidence tree with respect to $\otimes$.*

Proof. By Definition 24, (σ, μ, Ψ) is a fuzzy incidence cycle with respect to $\otimes$ if and only if there is no unique $(x, xy) \in I$ such that $\Psi(x, xy) = \wedge\{\Psi(z, zw) | (z, zw) \in I\}$ and (σ, μ, Ψ) is not a unique fuzzy incidence tree with respect to $\otimes$. Since (σ, μ, Ψ) is a fuzzy incidence cycle with respect to $\wedge$, there is no unique $(x, xy) \in I$ such that $\Psi(x, xy) = \wedge\{\Psi(z, zw) | (z, zw) \in I\}$. Hence the desired result holds. □

Let $G = (\sigma, \mu, \Psi)$ be a fuzzy incidence graph and let w be a vertex. Otherwise, let $H = (\tau, \nu, \Omega)$ be the fuzzy incidence subgraph defined by $\tau(w) = 0, \tau = \sigma$. If not, then $\nu(wz) = 0$ all $z \in \sigma^*$ and $\nu = \mu$. If not this, then $\Omega(w, wz) = 0$ for all $z \in \sigma^*$and $\Omega = \Psi$.

Theorem 5. *Suppose that $G = (\sigma, \mu, \Psi)$ is a fuzzy incidence graph with respect to $\otimes$. Suppose that (σ, μ, Ψ) is an incidence cycle. If a vertex is an incidence cutvertex, then it is a common vertex to incidence cutpairs.*

Proof. Let w be an incidence cutvertex of G. u, v such that $u \neq w \neq v$ such that w is on every strongest u–v incidence path since $\Omega^{\infty}_{\otimes}(u, uv) < \Psi^{\infty}_{\otimes}(u, uv)$. Since G is an incidence cycle, there exists only one strongest incidence path u–v incidence path containing w and all its pairs are incidence cutpairs. Thus, w is a common vertex of two incidence cutpairs. □

Example 7. *The converse of the previous result is not true. Let G be as defined in Example 3. Then z is a common vertex of the incidence bridges (x, xz) and (y, yz), but z is not an incidence cutvertex since its deletion does not reduce the strength of connectedness between any pair of vertices and edges.*

Theorem 6. *If (u, uv) is an incidence cutpair, then $\Psi^{\infty}_{\otimes}(u, uv) = \Psi(u, uv)$.*

Proof. Suppose that (u, uv) is an incidence cutpair and that $\Psi^{\infty}_{\otimes}(u, uv) > \Psi(u, uv)$. Then there is a strongest $u - uv$ incidence path P with incidence strength greater than $\Psi(u, uv)$ and all pairs of this strongest incidence path have incidence strength greater than $\Psi(u, uv)$. x, y such that $\Psi'^{\infty}_{\otimes}(u, uv) < \Psi^{\infty}_{\otimes}(u, uv)$ since (u, uv) is an incidence cutpair. Let Q be a strongest incidence path from x to xy including (u, uv). Then $Q - (u, uv)$ together with P is an incidence path from x to xy that is stronger than Q, a contradiction. □

Definition 25. *Suppose that $G = (\sigma, \mu, \Psi)$ is a fuzzy incidence graph with respect to $\otimes$. G is called **incidence complete** if $\forall(u, uv)$ in $G, \Psi(u, uv) = \sigma(u) \otimes \mu(uv)$.*

Proposition 6. *Suppose that $G = (\sigma, \mu, \Psi)$ is a complete fuzzy incidence graph with respect to $\otimes$. Then*

(1) $\Psi^{\infty}_{\otimes}(u, uv) = \Psi(u, uv)$ *for all (u, uv) in G;*

(2) *G has no incidence cutvertices and no incidence bridges.*

Proof. (1) Let $u \in V, uv \in E$. Since G is incidence complete, $\Psi(u, uv) = \sigma(u) \otimes \mu(uv)$. If P:$u, (u, uv_1), uv_1, (v_1, v_1v_2)..., (u, uv), uv$ is an incidence path from u to uv, then the incidence strength of P is of the form, $\sigma(u) \otimes \mu(uv_1) \otimes \sigma(v_1) \otimes \mu(v_1v_2) \otimes ... \otimes \sigma(v_n) \otimes \mu(uv_n) \leq \sigma(u) \otimes \mu(uv_n) = \Psi(u, uv)$, where $v_n = v$.

(2) Suppose w is an incidence cutvertex (with respect to $\otimes$). Then $\Psi'^{\infty}_{\otimes}(u, uv) < \Psi^{\infty}_{\otimes}(u, uv)$ for some u, v such that $u \neq w \neq v$, where $\Psi'(w, wz) = 0$ for all z and $\Psi' = \Psi$ otherwise. However, by (1), $\Psi(u, uv) = \Psi^{\infty}_{\otimes}(u, uv)$, which is impossible.

Suppose wz is an incidence bridge. Then $\Psi'^{\infty}_{\otimes}(u, uv) < \Psi^{\infty}_{\otimes}(u, uv)$ for some u, v such that $\Psi'(w, wz) = 0 = \Psi'(z, wz)$ and $\Psi' = \Psi$ elsewhere. However, by (1), $\Psi(u, uv) = \Psi^{\infty}_{\otimes}(u, uv)$, which is impossible. □

Proposition 7. *Let $G = (\sigma, \mu, \Psi)$ be a fuzzy incidence tree with respect to $\otimes$. Then G is not incidence complete.*

Proof. Suppose G is complete. Since G is a fuzzy incidence tree with respect to $\otimes$, (σ, μ, Ψ) has a partial fuzzy incidence spanning subgraph $F = (\tau, \nu, \Omega)$, which is an incidence tree and $\forall (x, xy)$ not in $F, \Psi(x, xy) < \Omega^{\infty}_{\otimes}(x, xy)$, but $\Omega^{\infty}_{\otimes}(x, xy) \leq \Psi^{\infty}_{\otimes}(x, xy)$ and so $\Psi(x, xy) < \Psi^{\infty}_{\otimes}(x, xy)$, a contradiction of (1) of the previous proposition. □

Example 8. *Consider the fuzzy incidence graph of Example 3. Then all pairs are incidence cutpairs, and z is a common vertex of two incidence cutpairs but is not an incidence cutvertex since, if z is deleted, both (z, zx) and (z, zy) are deleted and so there are no two elements from $V \cup E$ whose incidence connectivity has been reduced by the deletion of z.*

Theorem 7. *Let $G = (\sigma, \mu, \Psi)$ be a fuzzy incidence tree with respect $\otimes$. Then the internal vertices of the partial fuzzy incidence spanning subgraph $F = (\tau, \nu, \Omega)$ are the cut vertices of G.*

Proof. Let w be a vertex in G, which is not an end vertex of F. Then there are vertices x, y such that (w, xw) and (w, yw) are in F. Since F is an incidence tree, there is a unique incidence path in F from xw to yw. This incidence path must be $xw, (w, xw), w, (w, yw), yw$. If there is no incidence path from xw to yw, then the removal of w disconnects G, and w is thus an incidence cutvertex in G. Suppose there is an incidence path P in $G - w$ connecting xw and yw, since $xw, (w, xw), w, (w, yw), yw$ is the only incidence in F connecting xw, and yw, P must contain a pair (u, uv) not in F. Thus, $\Psi(u, uv) < \Omega^{\infty}_{\otimes}(u, uv) \leq \Psi^{\infty}_{\otimes}(u.uv)$. Hence, w is an incidence cutvertex. □

Theorem 8. *Let $G = (\sigma, \mu, \Psi)$ be a fuzzy incidence graph with respect to $\otimes$. Then G is a fuzzy incidence forest with respect to $\otimes$ if and only if in any incidence cycle of G, there is a pair (x, xy) such that $\Psi(x, xy) < \Psi'^{\infty}_{\otimes}(x, xy)$, where $G' = (\sigma, \mu, \Psi')$ is the partial fuzzy incidence subgraph obtained by deletion of the pair (x, xy) from G.*

Proof. Suppose (x, xy) is a pair in an incidence cycle C that satisfies the condition of the theorem and which is such that $\Psi(x, xy)$ is the smallest of all such pairs. If no such incidence cycle exists, then G is an incidence forest and so is a fuzzy incidence forest. The partial fuzzy incidence subgraph $G \backslash (x, xy)$ satisfies the condition of the theorem. If there are no incidence cycles in $G \backslash (x, xy)$, we repeat this process until we obtain a partial fuzzy incidence subgraph F of G without incidence cycles. Clearly, F is an incidence forest. Let (x, xy) be an incidence cutpair of G not in F. Then (x, xy) is a pair that was previously deleted and there is an incidence path from x to xy that is stronger than $\Psi(x, xy)$ and does not involve (x, xy) or any pairs deleted before the deletion of (x, xy). If this incidence path involves pairs that were deleted after (x, xy), it can be diverted around them using a stronger incidence path and hence stronger pairs. If this incidence path uses a pair that was deleted after (x, xy), the weakest such pair can be diverted around using a stronger incidence path and hence one with stronger pairs. This process ends in a finite number of steps resulting in F. Since $\Psi(x, xy) < \Psi'^{\infty}_{\otimes}(x, xy)$, G is a fuzzy incidence forest with respect to $\otimes$. □

Conversely, suppose G is a fuzzy incidence forest. Let C be any incidence cycle in G. Then there is a pair (x, xy) of C not in F, where F is a partial fuzzy incidence spanning subgraph of G. Thus, $\Psi(x, xy) < \Omega^{\infty}_{\otimes}(x, xy) \leq \Psi'^{\infty}_{\otimes}(x, xy)$, where $F = (\tau, \nu, \Omega)$ and $\Psi'(x, xy) = 0$; otherwise, $\Psi' = \Psi$. Thus, G is a fuzzy incidence forest.

Example 9. *Proposition 2.6 in [11] does not hold. Let $G = (\sigma, \mu, \Psi)$ be a fuzzy incidence forest with respect to $\otimes$ of Example 3. Then there is at most one strongest incidence between any two vertices, namely the pair itself. However, G is not a fuzzy incidence forest since $\Psi'^{\infty}_{\otimes}(x, xy) = 0.16 < 0.2 = \Psi(x, xy)$, where $\Psi'(x, xy) = 0$ and $\Psi' = \Psi$.*

Theorem 9. *Let $G = (\sigma, \mu, \Psi)$ be a fuzzy incidence forest with respect to $\otimes$. Then the pairs of $F = (\tau, \nu, \Omega)$ are just the incidence cutpairs of G.*

Proof. A pair (x, xy) not in F cannot be an incidence cutpair in G since $\Psi(x, xy) < \Omega^{\infty}_{\otimes}(x, xy)$. Suppose that a pair (x, xy) is a pair in F. Suppose (x, xy) is not an incidence cutpair of G. Then there is an incidence path P in G from x to xy not involving (x, xy) and of incidence strength greater than or equal to $\Psi(x, xy)$. Since $P \cup (x, xy)$ is an incidence cycle, P must involve pairs not in F, since F is an incidence forest and has no incidence cycles. Let (u, uv) be a pair in P not in F. Now (u, uv) can be replaced by an incidence path P' in F from u to uv of incidence strength greater than $\Psi(u, uv)$. Now P' cannot contain (x, xy) since all its pairs are strictly greater than $\Psi(u, uv) \geq \Psi(x, xy)$. Thus, by replacing each such pair (u, uv) by a P', we obtain an incidence cycle in F, contradicting the fact that F is an incidence forest. □

Theorem 10. *Let $G = (\sigma, \mu, \Psi)$ be a fuzzy incidence graph. Then G is a fuzzy incidence forest if and only if in any fuzzy incidence cycle of G, there is (x, xy) such that $\Psi(x, xy) < \Psi'^{\infty}(x, xy)$, where $G' = (\sigma, \mu, \Psi')$ is the fuzzy incidence subgraph obtained by deletion of (x, xy) from G and $\Psi' = \Psi$ restricted to $(V \times E) \backslash \{(x, xy)\}$ (or $\Psi'(x, xy) = 0$).*

Proof. If there are no incidence cycles, the result is trivially true. Suppose (x, xy) is an incidence pair in G that belongs to a fuzzy incidence cycle such that $\Psi(x, xy) < \Psi'^{\infty}(x, xy)$. Let it be the pair with the least value among all such pairs $(u, uv) \in \mathrm{Supp}(\Psi)$. Delete (x, xy). If there are other incidence cycles, remove incidence pairs in a similar way. At each step, the incidence pair deleted will not have lesser incidence strength than those deleted earlier. After deletion, the remaining fuzzy incidence subgraph is a fuzzy incidence forest F. Therefore, there is an incidence path P from x to xy with more incidence strength than $\Psi(x, xy)$ and not containing (x, xy). If incidence pairs deleted earlier are in P, then we can bypass them using an incidence path with more incidence strength. □

Conversely, if G is a fuzzy incidence forest and C any incidence cycle, then by definition, there is an (x, xy) of C not in F such that $\Psi(x, xy) < \Omega^{\infty}(x, xy) \leq \Psi'^{\infty}(x, xy)$, where F is as in the definition of a fuzzy incidence forest.

Example 10. *Example 3 shows that following statement is not true. If there is at most one path with the most incidence strength between any vertex and edge of the fuzzy incidence graph $G = (\sigma, \mu, \Psi)$, then G is a fuzzy incidence forest.*

Theorem 11. *Let $G = (\sigma, \mu, \Psi)$ be an incidence cycle. Then G is a fuzzy incidence cycle if and only if G is not a fuzzy incidence tree.*

Proof. Suppose $G = (\sigma, \mu, \Psi)$ is a fuzzy incidence cycle. Then there is at least two incidence pairs $(x_1, x_1y_2), (x_2, x_2y_2)$ with $\Psi(x_1, x_1y_1) = \Psi(x_2, x_2y_2) = \wedge\{\Psi(z, zw) | z \in V, zw \in \text{Supp}(\mu), (z, zw) \in \text{Supp}(\Psi)\}$. Let (σ, μ, Ω) be a spanning fuzzy incidence tree in G. Then there is a $u \in V, uv \in E$ such that $\text{Supp}(\Psi) \backslash \text{Supp}(\Omega) = \{(u, uv)\}$. Hence, there is no incidence path in (σ, μ, Ω) between u and uv of greater incidence strength than $\Psi(u, uv)$. Thus, G is not a fuzzy incidence tree. □

Conversely, suppose that G is not a fuzzy incidence tree. Since G is an incidence cycle, it follows that $\forall(u, uv)$ in $\text{Supp}(\Psi)$, there is a fuzzy incidence spanning subgraph (σ, μ, Ω), which is an incidence tree, $\Omega(u, uv) = 0, \Omega^\infty(u, uv) \leq \Psi(u, uv)$, and $\Omega(x, xy) = \Psi(x, xy) \forall (x, xy) \in \text{Supp}(\Psi) \backslash \{(u, vw)\}$. Therefore, Ψ does not uniquely attain $\wedge\{\Psi(x, xy) | (x, xy) \in \text{Supp}(\Psi)\}$. Hence, G is a fuzzy incidence cycle.

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