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Article

Nilpotent Fuzzy Subgroups

Elaheh Mohammadzadeh [1], **Rajab Ali Borzooei** [2,*]

[1] Department of Mathematics, Payame Noor University, Tehran 19395-3697, Iran; mohamadzadeh36@gmail.com
[2] Department of Mathematics, Shahid Beheshti University, G. C., Tehran 19395-3697, Iran
* Correspondence: borzooei@sbu.ac.ir

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Abstract: In this paper, we introduce a new definition for nilpotent fuzzy subgroups, which is called the good nilpotent fuzzy subgroup or briefly g-nilpotent fuzzy subgroup. In fact, we prove that this definition is a good generalization of abstract nilpotent groups. For this, we show that a group G is nilpotent if and only if any fuzzy subgroup of G is a g-nilpotent fuzzy subgroup of G. In particular, we construct a nilpotent group via a g-nilpotent fuzzy subgroup. Finally, we characterize the elements of any maximal normal abelian subgroup by using a g-nilpotent fuzzy subgroup.

Keywords: nilpotent group; nilpotent fuzzy subgroup; generalized nilpotent fuzzy subgroup

1. Introduction

Applying the concept of fuzzy sets of Zadeh [1] to group theory, Rosenfeld [2] introduced the notion of a fuzzy subgroup as early as 1971. Within a few years, it caught the imagination of algebraists like wildfire and there seems to be no end to its ramifications. With appropriate definitions in the fuzzy setting, most of the elementary results of group theory have been superseded with a startling generalized effect (see [3–5]). In [6] Dudek extended the concept of fuzzy sets to the set with one n-ary operation i.e., to the set G with one operation on $f : G \longrightarrow G$, where $n \geq 2$. Such defined groupoid will be denoted by (G, f). Moreover, he introduced the notion of a fuzzy subgroupoid of an n-ary groupoid. Specially, he proved that if every fuzzy subgroupoid μ defined on (G, f) has the finite image, then every descending chain of subgroupoids of (G, f) terminates at finite step. One of the important concept in the study of groups is the notion of nilpotency. In [7] Kim proposed the notion of a nilpotent fuzzy subgroup. There, he attached to a fuzzy subgroup an ascending series of subgroups of the underlying group to define nilpotency of the fuzzy subgroup. With this definition, the nilpotence of a group can be completely characterized by the nilpotence of its fuzzy subgroups. Then, in [8] Guptaa and Sarmahas, defined the commutator of a pair of fuzzy subsets of a group to generate the descending central chain of fuzzy subgroups of a given fuzzy subgroup and they proposed a new definition of a nilpotent fuzzy subgroup through its descending central chain. Specially, They proved that every Abelian (see [9]) fuzzy subgroup is nilpotent. There are many natural generalizations of the notion of a normal subgroup. One of them is subnormal subgroup. The new methods are important to guarantee some properties of the fuzzy sets; for example, see [10]. In [3] Kurdachenko and et all formulated this concept for fuzzy subgroups to prove that if every fuzzy subgroup of γ is subnormal in γ with defect at most d, then γ is nilpotent ([3] Corollary 4.6). Finally in [11,12] Borzooei et. al. defind the notions of Engel fuzzy subgroups (subpolygroups) and investigated some related results. Now, in this paper we define the ascending series differently with Kim's definition. We then propose a definition of a nilpotent fuzzy subgroup through its ascending central series and call it g-nilpotent fuzzy subgroups. Also, we show that each g-nilpotent fuzzy subgroup is nilpotent. Moreover, we get the main results of nilpotent fuzzy subgroups with our definition. Basically this definition help us with the fuzzification of much more properties of nilpotent groups. Furthermore, we prove that for a fuzzy subgroup μ of G, $\{x \in G \mid \mu([x, y_1, ..., y_n]) = \mu(e) \ \ for \ \ any \ \ y_1, ..., y_n \in G\}$ is equal to the $n-$th term of ascending series where $[x, y_1] = x^{-1}y_1^{-1}xy_1$ and $[x, y_1, ..., y_n] = [[x, y_1, ..., y_{n-1}], y_n]$. Therefore, we have a complete analogy concept of nilpotent groups of an abstract group. Specially, we prove that a finite maximal normal subgroup can control the g-nilpotent fuzzy subgroup and makes it finite.

2. Preliminary

Let G be any group and $x, y \in G$. Define the n-commutator $[x,_n y]$, for any $n \in \mathbb{N}$ and $x, y \in G$, by $[x,_0 y] = x$, $[x,_1 y] = x^{-1}y^{-1}xy$ and $[x,_n y] = [[x,_{n-1} y], y]$ also, for any $y_1, ..., y_n \in G$, $[x, y_1, ..., y_n] = [[x, y_1, ..., y_{n-1}], y_n]$. For any $x, g \in G$, we consider $x^g = g^{-1}xg$ and $[x, y] = [x,_1 y]$.

Theorem 1. *[13] Let G be a group and $x, y, z \in G$. Then*
(1) $[x, y] = [y, x]^{-1}$,
(2) $[x.y, z] = [x, z]^y.[y, z]$ and $[x, y.z] = [x, z].[x, y]^z$,
(3) $[x, y^{-1}] = ([x, y]^{y^{-1}})^{-1}$ and $[x^{-1}, y] = ([x, y]^{x^{-1}})^{-1}$.
Note that $x^g = x.[x, g]$.

Definition 1. *[13] Let $X_1, X_2, ...$ be nonempty subsets of a group G. Define the commutator subgroup of X_1 and X_2 by*

$$[X_1, X_2] = \langle [x_1, x_2] \mid x_1 \in X_1, x_2 \in X_2 \rangle.$$

More generally, define

$$[X_1, ..., X_n] = [[X_1, ..., X_{n-1}], X_n]$$

where $n \geq 2$ and $[X_1] = \langle X_1 \rangle$. Also recall that $X_1^{X_2} = \langle x_1^{x_2} \mid x_1 \in X_1, x_2 \in X_2 \rangle$

Definition 2. *[1] A fuzzy subset μ of X is a function $\mu : X \to [0, 1]$.*

Also, for fuzzy subsets μ_1 and μ_2 of X, then μ_1 is *smaller* than μ_2 and write $\mu_1 \leq \mu_2$ iff for all $x \in X$, we have $\mu_1(x) \leq \mu_2(x)$. Also, $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$, for any μ_1, μ_2 are defined as follows:
$(\mu_1 \vee \mu_2)(x) = max\{\mu_1(x), \mu_2(x)\}$, $(\mu_1 \wedge \mu_2)(x) = min\{\mu_1(x), \mu_2(x)\}$, for any $x \in X$.

Definition 3. *[14] Let f be a function from X into Y, and μ be a fuzzy subset of X. Define the fuzzy subset $f(\mu)$ of Y, for any $y \in Y$, by*

$$(f(\mu))(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} \mu(x), & f^{-1}(y) \neq \phi \\ 0, & otherwise \end{cases}$$

Definition 4. *[2] Let μ be a fuzzy subset of a group G. Then μ is called a fuzzy subgroup of G if for any $x, y \in G$; $\mu(xy) \geq \mu(x) \wedge \mu(y)$, and $\mu(x^{-1}) \geq \mu(x)$. A fuzzy subgroup μ of G is called normal if $\mu(xy) = \mu(yx)$, for any x, y in G. It is easy to prove that a fuzzy subgroup μ is normal if and only if $\mu(x) = \mu(y^{-1}xy)$, for any $x, y \in G$ (See [14]).*

Theorem 2. *[14] Let μ be a fuzzy subgroup of G. Then for any $x, y \in G$, $\mu(x) \neq \mu(y)$, implies $\mu(xy) = \mu(x) \wedge \mu(y)$. Moreover, for a normal subgroup N of G, fuzzy subset ξ of $\frac{G}{N}$ as the following definition:*

$$\xi(xN) = \bigvee_{z \in xN} \mu(z), \ \ for \ \ any \ \ x \in G$$

is a fuzzy subgroup of $\frac{G}{N}$.

Definition 5. *[14] Let* μ *be a fuzzy subset of a semigroup* G. *Then* $Z(\mu)$ *is define as follows:*

$$Z(\mu) = \{x \in G \mid \mu(xy) = \mu(yx) \ \textit{and} \ \mu(xyz) = \mu(yxz), \ \textit{for any } y, z \in G\}$$

If $Z(\mu) = G$, *then* μ *is called a commutative fuzzy subset of* G.

Note that since $\mu(xy) = \mu(yx)$ then we have $\mu(xyz) = \mu(x(yz)) = \mu((yz)x) = \mu(yzx)$.

Theorem 3. *[14] Let* μ *be a fuzzy subset of a semigroup* G. *If* $Z(\mu)$ *is nonempty, then* $Z(\mu)$ *is a subsemigroup of* G. *Moreover, if* G *is a group, then* $Z(\mu)$ *is a normal subgroup of* G.

We recall the notion of the ascending central series of a fuzzy subgroup and a nilpotent fuzzy subgroup of a group [14]. Let μ be a fuzzy subgroup of a group G and $Z^0(\mu) = \{e\}$. Clearly $\{e\}$ is a normal subgroup of G. Let π_0 be the natural homomorphism of G onto $\frac{G}{Z^0(\mu)}$. It is clear that $\pi_0 = I$. Suppose that $Z^1(\mu) = \pi_0^{-1}(Z(\pi_0(\mu)))$. Since $Z(\pi_0(\mu))$ is a normal subgroup of $\frac{G}{Z^0(\mu)}$, then it is clear that $Z^1(\mu)$ is a normal subgroup of G. Also we see that $Z^1(\mu) = Z(\mu)$. Now let π_1 be the natural homomorphism of G onto $\frac{G}{Z^1(\mu)}$ and $Z^2(\mu) = \pi_1^{-1}(Z(\pi_1(\mu)))$. Since $\pi_1(\mu)$ is a fuzzy subgroup of $\frac{G}{Z^1(\mu)}$, then $Z(\pi_1(\mu))$ is a normal subgroup of $\frac{G}{Z^1(\mu)}$, which implies that $Z^2(\mu)$ is a normal subgroup of G. Similarly suppose that $Z^i(\mu)$ has been defined and so $Z^i(\mu)$ is a normal subgroup of G, for $i \in \mathbb{N} \cup \{0\}$. Let π_i be the natural homomorphism of G onto $\frac{G}{Z^i(\mu)}$ and $Z^{i+1}(\mu) = \pi_i^{-1}(Z(\pi_i(\mu)))$. Then $Z^{i+1}(\mu)$ is a normal subgroup of G. Since $1_{\frac{G}{Z^i(\mu)}} \subseteq Z(\pi_i(\mu)$, then $\pi_i^{-1}(1_{\frac{G}{Z^i(\mu)}}) \subseteq \pi_i^{-1}(Z(\pi_i(\mu)))$. Therefore, $Ker(\pi_i) = Z^i(\mu) \subseteq Z^{i+1}(\mu)$, for $i = 0, 1, \ldots$.

Definition 6. *[14] Let* μ *be a fuzzy subgroup of a group* G. *The ascending central series of* μ *is defined to be the ascending chain of normal subgroups of* G *as follows:*

$$Z^0(\mu) \subseteq Z^1(\mu) \subseteq Z^2(\mu) \ldots.$$

Now the fuzzy subgroup μ *of* G *is called nilpotent if there exists a nonnegative integer* m, *such that* $Z^m(\mu) = G$. *The smallest such integer is called the class of* μ.

Theorem 4. *[14] Let* μ *be a fuzzy subgroup of a group* G, $i \in \mathbb{N}$ *and* $x \in G$. *If* $xyx^{-1}y^{-1} \in Z^{i-1}(\mu)$, *for any* $y \in G$, *then* $x \in Z^i(\mu)$. *Moreover, if* $T = \{x \in G \mid \mu(xyx^{-1}y^{-1}) = \mu(e), \textit{for any } y \in G\}$, *then* $T = Z(\mu)$.

Let G be a group. We know that $Z(G)$ is a normal subgroup of G. Let $Z_2(G)$ be the inverse image of $Z(\frac{G}{Z(G)})$, under the canonical projection $G \longrightarrow \frac{G}{Z(G)}$. Then $Z_2(G)$ is normal in G and contains $Z(G)$. Continue this process by defining inductively, $Z_1(G) = Z(G)$ and $Z_i(G)$ is the inverse image of $Z(\frac{G}{Z_{i-1}(G)})$ under the canonical projection $G \longrightarrow \frac{G}{Z_{i-1}(G)}$ for any $i \in \mathbb{N}$. Thus we obtain a sequence of normal subgroups of G, called the ascending central series of G that is, $\{e\} = Z_0(G) \subseteq Z_1(G) \subseteq Z_2(G) \subseteq \ldots$. The other definition is as follows [13]: Let G be a group and $Z_0(G) = \{e\}$. It is clear that $\{e\}$ is a normal subgroup of G. Put $\frac{Z_1(G)}{\{e\}} = Z(\frac{G}{\{e\}})$. Then $Z_1(G) = Z(G)$ is a normal subgroup of G. Similarly for any integer $n > 1$, put $\frac{Z_n(G)}{Z_{n-1}(G)} = Z(\frac{G}{Z_{n-1}(G)})$. Then $Z_i(\mu)$ is called the *i-th center* of group G. Moreover, $\{e\} = Z_0(G) \subseteq Z_1(G) \subseteq Z_2(G) \subseteq \ldots$ is called upper central series of G. These two definitions are equivalent since, $\pi(Z_2(G)) = \pi(\pi^{-1}(Z(\frac{G}{Z(G)}))) = Z(\frac{G}{Z(G)})$. Thus $\frac{Z_2(G)}{Z(G)} = Z(\frac{G}{Z(G)})$. Similarly we get the result for any $n \in \mathbb{N}$.

Theorem 5. *[13] Let G be a group and $n \in \mathbb{N}$. Then*
(i) $x \in Z_n(G)$ if and only if for any $y_i \in G$ where $1 \le i \le n$, $[x, y_1, ..., y_n] = e$,
(ii) $[Z_n(G), G] \subseteq Z_{n-1}(G)$.
(iii) Class of nilpotent groups is closed with respect to subgroups and homomorphic images.

Notation. From now on, in this paper we let G be a group.

3. Good Nilpotent Fuzzy Subgroups

One of the important concept in the study of groups is the notion of nilpotency. It was introduced for fuzzy subgroups, too (See [14]). Now, in this section we give a new definition of nilpotent fuzzy subgroups which is similar to one in the abstract group theory. It is a good generation of the last one. With this nilpotency we get some new main results.

Let μ be a fuzzy subgroup of G. Put $Z_0(\mu) = \{e\}$. Clearly $Z_0(\mu) \trianglelefteq G$. Let $Z_1(\mu) = \{x \in G \mid \mu([x, y]) = \mu(e)$, for any $y \in G\}$. Now using Theorems 4, we have $Z_1(\mu) = Z(\mu)$ is a normal subgroup of G. We define a subgroup $Z_2(\mu)$ of G such that $\frac{Z_2(\mu)}{Z_1(\mu)} = Z(\frac{G}{Z_1(\mu)})$; Since $Z_1(\mu) \trianglelefteq G$ then $Z_1(\mu) \trianglelefteq Z_2(\mu)$. We show that $[Z_2(\mu), G] \subseteq Z_1(\mu)$. For this let $x \in Z_2(\mu)$ and $g \in G$. Thus $xZ_1(\mu) \in \frac{Z_2(\mu)}{Z_1(\mu)} = Z(\frac{G}{Z_1(\mu)})$, which implies that $[xZ_1(\mu), gZ_1(\mu)] = Z_1(\mu)$ for any $g \in G$. Therefore $[x, g] \in Z_1(\mu)$. Hence $[Z_2(\mu), G] \subseteq Z_1(\mu)$. Therefore $x^g = x[x, g] \in Z_2(\mu)$. Thus $Z_2(\mu) \trianglelefteq G$. Similarly for $k \ge 2$ we define a normal subgroup $Z_k(\mu)$ such that $\frac{Z_k(\mu)}{Z_{k-1}(\mu)} = Z(\frac{G}{Z_{k-1}(\mu)})$. It is clear that $Z_0(\mu) \subseteq Z_1(\mu) \subseteq Z_2(\mu) \subseteq ....$

Definition 7. *A fuzzy subgroup μ of G is called a good nilpotent fuzzy subgroup of G or briefly g-nilpotent fuzzy subgroup of G if there exists a none negative integer n, such that $Z_n(\mu) = G$. The smallest such integer is called the class of μ.*

Example 1. *Let $D_3 = \langle a, b; a^3 = b^2 = e, ba = a^2b\rangle$ be the dihedral group with six element and $t_0, t_1 \in [0, 1]$ such that $t_0 > t_1$. Define a fuzzy subgroup μ of D_3 as follows:*

$$\mu(x) = \begin{cases} t_0 & if \quad x \in < a > \\ t_1 & if \quad x \notin < a > \end{cases}$$

Then $(D_3\backslash\langle a\rangle)(D_3\backslash\langle a\rangle) = \langle a\rangle$, $(\langle a\rangle)(D_3\backslash\langle a\rangle) = (D_3\backslash\langle a\rangle)$, $(D_3\backslash\langle a\rangle)(\langle a\rangle) = (D_3\backslash\langle a\rangle)$ and $(\langle a\rangle)(\langle a\rangle) = (\langle a\rangle)$. Now, we show that $Z_1(\mu) = D_3$. If $x \in \langle a\rangle$ and $y \notin \langle a\rangle$, then $xy \notin \langle a\rangle$. Thus by the above relations, we have $[x, y] = x^{-1}y^{-1}xy = (yx)^{-1}(xy) \in \langle a\rangle$, which implies that $\mu[x, y] = t_0 = \mu(e)$. Similarly, for the cases $x \notin \langle a\rangle$ and $y \in \langle a\rangle$ or $x, y \in \langle a\rangle$ or $x, y \notin \langle a\rangle$, we have $\mu[x, y] = \mu(e)$. Hence for any $x, y \in D_3$, $\mu[x, y] = \mu(e)$ and so by Theorem 4, $Z(\mu) = D_3$. Now, since $Z_1(\mu) = Z(\mu)$, we get μ is g-nilpotent fuzzy subgroup.

In the following we see that for $n \in \mathbb{N}$, each normal subgroup $Z_n(\mu)$, in which is defined by $\frac{Z_{n+1}(\mu)}{Z_n(\mu)} = Z(\frac{G}{Z_n(\mu)})$ is equal to $\{x \in G \mid \mu([x, y_1, ..., y_n]) = \mu(e),\ \ for\ \ any\ \ y_1, y_2, ..., y_n \in G\}$.

Lemma 1. *Let μ be a fuzzy subgroup of G. Then for $k \in \mathbb{N}$*

$$Z_k(\mu) = \{x \in G \mid \mu([x, y_1, ..., y_k]) = \mu(e),\ \ for\ \ any\ \ y_1, y_2, ..., y_k \in G\}.$$

Proof. We prove it by induction on k. If $k = 1$, then by definition of $Z_1(\mu)$ we have $Z_1(\mu) = \{x \in G \mid \mu([x,y]) = \mu(e)$ for any $y \in G\}$. Now let $k = n+1$, and the result is true for $k \leq n$. Then

$$\begin{aligned} x \in Z_{n+1}(\mu) &\iff xZ_n(\mu) \in \frac{Z_{n+1}(\mu)}{Z_n(\mu)} = Z(\frac{G}{Z_n(\mu)}) \\ &\iff [xZ_n(\mu), y_1 Z_n(\mu)] = Z_n(\mu), \ \ for \ \ any \ \ y_1 \in G \\ &\iff [x,y_1]Z_n(\mu) = Z_n(\mu), \ \ for \ \ any \ \ y_1 \in G \\ &\iff [x,y_1] \in Z_n(\mu), \ \ for \ \ any \ \ y_1 \in G \\ &\iff \mu([[x,y_1],y_2,...,y_{n+1}]) = \mu(e), \ \ for \ \ any \ \ y_1,...,y_{n+1} \in G. \end{aligned}$$

This complete the proof. □

Theorem 6. *Any g-nilpotent fuzzy subgroup of G is a nilpotent fuzzy subgroup.*

Proof. Let fuzzy subgroup μ of G be g-nilpotent. Since $Z_1(\mu) = Z(\mu) = Z^1(\mu)$, for $n = 1$, the proof is true. Now let $Z_{n+1}(\mu) = G$. Then by Lemma 1, $\{x \mid \mu([x,y_1,...,y_{n+1}]) = \mu(e) \ \ for \ \ any \ \ y_1,y_2,...,y_{n+1} \in G\} = G$. We should prove that $Z^{n+1}(\mu) = G$. Let $x \in G$. Then $\mu([x,y_1,...,y_{n+1}]) = \mu(e)$, for any $y_1,...,y_{n+1} \in G$. Therefore by Theorem 4, $[x,y_1,...,y_n] \in Z(\mu)$. Consequently, by Theorem 4, $[x,y_1,...,y_{n-1}] \in Z^2(\mu)$. Similarly, by using k-times Theorem 4, we have $x \in Z^{n+1}(\mu)$ and so $Z^{n+1}(\mu) = G$. Therefore μ is a nilpotent fuzzy subgroup of G. □

Theorem 7. *Let μ be a fuzzy subgroup of G. Then μ is commutative if and only if μ is g-nilpotent fuzzy subgroup of class 1.*

Proof. ($\Rightarrow$) Let μ be commutative. Then $Z(\mu) = G$. Since $Z_1(\mu) = Z(\mu)$, then $Z_1(\mu) = G$ which implies that μ is g-nilpotent of class 1.
($\Leftarrow$) If μ is g-nilpotent of class 1, then $Z_1(\mu) = G$. Hence $Z_1(\mu) = Z(\mu) = G$. Therefore, μ is commutative. □

Notation. If μ is a fuzzy subgroup of G, then $Z_{k-1}(\frac{G}{Z(\mu)})$ means the $(k-1)$-th center of $\frac{G}{Z(\mu)}$ ([15]).

Next we see that a g-nilpotent fuzzy subgroup of G makes the g-nilpotent fuzzy subgroup of $\frac{G}{Z(\mu)}$. For this, we need the following two Lemmas.

Lemma 2. *Let μ be a fuzzy subgroup of G. Then for any $k \in \mathbb{N}$, $\frac{Z_k(\mu)}{Z(\mu)} = Z_{k-1}(\frac{G}{Z(\mu)})$.*

Proof. First we recall that for $i \in \mathbb{N}$, $x \in Z_i(G)$ if and only if $[x,y_1,...,y_i] = e$, for any $y_1,y_2,...,y_i \in G$ (See [13]). Hence

$$\begin{aligned} xZ(\mu) \in Z_{k-1}(\frac{G}{Z(\mu)}) &\iff [xZ(\mu), y_1Z(\mu),...,y_{k-1}Z(\mu)] = Z(\mu), \ \ for \ \ any \ \ y_1,...,y_{k-1} \in G \\ &\iff [x,y_1,...,y_{k-1}]Z(\mu) = Z(\mu), \ \ for \ \ any \ \ y_1,...,y_{k-1} \in G \\ &\iff [x,y_1,...,y_{k-1}] \in Z(\mu), \ \ for \ \ any \ \ y_1,...,y_{k-1} \in G \\ &\iff \mu[x,y_1,...,y_{k-1},y_k] = \mu(e), \ \ for \ \ any \ \ y_k \in G \ \ (by \ \ Theorem \ \ 4) \\ &\iff x \in Z_k(\mu) \ \ (by \ \ Lemma \ \ 1) \\ &\iff xZ(\mu) \in \frac{Z_k(\mu)}{Z(\mu)}. \end{aligned}$$

Therefore $\frac{Z_k(\mu)}{Z(\mu)} = Z_{k-1}(\frac{G}{Z(\mu)})$. $\square$

Lemma 3. *Let μ be a fuzzy subgroup of G, $H = \frac{G}{Z(\mu)}$, $\overline{\mu}$ be a fuzzy subgroup of H and $N = Z(\overline{\mu})$. If H is nilpotent, then $\frac{H}{N}$ is nilpotent, too.*

Proof. Let H be nilpotent of class n, that is $Z_n(H) = H$. We will prove that there exist $m \leq n$ such that $Z_m(\frac{H}{N}) = \frac{H}{N}$. For this by Theorem 5, since $\frac{H}{N}$ is a homomorphic image of H, we get $\frac{H}{N}$ is nilpotent of class at most m. $\square$

Theorem 8. *Let μ be a fuzzy subgroup of G and $\overline{\mu}$ be a fuzzy subgroup of $\frac{G}{Z(\mu)}$. If μ is a g-nilpotent fuzzy subgroup of class n, then $\overline{\mu}$ is a g-nilpotent fuzzy subgroup of class m, where $m \leq n$.*

Proof. Let μ be a g-nilpotent fuzzy subgroup of class n. Then $Z_n(\mu) = G$. Now we show that there exists $m \leq n$, such that $Z_m(\overline{\mu}) = \frac{G}{Z(\mu)}$. By Lemma 2, $Z_n(\mu) = G \Longleftrightarrow \frac{G}{Z(\mu)} = \frac{Z_n(\mu)}{Z(\mu)} = Z_{n-1}(\frac{G}{Z(\mu)})$, and similarly (put m instead of n and $\overline{\mu}$ instead of μ),

$$Z_m(\overline{\mu}) = \frac{G}{Z(\mu)} \Longleftrightarrow Z_{m-1}(\frac{\frac{G}{Z(\mu)}}{Z(\overline{\mu})}) = \frac{\frac{G}{Z(\mu)}}{Z(\overline{\mu})}$$

Consequently, it is enough to show that if $Z_{n-1}(\frac{G}{Z(\mu)}) = \frac{G}{Z(\mu)}$, then

$$Z_{m-1}(\frac{\frac{G}{Z(\mu)}}{Z(\overline{\mu})}) = \frac{\frac{G}{Z(\mu)}}{Z(\overline{\mu})}$$

It follows by Lemma 3 (put $H = \frac{G}{Z(\mu)}$ in Lemma 3).
$\square$

We now consider homomorphic images and the homomorphic pre-image of g-nilpotent fuzzy subgroups.

Theorem 9. *Let H be a group, $f : G \longrightarrow H$ be an epimorphism and μ be a fuzzy subgroup of G. If μ is a g-nilpotent fuzzy subgroup, then $f(\mu)$ is a g-nilpotent fuzzy subgroup.*

Proof. First, we show that $f(Z_i(\mu)) \subseteq Z_i(f(\mu))$, for any $i \in \mathbb{N}$. Let $i \in \mathbb{N}$. Then $x \in f(Z_i(\mu))$ implies that $x = f(u)$, for some $u \in Z_i(\mu)$. Since f is epimorphism, hence for any $y_1, ..., y_n \in H$ we get $y_i = f(v_i)$ for some $v_i \in G$ where $1 \leq i \leq n$. Therefore $[x, y_1, ..., y_n] = [f(u), f(v_1), ..., f(v_n)]$ which implies that

$$(f(\mu))([x, y_1, ..., y_n]) = \bigvee_{f(z)=[x,y_1,...,y_n]} \mu(z) = \bigvee_{f(z)=f([u,v_1,...,v_n])} \mu(z)$$

Now, since $u \in Z_i(\mu)$, by Lemma 1, we get $\mu([u, v_1, ..., v_n]) = \mu(e_G)$. Therefore,

$$(f(\mu))([x, y_1, ..., y_n]) = \mu(e_G) = (f(\mu))(e_H)$$

Hence by Lemma 1, $x \in Z_i(f(\mu))$. Consequently, $f(Z_i(\mu)) \subseteq Z_i(f(\mu))$. Hence if μ is g-nilpotent, then there exists nonnegative integer n such that $Z_n(\mu) = G$ which implies that $f(Z_n(\mu)) = f(G)$. Therefore $Z_n(f(\mu)) = H$ which implies that $f(\mu)$ is g-nilpotent. $\square$

Theorem 10. *Let H be a group, $f : G \longrightarrow H$ be an epimorphism and ν be a fuzzy subgroup of H. Then ν is a g-nilpotent fuzzy subgroup if and only if $f^{-1}(\nu)$ is a g-nilpotent fuzzy subgroup.*

Proof. First, we show that $Z_i(f^{-1}(\nu)) = f^{-1}(Z_i(\nu))$, for any $i \in \mathbb{N}$. Now, let $i \in \mathbb{N}$. Then by Lemma 1,

$$\begin{aligned} x \in Z_i(f^{-1}(\nu)) &\iff (f^{-1}(\nu))([x, x_1, ..., x_i]) = (f^{-1}(\nu))(e), \ \ for \ \ any \ \ x_1, x_2, ..., x_i \in G \\ &\iff \nu([f(x), ..., f(x_i)]) = \nu(e), \ \ for \ \ any \ \ x_1, x_2, ..., x_i \in G \\ &\iff f(x) \in Z_i(\nu), \\ &\iff x \in f^{-1}(Z_i(\nu)). \end{aligned}$$

Hence ν is g-nilpotent if and only if there exists nonnegative integer n such that $Z_n(\nu) = H$ if and only if $f^{-1}(Z_n(\nu)) = f^{-1}(H)$ if and only if $Z_n(f^{-1}(\nu)) = G$ if and only if, $f^{-1}(\nu)$ is g-nilpotent. □

Proposition 1. *Let μ and ν be two fuzzy subgroups of G such that $\mu \subseteq \nu$ and $\mu(e) = \nu(e)$. Then $Z(\mu) \subseteq Z(\nu)$.*

Proof. Let $x \in Z(\mu)$. Then $\mu([x, y]) = \mu(e)$, for any $y \in G$. Since

$$\nu(e) = \mu(e) = \mu([x, y]) \leq \nu([x, y]) \leq \nu(e).$$

hence $\nu(e) = \nu([x, y])$ and so $x \in Z(\nu)$. Therefore $Z(\mu) \subseteq Z(\nu)$. □

Lemma 4. *Let μ be a fuzzy subgroup of G and $i > 1$. Then for any $y \in G$, $[x, y] \in Z_{i-1}(\mu)$ if and only if $x \in Z_i(\mu)$.*

Proof. ($\Longrightarrow$) Let $[x, y] \in Z_{i-1}(\mu)$. Then by Lemma 1, $\mu([[x, y], y_1, ..., y_{i-1}]) = \mu(e)$ for any $y, y_1, ..., y_{i-1} \in G$. Hence $x \in Z_i(\mu)$.
($\Longleftarrow$) The proof is similar. □

In the following we see a relation between nilpotency of a group and its fuzzy subgroups.

Theorem 11. *G is nilpotent if and only if any fuzzy subgroup μ of G is a g-nilpotent fuzzy subgroup.*

Proof. ($\Longrightarrow$) Let G be nilpotent of class n and μ be a fuzzy subgroup of G. Since $Z_n(G) = G$, it is enough to prove that for any nonnegative integer i, $Z_i(G) \subseteq Z_i(\mu)$. For i=0 or 1, the proof is clear. Let for $i > 1$, $Z_i(G) \subseteq Z_i(\mu)$ and $x \in Z_{i+1}(G)$. Then for any $y \in G$, $[x, y] \in Z_i(G) \subseteq Z_i(\mu)$ and so by Lemma 4, $x \in Z_{i+1}(\mu)$. Hence $Z_{i+1}(G) \subseteq Z_{i+1}(\mu)$, for any $i \geq 0$, and this implies that $Z_n(\mu) = G$. Therefore, μ is g-nilpotent.
($\Longleftarrow$) Let any fuzzy subgroups of G be g-nilpotent. Suppose that fuzzy set μ on G is defined as follows:

$$\mu(x) = \begin{cases} 1 & if \quad x \in Z_0(G) \\ \frac{1}{i+1} & if \quad x \in Z_i(G) - Z_{i-1}(G) \\ 0 & otherwise \end{cases}$$

We show that $Z_i(\mu) \subseteq Z_i(G)$, for any nonnegative integer i. For $i = 0$, the result is immediate. If $i = 1$ and $x \in Z_1(\mu)$, then $\mu([x, y]) = \mu(e) = 1$ for any $y \in G$. By definition of μ, $[x, y] \in Z_0(G) = \{e\}$ and so $x \in Z_1(G)$. Now let $Z_{i-1}(\mu) \subseteq Z_{i-1}(G)$, for $i \geq 2$. Then by Lemma 4, $x \in Z_i(\mu)$ implies that for any $y \in G$; $[x, y] \in Z_{i-1}(\mu) \subseteq Z_{i-1}(G)$. Hence, for any $y, y_1, ..., y_{i-1} \in G$, $[x, y, y_1, ..., y_{i-1}] = e$ which implies that $x \in Z_i(G)$. Thus by induction on i, $Z_i(\mu) \subseteq Z_i(G)$, for any nonnegative integer i. Now since $Z_i(G) \subseteq Z_i(\mu)$ for any nonnegative integer i, then $Z_i(\mu) = Z_i(G)$. Now by the hypotheses there exist $n \in \mathbb{N}$ such that $G = Z_n(\mu) = Z_n(G)$. Hence, G is nilpotent. □

Theorem 12. *Let fuzzy subgroups μ_1 and μ_2 of G be g-nilpotent fuzzy subgroups. Then the fuzzy set $\mu_1 \times \mu_2$ of $G \times G$ is a g-nilpotent fuzzy subgroup, too.*

Proof. Let $\mu = \mu_1 \times \mu_2$. It is clear that μ is fuzzy subgroup of G. So we show that μ is g-nilpotent. It is enough to show that $Z_n(\mu_1 \times \mu_2) = G \times G$, for $n \in \mathbb{N}$. Suppose that $(x, y) \in G \times G$. Then there exist $n_1, n_2 \in \mathbb{N}$ such that $Z_{n_1}(\mu_1) = G$ and $Z_{n_2}(\mu_2) = G$. Hence for any $x_1 ..., x_n, y_1 ..., y_n \in G$, $\mu_1([x, x_1 ..., x_n]) = \mu(e)$ and $\mu_2([y, y_1 ..., y_n]) = \mu(e)$ for $n = max\{n_2, n_1\}$. Then

$$(\mu_1 \times \mu_2)([(x, y), ..., (x_n, y_n)]) = min\{\mu_1[x, x_1 ..., x_n], \mu_2[y, y_1 ..., y_n]\} = (\mu_1 \times \mu_2)(e, e).$$

Therefore, $Z_n(\mu_1 \times \mu_2) = G \times G$. $\square$

Definition 8. *Let μ be a normal fuzzy subgroup of G. For any $x, y \in G$, define a binary relation on G as follows*

$$x \sim y \Longleftrightarrow \mu(xy^{-1}) = \mu(e)$$

Lemma 5. *Binary relation $\sim$ in Definition 8, is a congruence relation.*

Proof. The proof of reflexivity and symmetrically is clear. Hence, we prove the transitivity. Let $x \sim y$ and $y \sim z$, for $x, y, z \in G$. Then $\mu(xy^{-1}) = \mu(yz^{-1}) = \mu(e)$. Since μ is a fuzzy subgroup of G, then $\mu(xz^{-1}) \geq min\{\mu(xy^{-1}), \mu(yz^{-1})\} = \mu(e)$. Hence $\mu(xz^{-1}) = \mu(e)$ and so $x \sim z$. Therefore $\sim$ is an equivalence relation. Now let $x \sim y$ and $z \in G$. Then $\mu((xz)(yz)^{-1}) = \mu(xy^{-1}) = \mu(e)$ and so $xz \sim yz$. Since μ is normal, we get $\mu((zx)(zy)^{-1}) = \mu((zy)^{-1}(zx)) = \mu(y^{-1}x) = \mu(xy^{-1}) = \mu(e)$ and so $zx \sim zy$. Therefore, $\sim$ is a congruence relation on G. $\square$

Notation. For the congruence relation in Definition 8, for any $x \in G$, the equivalence class containing x is denoted by $x\mu$, and $\frac{G}{\mu} = \{x\mu \mid x \in G\}$. It is easy to prove that $\frac{G}{\mu}$ by the operation $(x\mu).(y\mu) = xy\mu$ for any $x\mu, y\mu \in \frac{G}{\mu}$ is a group, where $e\mu$ is unit of $\frac{G}{\mu}$ and $(x\mu)^{-1} = x^{-1}\mu$, for any $x\mu \in \frac{G}{\mu}$.

Theorem 13. *Let μ be a normal fuzzy subgroup of G. Then μ is a g-nilpotent fuzzy subgroup if and only if $\frac{G}{\mu}$ is a nilpotent group.*

Proof. ($\Longrightarrow$) Let μ be a g-nilpotent fuzzy subgroup of G. First we show that for any $n \in \mathbb{N}$ and $x_1, ..., x_n \in G$, $[x, x_1, ..., x_n]\mu = [x\mu, x_1\mu, ..., x_n\mu]$. For $n = 1$, we have

$$[x, x_1]\mu = (x^{-1}\mu).((x_1)^{-1}\mu).(x\mu).(x_1\mu) = [x\mu, x_1\mu]$$

Now assume that it is true for $n - 1$. By hypotheses of induction, we have

$$\begin{aligned} [x, x_1, ..., x_n]\mu &= ([x, x_1, ..., x_{n-1}]^{-1}\mu).(x_n^{-1}\mu).([x, x_1, ..., x_{n-1}]\mu).(x_n\mu) \\ &= ([x\mu, x_1\mu ..., x_{n-1}\mu]^{-1}).(x_n^{-1}\mu).([x\mu, x_1\mu ..., x_{n-1}\mu]).(x_n\mu) \\ &= [x\mu, x_1\mu, ..., x_n\mu]. \end{aligned}$$

Therefore, if μ is a g-nilpotent fuzzy subgroup then there exist $n \in \mathbb{N}$; $Z_n(\mu) = G$, which implies by Lemma 1, that

$$\{x \in G \mid \mu[x, x_1, ..., x_n] = \mu(e) \;\; for \;\; any \;\; x_1, x_2, x_3, ..., x_n \in G\} = G \, , (I)$$

Also $\mu(x) = \mu(e)$ if and only if $x \sim e$ if and only if $x\mu = e\mu$, (II). Thus, by (I) and (II) we have

$$\begin{aligned} \frac{G}{\mu} &= \{x\mu \mid x \in G\} = \{x\mu \mid \mu[x, x_1, ..., x_n] = \mu(e),\ \forall\ x_1, x_2, x_3, ..., x_n \in G\} \\ &= \{x\mu \mid [x\mu, x_1\mu, ..., x_n\mu] = e\mu,\ \forall x_1, x_2, x_3, ..., x_n \in G\} = Z_n(\frac{G}{\mu}) \end{aligned}$$

Consequently $\frac{G}{\mu}$ is a nilpotent group of class n.
$(\Longleftarrow)$ If $\frac{G}{\mu}$ is a nilpotent group of class n, then

$$\frac{G}{\mu} = Z_n(\frac{G}{\mu}) = \{x\mu \mid [x\mu, x_1\mu, ..., x_n\mu] = e\mu, \forall\ x_1, x_2, x_3, ..., x_n \in G\}\}$$

Thus for $x \in G$ we have $x\mu \in \frac{G}{\mu}$. Therefore $[x\mu, x_1\mu, ..., x_n\mu] = e\mu$ for any $x_1, x_2, x_3, ..., x_n \in G$ which implies by (II) that $\mu[x, x_1, ..., x_n] = \mu(e)$. Thus, by Lemma 1, $x \in Z_n(\mu)$. Thus $G = Z_n(\mu)$ and so μ is g-nilpotent. $\square$

Theorem 14. *Let μ be a fuzzy subgroup of G and $\mu_* = \{x \mid \mu(x) = \mu(e)\}$ be a normal subgroup of G. If $\frac{G}{\mu_*}$ is a nilpotent group, then μ is a g-nilpotent fuzzy subgroup.*

Proof. Let $\frac{G}{\mu_*}$ be a nilpotent group and $\pi : G \longrightarrow \frac{G}{\mu_*}$ be the natural epimomorphism. Since

$$\begin{aligned} z \in \pi^{-1}(\pi(x)) &\iff \pi(z) = \pi(x) \Longleftrightarrow \pi(z^{-1}x) = e \Longleftrightarrow z^{-1}x \in ker\pi = \mu_* \\ &\iff \mu(z^{-1}x) = \mu(e) \Longleftrightarrow \mu(z) = \mu(x). \end{aligned}$$

hence for any $x \in G$,

$$\pi^{-1}(\pi(\mu))(x) = \pi(\mu)(\pi(x)) = \bigvee_{z\in\pi^{-1}(\pi(x))} \mu(z) = \bigvee_{\mu(z)=\mu(x)} \mu(z) = \mu(x),$$

and so $\pi^{-1}(\pi(\mu)) = \mu$. Now since $\frac{G}{\mu_*}$ is a nilpotent group and $\pi(\mu)$ is a fuzzy subgroup of $\frac{G}{\mu_*}$, then by Theorem 11, $\pi(\mu)$ is g-nilpotent and by Theorem 10, $\pi^{-1}(\pi(\mu)) = \mu$ is g-nilpotent. $\square$

Example 2. *In Example 1, $\mu(e) = t_0$ and so $\mu_* = \{x \mid \mu(x) = \mu(e)\} = \langle a \rangle$. Thus μ_* is a normal subgroup of D_3. Also $\frac{D_3}{\mu_*} \approx \mathbb{Z}_2$. Since $\mathbb{Z}_2$ is Abelian hence it is nilpotent and so by Theorem 14, μ is a g-nilpotent fuzzy subgroup.*

Theorem 15. *Let μ and ν be two fuzzy subgroups of G such that $\mu \subseteq \nu$ and $\mu(e) = \nu(e)$. If μ is a g-nilpotent fuzzy subgroup of class m , then ν is a g-nilpotent fuzzy subgroup of class n, where $n \leq m$.*

Proof. Let μ and ν be two fuzzy subgroups of G where $\mu \subseteq \nu$ and $\mu(e) = \nu(e)$. First, we show that for any $i \in \mathbb{N}$, $Z_i(\mu) \subseteq Z_i(\nu)$. By Theorem 1, for $i = 1$ the proof is clear. Let for $i \geq 2$, $Z_i(\mu) \subseteq Z_i(\nu)$ and $x \in Z_{i+1}(\mu)$. Then by Lemma 4, for any $y \in G$, $[x, y] \in Z_i(\mu) \subseteq Z_i(\nu)$. Thus, by Lemma 4, $x \in Z_{i+1}(\nu)$. Hence $Z_{i+1}(\mu) \subseteq Z_{i+1}(\nu)$. Now let μ be g-nilpotent of class m. Then $G = Z_m(\mu) \subseteq Z_m(\nu) \subseteq G$. Thus $G = Z_m(\nu)$, which implies that ν is g-nilpotent of class at most m. $\square$

Definition 9. *[4] Let μ be a fuzzy set of a set S. Then the lower level subset is*

$$\overline{\mu}_t = \{x \in S; \mu(x) \leq t\}, \quad where \quad t \in [0, 1].$$

Now fuzzification of $\overline{\mu}_t$ is the fuzzy set $A_{\overline{\mu}_t}$ defined by

$$A_{\overline{\mu}_t}(x) = \begin{cases} \mu(x) & if \quad x \in \overline{\mu}_t \\ 0 & otherwise \end{cases}$$

Clearly, $A_{\overline{\mu}_t} \subseteq \mu$ and $\overline{(A_{\overline{\mu}_t})_t} = \overline{\mu}_t$.

Corollary 1. *Let μ be a nilpotent fuzzy subgroup of G. Then $A_{\overline{\mu}_t}$ is nilpotent too.*

Proof. Let μ be a nilpotent fuzzy subgroup of G, since $A_{\overline{\mu}_t} \subseteq \mu$ then by Theorem 15, $A_{\overline{\mu}_t}$ is nilpotent. $\square$

In the following we see that our definition for terms of $Z_k(\mu)$, is equivalent to an important relation, which will be used in the main Lemma 7.

Lemma 6. *Let μ be a fuzzy subgroup of G. For $k \geq 2$, $\frac{Z_k(\mu)}{Z_{k-1}(\mu)} = Z(\frac{G}{Z_{k-1}(\mu)})$ if and only if $[Z_k(\mu), G] \subseteq Z_{k-1}(\mu)$.*

Proof. ($\Longrightarrow$) Let for $k \geq 2$, $\frac{Z_k(\mu)}{Z_{k-1}(\mu)} = Z(\frac{G}{Z_{k-1}(\mu)})$ and $w \in [Z_k(\mu), G]$. Then there exist $x \in Z_k(\mu)$ and $g \in G$ such that $w = [x, g]$. Since

$$\begin{aligned} x \in Z_k(\mu) &\Longrightarrow xZ_{k-1}(\mu) \in \frac{Z_k(\mu)}{Z_{k-1}(\mu)} = Z(\frac{G}{Z_{k-1}(\mu)}) \\ &\Longrightarrow [xZ_{k-1}(\mu), gZ_{k-1}(\mu)] = Z_{k-1}(\mu), \;\; for \;\; any \;\; g \in G \\ &\Longrightarrow [x, g]Z_{k-1}(\mu) = Z_{k-1}(\mu), \;\; for \;\; any \;\; g \in G \\ &\Longrightarrow [x, g] \in Z_{k-1}(\mu). \end{aligned}$$

hence $w \in Z_{k-1}(\mu)$.
($\Longleftarrow$) Let for $k \geq 2$, $[Z_k(\mu), G] \subseteq Z_{k-1}(\mu)$ and $xZ_{k-1}(\mu) \in \frac{Z_k(\mu)}{Z_{k-1}(\mu)}$. Hence $x \in Z_k(\mu)$. Since $[Z_k(\mu), G] \subseteq Z_{k-1}(\mu)$, for any $g \in G$, we have $[x, g] \in Z_{k-1}(\mu)$ which implies that $[xZ_{k-1}(\mu), gZ_{k-1}(\mu)] = Z_{k-1}(\mu)$ and so $xZ_{k-1}(\mu) \in Z(\frac{G}{Z_{k-1}(\mu)})$. Hence $\frac{Z_k(\mu)}{Z_{k-1}(\mu)} \subseteq Z(\frac{G}{Z_{k-1}(\mu)})$. Now, let $xZ_{k-1}(\mu) \in Z(\frac{G}{Z_{k-1}(\mu)})$. Then for any $g \in G$ we have, $[xZ_{k-1}(\mu), gZ_{k-1}(\mu)] = Z_{k-1}(\mu)$ which implies that $[x, g]Z_{k-1}(\mu) = Z_{k-1}(\mu)$ and so $[x, g] \in Z_{k-1}(\mu)$. Now by Lemma 1, $\mu([x, g, y_1, y_2 ..., y_{k-1}]) = \mu(e)$, for any $g, y_1, y_2 ..., y_{k-1} \in G$. Hence $x \in Z_k(\mu)$ and this implies that $xZ_{k-1}(\mu) \in \frac{Z_k(\mu)}{Z_{k-1}(\mu)}$. So $\frac{Z_k(\mu)}{Z_{k-1}(\mu)} \supseteq Z(\frac{G}{Z_{k-1}(\mu)})$. Therefore, $\frac{Z_k(\mu)}{Z_{k-1}(\mu)} = Z(\frac{G}{Z_{k-1}(\mu)})$. $\square$

Lemma 7. *Let μ be a g-nilpotent fuzzy subgroup of G of class $n \geq 2$ and N, be a nontrivial normal subgroup of G (i.e $1 \neq N \trianglelefteq G$). Then $N \cap Z(\mu) \neq 1$.*

Proof. Since μ is g-nilpotent, so there exist $n \geq 2$ such that $Z_n(\mu) = G$. Thus

$$1 = Z_0(\mu) \subseteq Z_1(\mu) \subseteq ... \subseteq Z_n(\mu) = G$$

Since $N \cap Z_n(\mu) = N \cap G = N \neq 1$, then there is $j \in \mathbb{N}$ such that $N \cap Z_j(\mu) \neq 1$. Let i be the smallest index such that $N \cap Z_i(\mu) \neq 1$ (so $N \cap Z_{i-1}(\mu) = 1$). Then we claim that $[N \cap Z_i(\mu), G] \subseteq N$. For this let $w \in [N \cap Z_i(\mu), G]$. Then there exists $x \in N \cap Z_i(\mu)$ and $g \in G$ such that $w = [x, g] = x^{-1}g^{-1}xg$. Since $N \trianglelefteq G$, then $w = x^{-1}x^g \in N$. Thus $[N \cap Z_i(\mu), G] \subseteq N$. Also since $x \in N \cap Z_i(\mu)$, by Lemma 6, $[x, g] \in [Z_i(\mu), G] \subseteq Z_{i-1}(\mu)$. Thus $[N \cap Z_i(\mu), G] \subseteq Z_{i-1}(\mu)$. Hence $[N \cap Z_i(\mu), G] \subseteq N \cap Z_{i-1}(\mu) = 1$. Therefore $N \cap Z_i(\mu) \leq Z(G) \leq Z(\mu)$ and so $N \cap Z_i(\mu) \leq N \cap Z(\mu) = 1$. Hence $N \cap Z_i(\mu) = 1$ which is a contradiction. Consequently $N \cap Z(\mu) \neq 1$. $\square$

The following theorem shows that for a g-nilpotent fuzzy subgroup μ each minimal normal subgroup of G is contained in $Z(\mu)$.

Theorem 16. *Let μ be a g-nilpotent fuzzy subgroup of G of class $n \geq 2$. If N is a minimal normal subgroup of G, then $N \leq Z(\mu)$.*

Proof. Since N and $Z(\mu)$ are normal subgroups of G, we get $N \cap Z(\mu) \trianglelefteq G$. Now since N is a minimal normal subgroup of G, $N \cap Z(\mu) \leq N$ and by Lemma 7, $1 \neq N \cap Z(\mu)$. we get $N \cap Z(\mu) = N$. Therefore $N \leq Z(\mu)$. □

Theorem 17. *Let μ be a g-nilpotent fuzzy subgroup of G and A be a maximal normal Abelian subgroup of G. If $\mu(x) = \mu(e)$ for any $x \in A$, and $\mu(x) \neq \mu(e)$ for any $x \in G - A$, then*

$$A = C_G(A) = \{x \in G \mid [x, a] = e, \ \ for \ \ any \ \ a \in A\}.$$

Proof. First, we prove that $C_G(A) \trianglelefteq G$. For this, let $x \in C_G(A)$ and $g \in G$. Then for all $a \in A$ we have $[x^g, a] = [x, a^{g^{-1}}]^g$. Since A is Abelian, then $a^{g^{-1}} = a$. Hence $x \in C_G(A)$ implies that $[x^g, a] = [x, a^{g^{-1}}]^g = [x, a]^g = e$ and so $x^g \in C_G(A)$. Thus $C_G(A) \trianglelefteq G$. Suppose $A \subsetneq C_G(A)$. Then $1 \neq \frac{C_G(A)}{A} \trianglelefteq \frac{G}{A}$. Let $\overline{\mu}$ be the fuzzy subgroup of $\frac{G}{A}$. Then by Lemma 7, $\frac{C_G(A)}{A} \cap Z(\overline{\mu}) \neq 1$. So there exists $A \neq gA \in \frac{C_G(A)}{A} \cap Z(\overline{\mu})$. Hence $g \in C_G(A)$ and $\overline{\mu}[gA, xA] = \overline{\mu}(eA)$ for any $x \in G$. Thus by Theorem 2, $\bigvee\limits_{a \in A} \mu([g, x]a) = \mu(e)$. Now if for some $a \in A$, $\mu([g, x]) = \mu(a)$, then by definition of μ, $[g, x] \in A$ and if for any $a \in A$, $\mu([g, x]) \neq \mu(a)$, then by Theorem 2, $\bigvee\limits_{a \in A} \mu([g, x]a) = \bigvee\limits_{a \in A} (\mu([g, x]) \wedge \mu(a)) = \mu([g, x])$. Thus $[g, x] \in A$. Now let $B = \langle A, g \rangle$. Then $A \subsetneq B \trianglelefteq G(B \trianglelefteq G$ since $A \trianglelefteq G$, and for $x \in G$ we have $g^x = g[g, x] \in B$). Moreover, since $g \in C_G(A)$, then B is Abelian. Therefore, B is a normal Abelian subgroup of G, which is a contradiction. Thus $A = C_G(A)$.
□

Now we show that with some conditions every g-nilpotent fuzzy subgroup is finite.

Corollary 2. *Let μ be a g-nilpotent fuzzy subgroup of G and A be a finite maximal normal Abelian subgroup of G. If $\mu(x) = \mu(e)$ for any $x \in A$, and $\mu(x) \neq \mu(e)$ for any $x \in G - A$, then μ is finite, too.*

Proof. Since $A \trianglelefteq G$, for $g \in G$ and $x \in A$ we have $x^g \in A$. Now let

$$\begin{aligned} \theta \quad : \quad & G \longrightarrow Aut(A) \\ & g \longrightarrow \theta_g : A \longrightarrow A \\ & \qquad\qquad x \longrightarrow x^g. \end{aligned}$$

We prove that θ is a homomorphism. Let $g_1, g_2 \in G$. Then $\theta(g_1 g_2) = \theta_{g_1 g_2}$. Thus for $x \in A$,

$$(\theta(g_1 g_2))(x) = (\theta_{g_1 g_2})(x) = x^{g_1 g_2} = x^{g_1}.x^{g_2} = (\theta(g_1))(x).(\theta(g_2))(x).$$

But $Ker(\theta) = \{g \in G \mid \theta(g) = I\}$, in which I is the identity homomorphism. Thus for any $x \in A$, $(\theta(g))(x) = I(x)$ which implies that $x^g = x$. Hence $g \in C_G(A)$. Therefore, $Ker(\theta) = C_G(A)$. By Theorem 17, $A = C_G(A)$. Thus $\frac{G}{Ker(\theta)} = \frac{G}{A}$ is embeded in $Aut(A)$. Now since A and so $Aut(A)$ are finite we get G is finite which implies that μ is finite. □

4. Conclusions

By the notion of a g-nilpotent fuzzy subgroup we can investigate on fuzzification of nilpotent groups. Moreover, since this is similar to group theory′ definition, it is much easier than before to study the properties of nilpotent fuzzy groups. Moreover, if we accept the definition of a g-nilpotent fuzzy subgroup, then one can verify, as we have done in Theorem 16, that for a g-nilpotent fuzzy subgroup μ each minimal normal subgroup of G is contained in the center of μ. We hope that these results inspire other papers on nilpotent fuzzy subgroups.

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