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Hypergraphs in m-Polar Fuzzy Environment

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Abstract: Fuzzy graph theory is a conceptual framework to study and analyze the units that are intensely or frequently connected in a network. It is used to study the mathematical structures of pairwise relations among objects. An m-polar fuzzy (mF, for short) set is a useful notion in practice, which is used by researchers or modelings on real world problems that sometimes involve multi-agents, multi-attributes, multi-objects, multi-indexes and multi-polar information. In this paper, we apply the concept of mF sets to hypergraphs, and present the notions of regular mF hypergraphs and totally regular mF hypergraphs. We describe the certain properties of regular mF hypergraphs and totally regular mF hypergraphs. We discuss the novel applications of mF hypergraphs in decision-making problems. We also develop efficient algorithms to solve decision-making problems.

Keywords: regular mF hypergraph; totally regular mF hypergraph; decision-making; algorithm; time complexity

1. Introduction

Graph theory has interesting applications in different fields of real life problems to deal with the pairwise relations among the objects. However, this information fails when more than two objects satisfy a certain common property or not. In several real world applications, relationships are more problematic among the objects. Therefore, we take into account the use of hypergraphs to represent the complex relationships among the objects. In case of a set of multiarity relations, hypergraphs are the generalization of graphs, in which a hypergraph may have more than two vertices. Hypergraphs have many applications in different fields including biological science, computer science, declustering problems and discrete mathematics.

In 1994, Zhang [1] proposed the concept of bipolar fuzzy set as a generalization of fuzzy set [2]. In many problems, bipolar information are used, for instance, common efforts and competition, common characteristics and conflict characteristics are the two-sided knowledge. Chen et al. [3] introduced the concept of m-polar fuzzy (mF, for short) set as a generalization of a bipolar fuzzy set and it was shown that 2-polar and bipolar fuzzy set are cryptomorphic mathematical notions. The framework of this theory is that "multipolar information" (unlike the bipolar information which gives two-valued logic) arise because information for a natural world are frequently from n factors ($n \geq 2$). For example, 'Pakistan is a good country'. The truth value of this statement may not be a real number in $[0,1]$. Being a good country may have several properties: good in agriculture, good in political awareness, good in regaining macroeconomic stability etc. Each component may be a real number in $[0,1]$. If n is the number of such components under consideration, then the truth value of the fuzzy statement is a n-tuple of real numbers in $[0,1]$, that is, an element of $[0,1]^n$. The perception of fuzzy graphs based on Zadeh's fuzzy relations [4] was introduced by Kauffmann [5]. Rosenfeld [6] described the fuzzy graphs structure. Later, some remarks were given by Bhattacharya [7] on fuzzy graphs. Several concepts on fuzzy graphs were introduced by Mordeson and Nair [8]. In 2011, Akram introduced the notion of bipolar fuzzy graphs in [9]. Li et al. [10] considered different algebraic operations on mF graphs.

In 1977, Kauffmann [5] proposed the fuzzy hypergraphs. Chen [11] studied the interval-valued fuzzy hypergraph. Generalization and redefinition of the fuzzy hypergraph were explained by Lee-Kwang and Keon-Myung [12]. Parvathi et al. [13] introduced the concept of intuitionistic fuzzy hypergraphs. Samanta and Pal [14] dealt with bipolar fuzzy hypergraphs. Later on, Akram et al. [15] considered certain properties of the bipolar fuzzy hypergraph. Bipolar neutrosophic hypergraphs with applications were presented by Akram and Luqman [16]. Sometimes information is multipolar, that is, a communication channel may have various signal strengths from the others due to various reasons including atmosphere, device distribution, mutual interference of satellites etc. The accidental mixing of various chemical substances could cause toxic gases, fire or explosion of different degrees. All these are components of multipolar knowledge which are fuzzy in nature. This idea motivated researchers to study mF hypergraphs [17]. Akram and Sarwar [18] considered transversals of mF hypergraphs with applications. In this research paper, we introduce the idea of regular and totally regular mF hypergraphs and investigate some of their properties. We discuss the new applications of mF hypergraphs in decision-making problems. We develop efficient algorithms to solve decision-making problems and compute the time complexity of algorithms. For other notations, terminologies and applications not mentioned in the paper, the readers are referred to [19–31].

In this paper, the following notations are used:

Table 1. Notations

Symbol	Definition
$H^* = (A^*, B^*)$	Crisp hypergraph
$H = (A, B)$	mF hypergraph
$H^D = (A^*, B^*)$	Dual mF hypergraph
$N(x)$	Open neighhourhood degree of a vertex in H
$N[x]$	Closed neighhourhood degree of a vertex in H
$\gamma(x_1, x_2)$	Adjacent level of two vertices
$\sigma(T_1, T_2)$	Adjacent level of two hyperedges

2. Notions of mF Hypergraph

Definition 1. *An mF set on a non-empty crisp set X is a function $A : X \to [0,1]^m$. The degree of each element $x \in X$ is denoted by $A(x) = (P_1oA(x), P_2oA(x), ..., P_moA(x))$, where $P_ioA : [0,1]^m \to [0,1]$ is the i-th projection mapping [3].*

Note that $[0,1]^m$ (m-th-power of $[0,1]$) is considered as a poset with the point-wise order $\leq$, where m is an arbitrary ordinal number (we make an appointment that $m = \{n|n < m\}$ when $m > 0$), $\leq$ is defined by $x < y \Leftrightarrow p_i(x) \leq p_i(y)$ for each $i \in m$ ($x, y \in [0,1]^m$), and $P_i : [0,1]^m \to [0,1]$ is the i-th projection mapping ($i \in m$). $\mathbf{1} = (1,1,...,1)$ is the greatest value and $\mathbf{0} = (0,0,...,0)$ is the smallest value in $[0,1]^m$.

Definition 2. *Let A be an mF subset of a non-empty fuzzy subset of a non-empty set X. An mF relation on A is an mF subset B of $X \times X$ defined by the mapping $B : X \times X \to [0,1]^m$ such that for all $x, y \in X$*

$$P_ioB(xy) \leq \inf\{P_ioA(x), P_ioA(y)\}$$

$1 \leq i \leq m$, where $P_ioA(x)$ denotes the i-th degree of membership of a vertex x and $P_ioB(xy)$ denotes the i-th degree of membership of the edge xy.

Definition 3. *An mF graph is a pair $G = (A, B)$, where $A : X \to [0,1]^m$ is an mF set in X and $B : X \times X \to [0,1]^m$ is an mF relation on X such that*

$$P_ioB(xy) \leq \inf\{P_ioA(x), P_ioA(y)\}$$

$1 \leq i \leq m$, *for all* $x, y \in X$ *and* $P_i oB(xy) = 0$ *for all* $xy \in X \times X - E$ *for all* $i = 1, 2, ..., m$. *A is called the mF vertex set of G and B is called the mF edge set of G, respectively* [3].

Definition 4. *An mF hypergraph on a non-empty set X is a pair* $H = (A, B)$ [17], *where* $A = \{\zeta_1, \zeta_2, \zeta_3, ..., \zeta_r\}$ *is a family of mF subsets on X and B is an mF relation on the mF subsets* ζ_j *such that*

1. $B(E_j) = B(\{x_1, x_2, ..., x_r\}) \leq \inf\{\zeta_j(x_1), \zeta_j(x_2), ..., \zeta_j(x_s)\}$, *for all* $x_1, x_2, ..., x_s \in X$.
2. $\bigcup_k supp(\zeta_k) = X$, *for all* $\zeta_k \in A$.

Example 1. *Let* $A = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\}$ *be a family of 4-polar fuzzy subsets on* $X = \{a, b, c, d, e, f, g\}$ *given in Table 2. Let B be a 4-polar fuzzy relation on* ζ_j'*s*, $1 \leq j \leq 5$, *given as,* $B(\{a, c, e\}) = (0.2, 0.4, 0.1, 0.3)$, $B(\{b, d, f\}) = (0.2, 0.1, 0.1, 0.1)$, $B(\{a, b\}) = (0.3, 0.1, 0.1, 0.6)$, $B(\{e, f\}) = (0.2, 0.4, 0.3, 0.2)$, $B(\{b, e, g\}) = (0.2, 0.1, 0.2, 0.4)$. *Thus, the 4-polar fuzzy hypergraph is shown in Figure 1.*

Table 2. 4-polar fuzzy subsets

$x \in X$	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5
a	(0.3,0.4,0.5,0.6)	(0,0,0,0)	(0.3,0.4,0.5,0.6)	(0,0,0,0)	(0,0,0,0)
b	(0,0,0,0)	(0.4,0.1,0.1,0.6)	(0.4,0.1,0.1,0.6)	(0,0,0,0)	(0.4,0.1,0.1,0.6)
c	(0.3,0.5,0.1,0.3)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)
d	(0,0,0,0)	(0.4,0.2,0.5,0.1)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)
e	(0.2,0.4,0.6,0.8)	(0,0,0,0)	(0,0,0,0)	(0.2,0.4,0.6,0.8)	(0.2,0.4,0.6,0.8)
f	(0,0,0,0)	(0.2,0.5,0.3,0.2)	(0,0,0,0)	(0.2,0.5,0.3,0.2)	(0,0,0,0)
g	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(0.3,0.5,0.1,0.4)

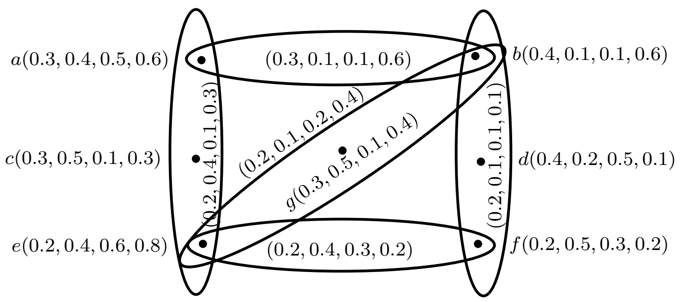

Figure 1. 4-Polar fuzzy hypergraph.

Example 2. *Consider a 5-polar fuzzy hypergraph with vertex set {a, b, c, d, e, f, g} whose degrees of membership are given in Table 3 and three hyperedges {a,b,c}, {b,d,e}, {b,f,g} such that* $B(\{a, b, c\}) = (0.2, 0.1, 0.3, 0.1, 0.2)$, $B(\{b, d, e\}) = (0.1, 0.2, 0.3, 0.4, 0.2)$, $B(\{b, f, g\}) = (0.2, 0.2, 0.3, 0.3, 0.2)$. *Hence, the 5-polar fuzzy hypergraph is shown in Figure 2.*

Table 3. 5-polar fuzzy subsets

$x \in X$	ζ_1	ζ_2	ζ_3
a	(0.2,0.1,0.3,0.1,0.3)	(0,0,0,0,0)	(0,0,0,0,0)
b	(0.2,0.3,0.5,0.6,0.2)	(0.2,0.3,0.5,0.6,0.2)	(0.2,0.3,0.5,0.6,0.2)
c	(0.3,0.2,0.4,0.5,0.2)	(0,0,0,0,0)	(0,0,0,0,0)
d	(0,0,0,0,0)	(0.6,0.2,0.2,0.3,0.3)	(0,0,0,0,0)
e	(0,0,0,0,0)	(0.4,0.5,0.6,0.7,0.3)	(0,0,0,0,0)
f	(0,0,0,0,0)	(0,0,0,0,0)	(0.1,0.2,0.3,0.4,0.4)
g	(0,0,0,0,0)	(0,0,0,0,0)	(0.2,0.4,0.6,0.8,0.4)

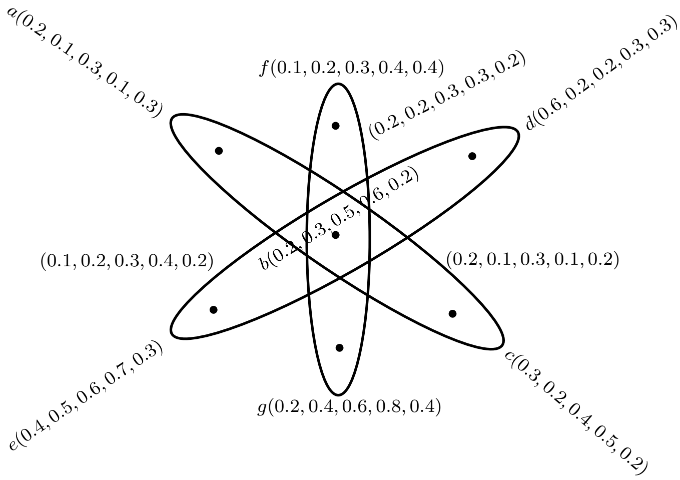

Figure 2. 5-Polar fuzzy hypergraph.

Definition 5. *Let $H = (A, B)$ be an mF hypergraph on a non-empty set X [17]. The dual mF hypergraph of H, denoted by $H^D = (A^*, B^*)$, is defined as*

1. *$A^* = B$ is the mF set of vertices of H^D.*
2. *If $|X| = n$ then, B^* is an mF set on the family of hyperedges $\{X_1, X_2, ..., X_n\}$ such that, X_i={$E_j \mid x_j \in E_j$, E_j is a hyperedge of H}, i.e., X_i is the mF set of those hyperedges which share the common vertex x_i and $B^*(X_i) = \inf\{E_j \mid x_j \in E_j\}$.*

Example 3. *Consider the example of a 3-polar fuzzy hypergraph $H = (A, B)$ given in Figure 3, where $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $E = \{E_1, E_2, E_3, E_4\}$. The dual 3-polar fuzzy hypergraph is shown in Figure 4 with dashed lines with vertex set $E = \{E_1, E_2, E_3, E_4\}$ and set of hyperedges $\{X_1, X_2, X_3, X_4, X_5, X_6\}$ such that $X_1 = X_3$.*

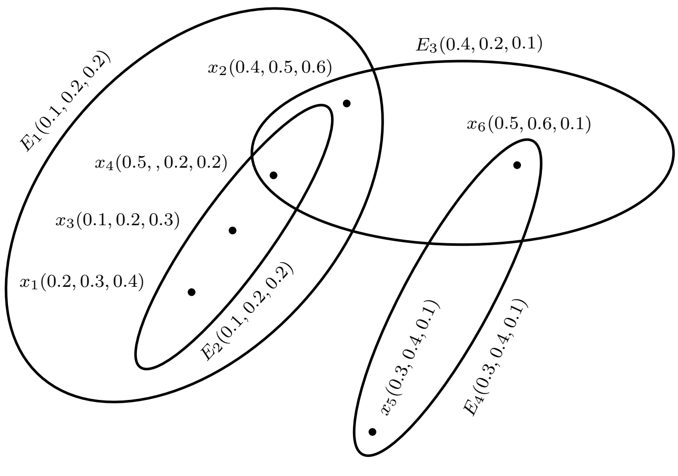

Figure 3. 3-Polar fuzzy hypergraph.

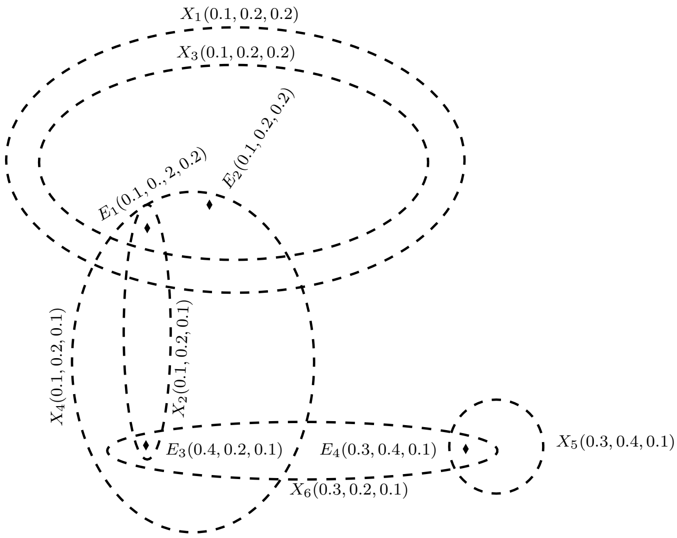

Figure 4. Dual 3-polar fuzzy hypergraph.

Definition 6. *The open neighourhood of a vertex x in the mF hypergraph is the set of adjacent vertices of x excluding that vertex and it is denoted by $N(x)$.*

Example 4. *Consider the 3-polar fuzzy hypergraph $H = (A, B)$, where $A = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ is a family of 3-polar fuzzy subsets on $X = \{a, b, c, d, e\}$ and B is a 3-polar fuzzy relation on the 3-polar fuzzy subsets ζ_i, where $\zeta_1 = \{(a, 0.3, 0.4, 0.5), (b, 0.2, 0.4, 0.6)\}$, $\zeta_2 = \{(c, 0.2, 0.1, 0.4), (d, 0.5, 0.1, 0.1), (e, 0.2, 0.3, 0.1)\}$, $\zeta_3 = \{(b, 0.1, 0.2, 0.4), (c, 0.4, 0.5, 0.6)\}$, $\zeta_4 = \{(a, 0.1, 0.3, 0.2), (d, 0.3, 0.4, 0.4)\}$. In this example, open neighourhood of the vertex a is b and d, as shown in Figure 5.*

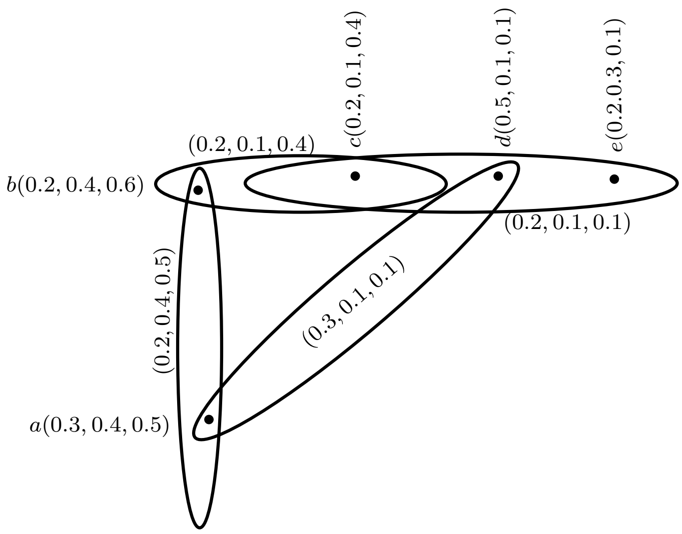

Figure 5. 3-Polar fuzzy hypergraph.

Definition 7. *The closed neighourhood of a vertex x in the mF hypergraph is the set of adjacent vertices of x including that vertex and it is denoted by $N[x]$.*

Example 5. *Consider the 3-polar fuzzy hypergraph $H = (A, B)$, where $A = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ is a family of 3-polar fuzzy subsets on $X = \{a, b, c, d, e\}$ and B is a 3-polar fuzzy relation on the 3-polar fuzzy subsets ζ_j, where $\zeta_1 = \{(a, 0.3, 0.4, 0.5), (b, 0.2, 0.4, 0.6)\}$, $\zeta_2 = \{(c, 0.2, 0.1, 0.4), (d, 0.5, 0.1, 0.1), (e, 0.2, 0.3, 0.1)\}$,*

$\zeta_3 = \{(b, 0.1, 0.2, 0.4), (c, 0.4, 0.5, 0.6)\}$, $\zeta_4 = \{(a, 0.1, 0.3, 0.2), (d, 0.3, 0.4, 0.4)\}$. *In this example, closed neighourhood of the vertex a is a, b and d, as shown in Figure 5.*

Definition 8. *Let $H = (A, B)$ be an mF hypergraph on crisp hypergraph $H^* = (A^*, B^*)$. If all vertices in A have the same open neighbourhood degree n, then H is called n-regular mF hypergraph.*

Definition 9. *The open neighbourhood degree of a vertex x in H is denoted by $deg(x)$ and defined by $deg(x) = (deg^{(1)}(x), deg^{(2)}(x), deg^{(3)}(x), \dots, deg^{(m)}(x))$, where*

$$deg^{(1)}(x) = \Sigma_{x \in N(x)} P_1 \circ \zeta_j(x),$$

$$deg^{(2)}(x) = \Sigma_{x \in N(x)} P_2 \circ \zeta_j(x),$$

$$deg^{(3)}(x) = \Sigma_{x \in N(x)} P_3 \circ \zeta_j(x),$$

$$\vdots$$

$$deg^{(m)}(x) = \Sigma_{x \in N(x)} P_m \circ \zeta_j(x).$$

Example 6. *Consider the 3-polar fuzzy hypergraph $H = (A, B)$, where $A = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ is a family of 3-polar fuzzy subsets on $X = \{a, b, c, d, e\}$ and B is a 3-polar fuzzy relation on the 3-polar fuzzy subsets ζ_j, where $\zeta_1 = \{(a, 0.3, 0.4, 0.5), (b, 0.2, 0.4, 0.6)\}$, $\zeta_2 = \{(c, 0.2, 0.1, 0.4), (d, 0.5, 0.1, 0.1), (e, 0.2, 0.3, 0.1)\}$, $\zeta_3 = \{(b, 0.1, 0.2, 0.4), (c, 0.4, 0.5, 0.6)\}$, $\zeta_4 = \{(a, 0.1, 0.3, 0.2), (d, 0.3, 0.4, 0.4)\}$. The open neighbourhood degree of a vertex a is $deg(a) = (0.5, 0.8, 1)$.*

Definition 10. *Let $H = (A, B)$ be an mF hypergraph on crisp hypergraph $H^* = (A^*, B^*)$. If all vertices in A have the same closed neighbourhood degree m, then H is called m-totally regular mF hypergraph.*

Definition 11. *The closed neighbourhood degree of a vertex x in H is denoted by $deg[x]$ and defined by $deg[x] = (deg^{(1)}[x], deg^{(2)}[x], deg^{(3)}[x], \dots, deg^{(m)}[x])$, where*

$$deg^{(1)}[x] = deg^{(1)}(x) + \wedge_j P_1 \circ \zeta_j(x),$$

$$deg^{(2)}[x] = deg^{(2)}(x) + \wedge_j P_2 \circ \zeta_j(x),$$

$$deg^{(3)}[x] = deg^{(3)}(x) + \wedge_j P_3 \circ \zeta_j(x),$$

$$\vdots$$

$$deg^{(m)}[x] = d_G^{(m)}(x) + \wedge_j P_m \circ \zeta_j(x).$$

Example 7. *Consider the 3-polar fuzzy hypergraph $H = (A, B)$, where $A = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ is a family of 3-polar fuzzy subsets on $X = \{a, b, c, d, e\}$ and B is a 3-polar fuzzy relation on the 3-polar fuzzy subsets ζ_j, where $\zeta_1 = \{(a, 0.3, 0.4, 0.5), (b, 0.2, 0.4, 0.6)\}$, $\zeta_2 = \{(c, 0.2, 0.1, 0.4), (d, 0.5, 0.1, 0.1), (e, 0.2, 0.3, 0.1)\}$, $\zeta_3 = \{(b, 0.1, 0.2, 0.4), (c, 0.4, 0.5, 0.6)\}$, $\zeta_4 = \{(a, 0.1, 0.3, 0.2), (d, 0.3, 0.4, 0.4)\}$. The closed neighbourhood degree of a vertex a is $deg[a] = (0.6, 1.1, 1.2)$.*

Example 8. *Consider the 3-polar fuzzy hypergraph $H = (A, B)$, where $A = \{\zeta_1, \zeta_2, \zeta_3\}$ is a family of 3-polar fuzzy subsets on $X = \{a, b, c, d, e\}$ and B is a 3-polar fuzzy relation on the 3-polar fuzzy subsets ζ_j, where*

$$\zeta_1\{(a, 0.5, 0.4, 0.1), (b, 0.3, 0.4, 0.1), (c, 0.4, 0.4, 0.3)\},$$

$$\zeta_2 = \{(a, 0.3, 0.1, 0.1), (d, 0.2, 0.3, 0.2), (e, 0.4, 0.6, 0.1)\},$$

$$\zeta_3 = \{(b, 0.3, 0.4, 0.3), (d, 0.4, 0.3, 0.4), (e, 0.4, 0.3, 0.1)\}.$$

By routine calculations, we can show that the above 3-polar fuzzy hypergraph is neither regular nor totally regular.

Example 9. *Consider the 4-polar fuzzy hypergraph $H = (A, B)$; define $X = \{a, b, c, d, e, f, g, h, i\}$ and $A = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6\}$, where*

$$\zeta_1 = \{(a, 0.4, 0.4, 0.4, 0.4), (b, 0.4, 0.4, 0.4, 0.4), (c, 0.4, 0.4, 0.4, 0.4)\},$$

$$\zeta_2 = \{(d, 0.4, 0.4, 0.4, 0.4), (e, 0.4, 0.4, 0.4, 0.4), (f, 0.4, 0.4, 0.4, 0.4)\},$$

$$\zeta_3 = \{(g, 0.4, 0.4, 0.4, 0.4), (h, 0.4, 0.4, 0.4, 0.4), (i, 0.4, 0.4, 0.4, 0.4)\},$$

$$\zeta_4 = \{(a, 0.4, 0.4, 0.4, 0.4), (d, 0.4, 0.4, 0.4, 0.4), (g, 0.4, 0.4, 0.4, 0.4)\},$$

$$\zeta_5 = \{(b, 0.4, 0.4, 0.4, 0.4), (e, 0.4, 0.4, 0.4, 0.4), (h, 0.4, 0.4, 0.4, 0.4)\},$$

$$\zeta_6 = \{(c, 0.4, 0.4, 0.4, 0.4), (f, 0.4, 0.4, 0.4, 0.4), (i, 0.4, 0.4, 0.4, 0.4)\}.$$

By routine calculations, we see that the 4-polar fuzzy hypergraph as shown in Figure 6 is both regular and totally regular.

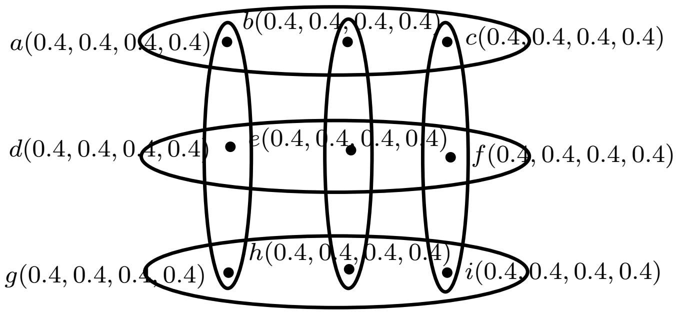

Figure 6. 4-Polar regular and totally regular fuzzy hypergraph.

Remark 1. 1. *For an mF hypergraph $H = (A, B)$ to be both regular and totally regular, the number of vertices in each hyperedge B_j must be the same. Suppose that $|B_j| = k$ for every j, then H is said to be k-uniform.*
2. *Each vertex lies in exactly the same number of hyperedges.*

Definition 12. *Let $H = (A, B)$ be a regular mF hypergraph. The order of a regular fuzzy hypergraph H is*

$$O(H) = (\Sigma_{x \in X} \wedge P_1 \circ \zeta_j(x), \Sigma_{x \in X} \wedge P_2 \circ \zeta_j(x), \cdots, \Sigma_{x \in X} \wedge P_m \circ \zeta_j(x)),$$

for every $x \in X$. The size of a regular mF hypergraph is $S(H) = \sum_j S(B_j)$, where

$$S(B_j) = (\Sigma_{x \in B_j} P_1 \circ \zeta_j(x), \Sigma_{x \in B_j} P_2 \circ \zeta_j(x), \cdots, \Sigma_{x \in B_j} P_m \circ \zeta_j(x)).$$

Example 10. *Consider the 4-polar fuzzy hypergraph $H = (A, B)$; define $X = \{a, b, c, d, e, f, g, h, i\}$ and $A = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6\}$, where*

$$\zeta_1 = \{(a, 0.4, 0.4, 0.4, 0.4), (b, 0.4, 0.4, 0.4, 0.4), (c, 0.4, 0.4, 0.4, 0.4)\},$$

$$\zeta_2 = \{(d, 0.4, 0.4, 0.4, 0.4), (e, 0.4, 0.4, 0.4, 0.4), (f, 0.4, 0.4, 0.4, 0.4)\},$$

$$\zeta_3 = \{(g, 0.4, 0.4, 0.4, 0.4), (h, 0.4, 0.4, 0.4, 0.4), (i, 0.4, 0.4, 0.4, 0.4)\},$$

$$\zeta_4 = \{(a, 0.4, 0.4, 0.4, 0.4), (d, 0.4, 0.4, 0.4, 0.4), (g, 0.4, 0.4, 0.4, 0.4)\},$$
$$\zeta_5 = \{(b, 0.4, 0.4, 0.4, 0.4), (e, 0.4, 0.4, 0.4, 0.4), (h, 0.4, 0.4, 0.4, 0.4)\},$$
$$\zeta_6 = \{(c, 0.4, 0.4, 0.4, 0.4), (f, 0.4, 0.4, 0.4, 0.4), (i, 0.4, 0.4, 0.4, 0.4)\}.$$

The order of H is, $O(H) = (3.6, 3.6, 3.6, 3.6)$ *and* $S(H) = (7.2, 7.2, 7.2, 7.2)$.

We state the following propositions without proof.

Proposition 1. *The size of a n-regular mF hypergraph is* $\frac{nk}{2}$, $|X| = k$.

Proposition 2. *If H is both n-regular and m-totally regular mF hypergraph , then* $O(H) = k(m - n)$, *where* $|X| = K$.

Proposition 3. *If H is both m-totally regular mF hypergraph , then* $2S(H) + O(H) = mk, |X| = K$.

Theorem 1. *Let* $H = (A, B)$ *be an mF hypergraph of a crisp hypergraph* H^*. *Then* $A : X \longrightarrow [0, 1]^m$ *is a constant function if and only if the following are equivalent:*
(a) H is a regular mF hypergraph,
(b) H is a totally regular mF hypergraph.

Proof. Suppose that $A : X \longrightarrow [0, 1]^m$, where $A = \{\zeta_1, \zeta_2, ..., \zeta_r\}$ is a constant function. That is, $P_i o\zeta_j(x) = c_i$ for all $x \in \zeta_j$, $1 \leq i \leq m$, $1 \leq j \leq r$.

$(a) \Rightarrow (b)$: Suppose that H is n-regular mF hypergraph. Then $deg(x) = n_i$, for all $x \in \zeta_j$. By using definition 11, $deg[x] = n_i + k_i$ for all $x \in \zeta_j$. Hence, H is a totally regular mF hypergraph.

$(b) \Rightarrow (a)$: Suppose that H is a m-totally regular mF hypergraph. Then $deg[x] = k_i$, for all $x \in \zeta_j$, $1 \leq j \leq r$.

$$\Rightarrow deg(x) + \wedge_j P_i o\zeta_j(x) = k_i \text{ for all } x \in \zeta_j,$$

$$\Rightarrow deg(x) + c_i = k_i \text{ for all } x \in \zeta_j,$$

$$\Rightarrow deg(x) = k_i - c_i \text{ for all } x \in \zeta_j.$$

Thus, H is a regular mF hypergraph. Hence (1) and (2) are equivalent.

Conversely, suppose that (1) and (2) are equivalent, i.e. H is regular if and only if H is totally regular. On contrary, suppose that A is not constant, that is, $P_i o\zeta_j(x) \neq P_i o\zeta_j(y)$ for some x and y in A. Let $H = (A, B)$ be a n-regular mF hypergraph; then

$$deg(x) \quad = \quad n_i \text{ for all } x \in \zeta_j(x).$$

Consider,

$$\begin{aligned} deg[x] &= deg(x) + \wedge_j P_i o\zeta_j(x) = n_i + \wedge_j P_i o\zeta_j(x), \\ deg[y] &= deg(y) + \wedge_j P_i o\zeta_j(y) = n_i + \wedge_j P_i o\zeta_j(y). \end{aligned}$$

Since $P_i o\zeta_j(x)$ and $P_i o\zeta_j(y)$ are not equal for some x and y in X, hence $deg[x]$ and $deg[y]$ are not equal, thus H is not a totally regular m-poalr fuzzy hypergraph, which is again a contradiction to our assumption.

Next, let H be a totally regulr mF hypergraph, then $deg[x] = deg[y]$.
That is,

$$\begin{aligned} deg(x) + \wedge_j P_i o\zeta_j(x) &= deg(y) + \wedge_j P_i o\zeta_j(y), \\ deg(x) - deg(y) &= \wedge_j P_i o\zeta_j(y) - \wedge_j P_i o\zeta_j(x). \end{aligned}$$

Since the right hand side of the above equation is nonzero, the left hand side of the above equation is also nonzero. Thus $deg(x)$ and $deg(y)$ are not equal, so H is not a regular mF hypergraph, which is again contradiction to our assumption. Hence, A must be constant and this completes the proof. □

Theorem 2. *If an mF hypergraph is both regular and totally regular, then $A : X \longrightarrow [0,1]^m$ is constant function.*

Proof. Let H be a regular and totally regular mF hypergraph. Then

$$deg(x) \quad = \quad n_i \text{ for all } x \in \zeta_j(x),$$

and

$$\begin{aligned} deg[x] &= k_i \text{ for all } x \in \zeta_j(x), \\ \Leftrightarrow deg(x) + \wedge_j P_i o\zeta_j(x) &= k_i, \text{ for all } x \in \zeta_j(x), \\ \Leftrightarrow n_1 + \wedge_j P_i o\zeta_j(x) &= k_i, \text{ for all } x \in \zeta_j(x), \\ \Leftrightarrow \wedge_j P_i o\zeta_j(x) &= k_i - n_i, \text{ for all } x \in \zeta_j(x), \\ \Leftrightarrow P_i o\zeta_j(x) &= k_i - n_i, \text{ for all } x \in \zeta_j(x). \end{aligned}$$

Hence, $A : X \longrightarrow [0,1]^m$ is a constant function. □

Remark 2. *The converse of Theorem 1 may not be true, in general. Consider a 3-polar fuzzy hypergraph $H = (A, B)$, define $X = \{a, b, c, d, e\}$,*

$$\zeta_1 = \{(a, 0.2, 0, 2, 0.2), (b, 0.2, 0.2, 0.2), (c, 0.2, 0.2, 0.2)\},$$

$$\zeta_2 = \{(a, 0.2, 0, 2, 0.2), (d, 0.2, 0.2, 0.2)\},$$

$$\zeta_3 = \{(b, 0.2, 0.2, 0.2), (e, 0.2, 0.2, 0.2)\},$$

$$\zeta_4 = \{(c, 0.2, 0.2, 0.2), (e, 0.2, 0.2, 0.2)\}.$$

Then $A : X \longrightarrow [0,1]^m$, where $A = \{\zeta_1, \zeta_2, ..., \zeta_r\}$ is a constant function. But $deg(a) = (0.6, 0.6, 0.6) \neq (0.4, 0.4, 0.4) = deg(e)$. Also $(deg[a] = (0.8, 0.8, 0.8) \neq (0.6, 0.6, 0.6) = deg[e])$. So H is neither regular nor totally regular mF hypergraph.

Definition 13. *An mF hypergraph $H = (A, B)$ is called complete if for every $x \in X$, $N(x) = \{x \mid x \in X - x\}$ that is, $N(x)$ contains all the remaining vertices of X except x.*

Example 11. *Consider a 3-polar fuzzy hypergraph $H = (A, B)$ as shown in Figure 7, where $X = \{a, b, c, d\}$ and $A = \{\zeta_1, \zeta_2, \zeta_3\}$, where $\zeta_1 = \{(a, 0.3, 0.4, 0.6), (c, 0.3, 0.4, 0.6)\}$, $\zeta_2 = \{(a, 0.3, 0.4, 0.6), (b, 0.3, 0.4, 0.6), (d, 0.3, 0.4, 0.6)\}$, $\zeta_3 = \{(b, 0.3, 0.4, 0.6), (c, 0.3, 0.4, 0.6), (d, 0.3, 0.4, 0.6)\}$. Then $N(a) = \{b, c, d\}$, $N(b) = \{a, c, d\}$, $N(c) = \{a, b, d\}$.*

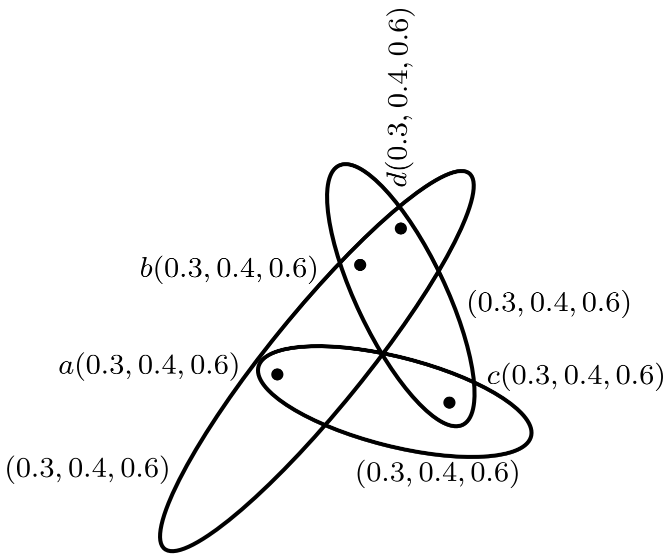

Figure 7. 3-Polar fuzzy hypergraph.

Remark 3. *For a complete mF hypergraph, the cardinality of $N(x)$ is the same for every vertex.*

Theorem 3. *Every complete mF hyprgraph is a totally regular mF hypergraph.*

Proof. Since given mF hypergraph H is complete, each vertex lies in exactly the same number of hyperedges and each vertex has the same closed neighborhood degree m. That is, $deg[x_1] = deg[x_2]$ for all $x_1, x_2 \in X$. Hence H is m-totally regular. □

3. Applications to Decision-Making Problems

Analysis of human nature and its culture has been entangled with the assessment of social networks for many years. Such networks are refined by designating one or more relations on the set of individuals and the relations can be taken from efficacious relationships, facets of some management and from a large range of others means. For super-dyadic relationships between the nodes, network models represented by simple graph are not sufficient. Natural presence of hyperedges can be found in co-citation, e-mail networks, co-authorship, web log networks and social networks etc. Representation of these models as hypergraphs maintain the dyadic relationships.

3.1. Super-Dyadic Managements in Marketing Channels

In marketing channels, dyadic correspondence organization has been a basic implementation. Marketing researchers and managers have realized that their common engagement in marketing channels is a central key for successful marketing and to yield benefits for the company. mF hypergraphs consist of marketing managers as vertices and hyperedges show their dyadic communication involving their parallel thoughts, objectives, plans, and proposals. The more powerful close relation in the research is more beneficial for the marketing strategies and the production of an organization. A 3-polar fuzzy network model showing the dyadic communications among the marketing managers of an organization is given in Figure 8.

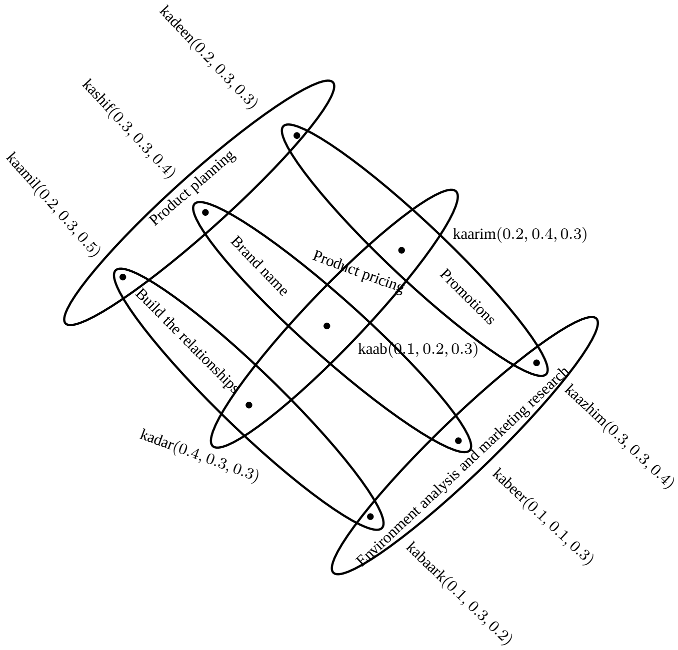

Figure 8. Super-dyadic managements in marketing channels.

The membership degrees of each person symbolize the percentage of its dyadic behaviour towards the other people of the same dyad group. The adjacent level between any pair of vertices illustrates how proficient their dyadic relationship is. The adjacent levels are given in Table 4.

Table 4. Adjacent levels

Dyad pairs	Adjacent level	Dyad pairs	Adjacent level
γ(Kadeen, Kashif)	(0.2,0.3,0.3)	γ(Kaarim, Kaazhim)	(0.2,0.3,0.3)
γ(Kadeen, Kaamil)	(0.2,0.3,0.3)	γ(Kaarim, Kaab)	(0.1,0.2,0.3)
γ(Kadeen, Kaarim)	(0.2,0.3,0.3)	γ(Kaarim, Kadar)	(0.2,0.3,0.3)
γ(Kadeen, Kaazhim)	(0.2,0.3,0.3)	γ(Kaab, Kadar)	(0.1,0.2,0.3)
γ(Kashif, Kaamil)	(0.2,0.3,0.4)	γ(Kaab, Kabeer)	(0.1,0.1,0.3)
γ(Kashif, Kaab)	(0.1,0.2,0.3)	γ(Kadar, Kabaark)	(0.1,0.3,0.2)
γ(Kashif, Kabeer)	(0.1,0.1,0.3)	γ(Kaazhim, Kabeer)	(0.1,0.1,0.3)
γ(Kaamil, Kadar))	(0.2,0.2,0.3)	γ(Kaazhim, Kabaark)	(0.1,0.3,0.2)
γ(Kaamil, Kabaark)	(0.1,0.3,0.2)	γ(Kabeer, Kabaark)	(0.1,0.1,0.2)

It can be seen that the most capable dyadic pair is (Kashif, Kaamil). 3-polar fuzzy hyperedges are taken as different digital marketing strategies adopted by the different dyadic groups of the same organization. The vital goal of this model is to determine the most potent dyad of digital marketing techniques. The six different groups are made by the marketing managers and the digital marketing strategies adopted by these six groups are represented by hyperedges. i.e., the 3-polar fuzzy hyperedges $\{T_1, T_2, T_3, T_4, T_5, T_6\}$ show the following strategies {Product pricing, Product planning, Environment analysis and marketing research, Brand name, Build the relationships, Promotions}, respectively. The exclusive effects of membership degrees of each marketing strategy towards the achievements of an organization are given in Table 5.

Table 5. Effects of marketing strategies.

Marketing Strategy	Profitable growth	Instruction manual for company success	Create longevity of the business
Product pricing	0.1	0.2	0.3
Product planning	0.2	0.3	0.3
Environment analysis and marketing research	0.1	0.2	0.2
Brand name	0.1	0.3	0.3
Build the relationships	0.1	0.3	0.2
Promotions	0.2	0.3	0.3

Effective dyads of market strategies enhance the performance of an organization and discover the better techniques to be adopted. The adjacency of all dyadic communication managements is given in Table 6.

Table 6. Adjacency of all dyadic communication managements

Dyadic strategies	Effects
σ(Product pricing, Product planning)	(0.1,0.2,0.3)
σ(Product pricing, Environment analysis and marketing research)	(0.1,0.2,0.2)
σ(Product pricing, Brand name)	(0.1,0.2,0.3)
σ(Product pricing, Build the relationships)	(0.1,0.2,0.2)
σ(Product pricing, Promotions)	(0.1,0.2,0.3)
σ(Product planning, Environment analysis and marketing research)	(0.1,0.2,0.2)
σ(Product planning, Brand name)	(0.1,0.3,0.3)
σ(Product planning, Build the relationships)	(0.1,0.3,0.2)
σ(Product planning, Promotions)	(0.2,0.3,0.3)
σ(Environment analysis and marketing research, Brand name)	(0.1,0.2,0.2)
σ(Environment analysis and marketing research, Build the relationships)	(0.1,0.2,0.2)
σ(Environment analysis and marketing research, Promotions)	(0.1,0.2,0.2)
σ(Brand name, Build the relationships)	(0.1,0.3,0.2)
σ(Brand name, Promotions)	(0.1,0.3,0.3)
σ(Build the relationships, Promotions)	(0.1,0.3,0.2)

The most dominant and capable marketing strategies adopted mutually are Product planning and Promotions. Thus, to increase the efficiency of an organization, dyadic managements should make powerful planning for products and use the promotions skill to attract customers to purchase their products. The membership degrees of this dyad is (0.2, 0.3, 0.3) which shows that the amalgamated effect of this dyad will increase the profitable growth of an organization up to 20%, instruction manual for company success up to 30%, create longevity of the business up to 30% . Thus, to promote the performance of an organization, super dyad marketing communications are more energetic. The method of determining the most effective dyads is explained in the following algorithm.

Algorithm 1

Input: The membership values $A(x_i)$ of all nodes (marketing managers) $x_1, x_2, ..., x_n$.
Input: The membership values $B(T_i)$ of all hyperedges $T_1, T_2, ..., T_r$.
Find the adjacent level between nodes x_i and x_j as,
do i from $1 \to n-1$
  do j from $i+1 \to n$
    do k from $1 \to r$
      if $x_i, x_j \in E_k$ **then**
        $\gamma(x_i, x_j) = \max_k \inf\{A(x_i), A(x_j)\}$.
      end if
    end do
  end do

12. **end do**
13. Find the best capable dyadic pair as $\max_{i,j} \gamma(x_i, x_j)$.
14. **do** i from $1 \to r-1$
15.   **do** j from $i+1 \to r$
16.   **do** k from $1 \to r$
17.   **if** $x_k \in T_i \cap T_j$ **then**
18. $\sigma(T_i, T_j) = \max_k \inf\{B(T_i), B(T_j)\}$.
19.   **end if**
20.   **end do**
21.   **end do**
22. **end do**
23. Find the best effective super dyad management as $\max_{i,j} \sigma(T_i, T_j)$.

Description of Algorithm 1: Lines 1 and 2 pass the input of $m-$polar fuzzy set A on n vertices $x_1, x_2, \ldots, x_n$ and $m-$polar fuzzy relation B on r edges $T_1, T_2, ..., T_r$. Lines 3 to 12 calculate the adjacent level between each pair of nodes. Line 14 calculates the best capable dyadic pair. The loop initializes by taking the value $i = 1$ of do loop which is always true, i.e., the loop runs for the first iteration. For any ith iteration of do loop on line 3, the do loop on line 4 runs $n - i$ times and, the do loop on line 5 runs r times. If there exists a hyperedge E_k containing x_i and x_j then, line 7 is executed otherwise the if conditional terminates. For every ith iteration of the loop on line 3, this process continues n times and then increments i for the next iteration maintaining the loop throughout the algorithm. For $i = n - 1$, the loop calculates the adjacent level for every pair of distinct vertices and terminates successfully at line 12. Similarly, the loops on lines 13, 14 and 15 maintain and terminate successfully.

3.2. m-Polar Fuzzy Hypergraphs in Work Allotment Problem

In customer care centers, availability of employees plays a vital role in solving customer problems. Such a department should ensure that the system has been managed carefully to overcome practical difficulties. A lot of customers visit such centers to find a solution of their problems. In this part, focus is given to alteration of duties for the employees taking leave. The problem is that employees are taking leave without proper intimation and alteration. We now show the importance of m-polar fuzzy hypergraphs for the allocation of duties to avoid any difficulties.

Consider the example of a customer care center consisting of 30 employees. Assuming that six workers are necessary to be available at their duties. We present the employees as vertices and the degree of membership of each employee represents the work load, percentage of available time and number of workers who are also aware of the employee's work type. The range of values for present time and the workers, knowing the type of work is given in Table 7 and Table 8.

Table 7. Range of membership values of table time.

Time	Membership value
5 hours	0.40
6 hours	0.50
8 hours	0.70
10 hours	0.90

Table 8. Workers knowing the work type.

Workers	Membership value
3	0.40
4	0.60
5	0.80
6	0.90

The degree of membership of each edge represents the common work load, percentage of available time and number of workers who are also aware of the employee's work type. This phenomenon can be represented by a 3-polar fuzzy graph as shown in Figure 9.

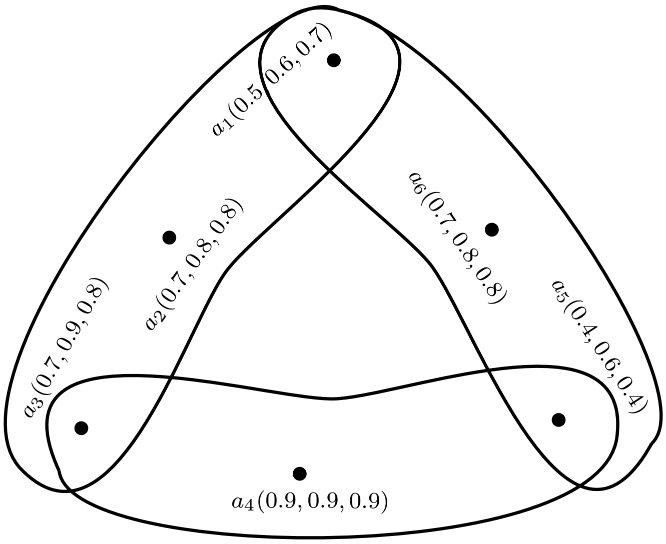

Figure 9. 3-Polar fuzzy graph.

Using Algorithm 2, the strength of allocation and alteration of duties among employees is given in Table 9.

Table 9. Alteration of duties.

Workers	$A(a_i, a_j)$	$S(a_i, a_j)$
a_1, a_2	(0.7,0.8,0.8)	0.77
a_1, a_3	(0.7,0.9,0.8)	0.80
a_2, a_3	(0.5,0.7,0.7)	0.63
a_3, a_4	(0.7,0.6,0.8)	0.70
a_3, a_5	(0.7,0.9,0.8)	0.80
a_4, a_5	(0.9,0.9,0.9)	0.90
a_5, a_6	(0.7,0.8,0.8)	0.77
a_5, a_1	(0.5,0.6,0.7)	0.60
a_1, a_6	(0.6,0.8,0.5)	0.63

Column 3 in Table 9 shows the percentage of alteration of duties. For example, in case of leave, duties of a_1 can be given to a_3 and similarly for other employees.

The method for the calculation of alteration of duties is given in Algorithm 2.

Algorithm 2

1. Input: The n number of employees $a_1, a_2, \ldots, a_n$.
2. Input: The number of edges $E_1, E_2, \ldots, E_r$.
3. Input: The incident matrix B_{ij} where, $1 \leq i \leq n, 1 \leq j \leq r$.
4. Input the membership values of edges $\xi_1, \xi_2, \ldots, \xi_r$
5. **do** i from $1 \to n$
6.   **do** j from $1 \to n$
7.   **do** k from $1 \to r$
8.   **if** $a_i, a_j \in E_k$ **then**
9.   **do** t from $1 \to m$
10. $P_t \circ A(a_i, a_j) = |P_t \circ B_{ik} - P_t \circ B_{jk}| + P_t \circ \xi_k$
11.   **end do**

12. **end if**
13. **end do**
14. **end do**
15. **end do**
16. **do** i from $1 \to n$
17. **do** j from $1 \to n$
18. **if** $A(a_i, a_j) > 0$ **then**
19. $S(a_i, a_j) = \dfrac{P_1 \circ A(a_i, a_j) + P_2 \circ A(a_i, a_j) + \ldots + P_m \circ A(a_i, a_j)}{m}$
20. **end if**
21. **end do**
22. **end do**

Description of Algorithm 2: Lines 1, 2, 3 and 4 pass the input of membership values of vertices, hyperedges and an $m-$polar fuzzy adjacency matrix B_{ij}. The nested loops on lines 5 to 15 calculate the $rth, 1 \leq r \leq m$, strength of allocation and alteration of duties between each pair of employees. The nested loops on lines 16 to 22 calculate the strength of allocation and alteration of duties between each pair of employees. The net time complexity of the algorithm is $\mathrm{O}(n^2 rm)$.

3.3. Availability of Books in Library

A library in a college is a collection of sources of information and similar resources, made accessible to the student community for reference and examination preparation. A student preparing for a given examination will use the knowledge sources such as

1. Prescribed textbooks (A)
2. Reference books in syllabus (B)
3. Other books from library (C)
4. Knowledgeable study materials (D)
5. E-gadgets and internet (E)

It is important to consider the maximum availability of the sources which students mostly use. This phenomenon can be discussed using $m-$polar fuzzy hypergraphs. We now calculate the importance of each source in the student community.

Consider the example of five library resources $\{A, B, C, D, E\}$ in a college. We represent these sources as vertices in a $3-$polar fuzzy hypergraph. The degree of membership of each vertex represents the percentage of students using a particular source for exam preparation, percentage of faculty members using the sources and number of sources available. The degree of membership of each edge represents the common percentage. The $3-$polar fuzzy hypergraph is shown in Figure 10.

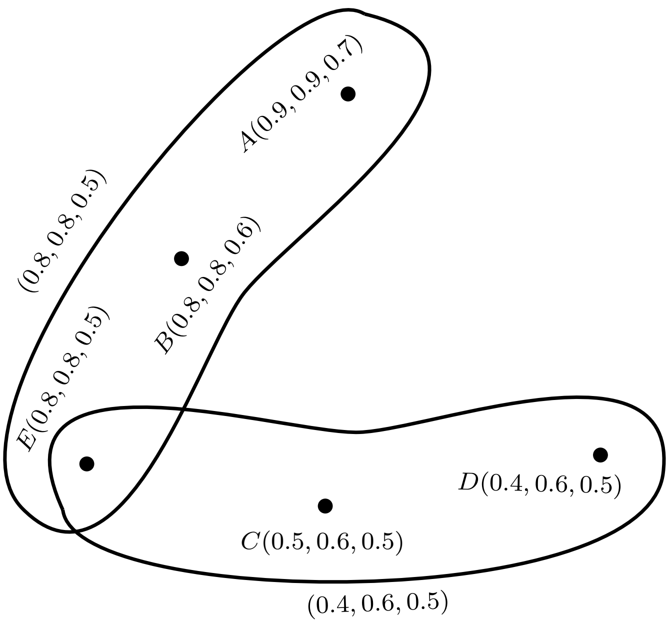

Figure 10. 3-Polar fuzzy hypergraph.

Using Algorithm 3, the strength of each library source in given in Table 10.

Table 10. Library sources.

Sources s_i	$T(s_i)$	$S(a_i, a_j)$
A	(1.7,1.7,1.4)	1.60
B	(1.6,1.6,1.1)	1.43
E	(1.6,1.6,1.0)	1.40
C	(0.9,1.2,1.0)	1.03
D	(0.8,1.2,1.0)	1.0

Column 3 in Table 10 shows that sources A and B are mostly used by students and faculty. Therefore, these should be available in maximum number. There is also a need to confirm the availability of source E to students and faculty.

The method for the calculation of percentage importance of the sources is given in Algorithm 3 whose net time complexity is O(nrm).

Algorithm 3

1. Input: The n number of sources $s_1, s_2, \ldots, s_n$.
2. Input: The number of edges $E_1, E_2, \ldots, E_r$.
3. Input: The incident matrix B_{ij} where, $1 \leq i \leq n, 1 \leq j \leq r$.
4. Input: The membership values of edges $\xi_1, \xi_2, \ldots, \xi_r$
5. **do** i from $1 \to n$
6. $\quad A(s_i) = \mathbf{1}$
7. $\quad C(s_i) = \mathbf{1}$
8. $\quad$ **do** k from $1 \to r$
9. $\qquad$ **if** $s_i \in E_k$ **then**
10. $\qquad\quad A(s_i) = \max\{A(s_i), \xi_k\}$
11. $\qquad\quad C(s_i) = \min\{C(s_i), B_{ik}\}$
12. $\qquad$ **end if**
13. $\quad$ **end do**
14. $\quad T(s_i) = C(s_i) + A(s_i)$
15. **end do**

16. **do** i from $1 \to n$
17. **if** $T(s_i) > \mathbf{0}$ **then**
18. $S(s_i) = \dfrac{P_1 \circ T(s_i) + P_2 \circ T(s_i) + \ldots + P_m \circ T(s_i)}{m}$
19. **end if**
20. **end do**

Description of Algorithm 3: Lines 1, 2, 3 and 4 pass the input of membership values of vertices, hyperedges and an $m-$polar fuzzy adjacency matrix B_{ij}. The nested loops on lines 5 to 15 calculate the degree of usage and availability of library sources. The nested loops on lines 16 to 20 calculate the strength of each library source.

4. Conclusions

Hypergraphs are generalizations of graphs. Many problems which cannot be handled by graphs can be solved using hypergraphs. mF graph theory has numerous applications in various fields of science and technology including artificial intelligence, operations research and decision making. An mF hypergraph constitutes a generalization of the notion of an mF fuzzy graph. mF hypergraphs play an important role in discussing multipolar uncertainty among several individuals. In this research article, we have conferred certain concepts of regular mF hypergraphs and applications of mF hypergraphs in decision-making problems. We aim to generalize our notions to (1) mF soft hypergraphs, (2) soft rough mF hypergraphs, (3) soft rough hypergraphs, and (4) intuitionistic fuzzy rough hypergraphs.

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